Wada, T.
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SCATTERING THEORY FOR TIME-DEPENDENT HARTREE-FOCK TYPE EQUATION

TAKESHI WADA

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1. Introduction

In this paper we consider the scattering problem for the following system of nonlinear Schrödinger equations with nonlocal interaction

$$
\begin{aligned}
(1)\qquad & i\frac{\partial}{\partial t}u_j &=& -\frac{1}{2}\Delta u_j + f_j(\vec{u}), \quad (t,x)\in \mathbf{R}\times\mathbf{R}^n,\\
(2)\qquad & u_j(0,x) &=& \phi_j(x), \qquad j=1,\cdots,N.
\end{aligned}
$$

Here Δ denotes the Laplacian in x,

$$
f_j(\vec{u}) = \sum_{k=1}^{N}(V*|u_k|^2)u_j - \sum_{k=1}^{N}[V*(u_j\bar{u}_k)]u_k,
$$

and $*$ denotes the convolution in $\mathbf{R}^n$. In this paper we treat the case $n\geq 2$ and $V(x)=|x|^{-\gamma}$ with $0<\gamma<n$.

The system (1)-(2) appears in the quantum mechanics as an approximation to a fermionic N-body system and is called the time-dependent Hartree-Fock type equation.

Throughout the paper we use the following notation:

$\mathbf{N}=\{1,2,3,\cdots\}$, $\nabla=(\partial/\partial x_1,\cdots,\partial/\partial x_n)$, $U(t)=\exp(it\Delta/2), M(t)=\exp(i|x|^2/2t)$, $J=U(t)xU(-t)=M(t)(it\nabla)M(-t)$. For $1\leq p\leq\infty$, $p'=p/(p-1)$, $\delta(p)=n/2-n/p$. $\|\cdot\|_p$ denotes the norm of $L^p(\mathbf{R}^n)$ (if $p=2$, we write $\|\cdot\|_2=\|\cdot\|$). For $1\leq q,r\leq\infty$ and for the interval $I\subset\mathbf{R}$, $\|\cdot\|_{q,r,I}$ denotes the norm of $L^r(I;L^q(\mathbf{R}^n))$, namely, $\|u\|_{q,r,I}=\left[\int_I\left(\int_{\mathbf{R}^n}|u(t,x)|^q dx\right)^{r/q}dt\right]^{1/r}$. For positive integers l and m, $\Sigma^{l,m}$ denotes the Hilbert space defined as

$$
\Sigma^{l,m}=\Big\{\psi\in L^2(\mathbf{R}^n); \|\psi\|_{\Sigma^{l,m}}=\big(\sum_{|\alpha|\leq l}\|\nabla^\alpha\psi\|^2+\sum_{|\beta|\leq m}\|x^\beta\psi\|^2\big)^{1/2}<\infty\Big\}.
$$

When we use N'th direct sums of various function spaces, we denote them by the same symbols and denote these elements by writing arrow over the letter, like $\vec{f}$.

Now we state our main theorem.

Theorem 1.1. (i) *Suppose that* $1 < \gamma < \min(4, n)$, *and* $l, m \in \mathbf{N}$. *Then for any* $\vec{\phi}^{(+)} \in \Sigma^{l,m}$, *there exists a unique* $\vec{\phi} \in \Sigma^{l,m}$ *such that*

$$\lim_{t \to +\infty} \|\vec{\phi}^{(+)} - U(-t)\vec{u}(t)\|_{\Sigma^{l,m}} = 0, \tag{3}$$

where $\vec{u}(t)$ *is the solution of* (1)-(2) *with* $U(-t)\vec{u}(t) \in C(\mathbf{R}; \Sigma^{l,m})$. *For any* $\vec{\phi}^{(-)} \in \Sigma^{l,m}$, *the same result as above holds valid with* $+\infty$ *replaced by* $-\infty$ *in* (3).

(ii) *Suppose that* $4/3 < \gamma < \min(4, n)$, *and* $l, m \in \mathbf{N}$. *And if* $\gamma \leq \sqrt{2}$, *suppose, in addition, that* $m \geq 2$. *Then for any* $\vec{\phi} \in \Sigma^{l,m}$, *there exist* $\vec{\phi}^{(\pm)} \in \Sigma^{l,m}$ *such that the solution of* (1)-(2) with $U(-t)\vec{u}(t) \in C(\mathbf{R}; \Sigma^{l,m})$ satisfies

$$\lim_{t \to \pm\infty} \|\vec{\phi}^{(\pm)} - U(-t)\vec{u}(t)\|_{\Sigma^{l,m}} = 0. \tag{4}$$

By Theorem 1.1 (i), if $1 < \gamma < \min(4, n)$, we can define the operator W_+ in $\Sigma^{l,m}$ as

$$W_+ : \vec{\phi}^{(+)} \longmapsto \vec{\phi},$$

which is called the wave operator. The operator W_- is defined similarly. Under the condition of $4/3 < \gamma < \min(4, n)$ ($m \geq 2$ if $\gamma \leq \sqrt{2}$), Theorem 1.1 (ii) implies the completeness of $W_\pm$, namely, $\mathrm{Range} W_\pm = \Sigma^{l,m}$.

There are many papers for the following equation

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + f(u), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n, \tag{5}$$

$$u(0, x) = \phi(x), \tag{6}$$

where

$$f(u) = [V * |u|^2]u = \int_{\mathbf{R}^n} |x - y|^{-\gamma} |u(t, y)|^2 dy \, u(t, x)$$

(see, for example, [5, 7, 8, 9, 12]). The equation (5)-(6) is called the Hartree type equation. For the scattering problem for (5)-(6), the following results are known (see [9]).

[A] *Suppose that* $1 < \gamma < \min(4, n)$, *and* $l, m \in \mathbf{N}$. *Then, for any* $\phi^{(+)} \in \Sigma^{l,m}$, *there exists a unique* $\phi \in \Sigma^{l,m}$ *such that*

$$\lim_{t \to +\infty} \|\phi^{(+)} - U(-t)u(t)\|_{\Sigma^{l,m}} = 0, \tag{7}$$

where $u(t)$ *is the solution of* (5)-(6) *with* $U(-t)u(t) \in C(\mathbf{R}; \Sigma^{l,m})$. *For any* $\phi^{(-)} \in \Sigma^{l,m}$, *the same result as above holds valid with* $+\infty$ *replaced by* $-\infty$ *in* (7).

[B] *Suppose that* $4/3 < \gamma < \min(4, n)$, *and* $l, m \in \mathbf{N}$. *Then, for any* $\phi \in \Sigma^{l,m}$, *there exist unique* $\phi^{(\pm)} \in \Sigma^{l,m}$ *such that the solution* $u(t)$ *of* (5)-(6) *with* $U(-t)u(t) \in C(\mathbf{R}; \Sigma^{l,m})$ *satisfies*

$$\lim_{t \to \pm\infty} \|\phi^{(\pm)} - U(-t)u(t)\|_{\Sigma^{l,m}} = 0. \tag{8}$$

Our main Theorem is the analogous results to [A], [B].

Since $U(t)$ is unitary in H^l, (4) implies that the asymptotic profiles of $\vec{u}(t)$ as $t \to \pm\infty$ are $U(t)\vec{\phi}^{(\pm)}$; and by the estimates

$$\|U(t)\vec{\phi}^{(\pm)}\|_p \leq (2\pi|t|)^{-\delta(p)} \|\vec{\phi}^{(\pm)}\|_{p'}, \quad 2 \leq p \leq \infty,$$

it is expected that

$$\|\vec{u}(t)\|_p = O(|t|^{-\delta(p)}) \tag{9}$$

as $t \to \pm\infty$. Indeed, in Corollary 4.1, we shall prove (9) for $p = \infty$ under the suitable condition for $\vec{\phi}$.

Conversely, if (9) holds for some p sufficiently large, We can prove Theorem 1.1(ii). Actually, in Propositions 3.1 and 3.2, we prove (9) for some $p > 2$. This decay estimate is the key point of our proof of the main theorem.

The proof of Theorems [B] is much more simple than our proof of Theorem 1.1 (ii). But we cannot apply the method in [9] for (5)-(6) to prove Theorem 1.1 (ii). So we shall use the method in our work [15] to prove the main theorem.

2. Preliminaries

First, we collect various inequalities which will be used in later sections.

Lemma 2.1. (The Gagliardo-Nirenberg inequality) *Let* $1 \leq q, r \leq \infty$ *and* j, m *be any integers satisfying* $0 \leq j < m$. *If* u *is any function in* $W^{m,q}(\mathbf{R}^n) \cap L^r(\mathbf{R}^n)$, *then*

$$\sum_{|\alpha|=j} \|\nabla^\alpha u\|_p \leq C \left(\sum_{|\beta|=m} \|\nabla^\beta u\|_q \right)^a \|u\|_r^{1-a} \tag{10}$$

where

$$\frac{1}{p} - \frac{j}{n} = a(\frac{1}{q} - \frac{m}{n}) + (1-a)\frac{1}{r}$$

for all a in the interval $j/m \le a \le 1$, where the constant C is independent of u, with the following exception: if $m - j - (n/q)$ is a nonnegative integer, then (10) *is asserted for $j/m \le a < 1$.*

For the proof of Lemma 2.1, see [3, 14].

Lemma 2.2. *Let $\alpha > 0$. Then*

$$\|(-\Delta)^{\alpha/2} fg\| \le C(\|(-\Delta)^{\alpha/2} f\| \; \|g\|_\infty + \|f\|_\infty \|(-\Delta)^{\alpha/2} g\|). \tag{11}$$

This lemma is essentially due to [4, 6]. The lemma is obtained as in the proof of Lemma 3.4 in [4] and Lemma 3.2 in [6], by using the theory of Besov space (for Besov space, see [1]).

Lemma 2.3. (The Hardy-Littlewood-Sobolev inequality) *Let $0 < \gamma < n, 1 < p, q < \infty$ and $1 + 1/p = \gamma/n + 1/q$. Then*

$$\| \, |x|^{-\gamma} * \phi\|_p \le C\|\phi\|_q. \tag{12}$$

For the proof, see [10, 13].

A pair (q, r) of real numbers is called admissible, if it satisfies the condition $0 \le \delta(q) = 2/r < 1$. Then

Lemma 2.4. *If a pair (q, r) is admissible, then for any $\psi \in L^2(\mathbf{R}^n)$, we have*

$$\|U(t)\psi\|_{q,r,\mathbf{R}} \le C\|\psi\|. \tag{13}$$

Lemma 2.5. *We put $(Gu)(t) = \int_{t_0}^{t} U(t-\tau)u(\tau)d\tau$. Let $I \subset \mathbf{R}$ be an interval containing t_0, and let pairs $(q_j, r_j), j = 1, 2$, be admissible. Then G maps $L^{r_1'}(I; L^{q_1'})$ into $L^{r_2}(I; L^{q_2})$ and satisfies*

$$\|Gu\|_{q_2,r_2,I} \le C\|u\|_{q_1',r_1',I}, \tag{14}$$

where C is independent of I.

For the proof of Lemmas 2.4 and 2.5, see [11, 16].

Next, we summarize the results for the Cauchy problem to (1)-(2) . We convert (1)-(2) into the integral equations

$$u_j(t) = U(t)\phi_j - i\int_0^t U(t-\tau)f_j(\vec{u}(\tau))d\tau, \quad j = 1, \cdots, N, \tag{15}$$

then

Proposition 2.1. (i) *Suppose that $n \geq 2$, $0 < \gamma < \min(4, n)$, and $l, m \in \mathbf{N}$. Then for any $\vec{\phi} \in H^l$, there exists a unique solution $\vec{u}(t) \in C(\mathbf{R}; H^l)$ of* (15). *The solution $\vec{u}(t)$ satisfies following equalities.*

$$(u_j(t), u_k(t)) = (\phi_j, \phi_k), \quad j, k = 1, \cdots, N, \tag{16}$$

especially,

$$\|u_j(t)\| = \|\phi_j\|, \quad j = 1, \cdots, N; \tag{17}$$

and

$$E(\vec{u}(t)) = E(\vec{\phi}), \tag{18}$$

where

$$\begin{aligned} E(\vec{\psi}) &= \sum_{j=1}^{N} \|\nabla\psi_j\|^2 + P(\vec{\psi}), \\ P(\vec{\psi}) &= \sum_{j,k=1}^{N} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |x-y|^{-\gamma}(|\psi_j(x)|^2|\psi_k(y)|^2 - \psi_j(x)\bar{\psi}_k(x)\psi_k(y)\bar{\psi}_j(y))dxdy; \end{aligned}$$

(ii) *Furthermore, if $\vec{\phi} \in \Sigma^{l,m}$, then $U(-t)\vec{u}(t) \in C(\mathbf{R}; \Sigma^{l,m})$, and the solution $\vec{u}(t)$ satisfies*

$$\sum_{j=1}^{N} \|xU(-t)u_j(t)\|^2 + t^2P(\vec{u}(t)) = \sum_{j=1}^{N} \|x\psi_j\|^2 + (2-\gamma)\int_0^t \tau P(\vec{u}(\tau))d\tau. \tag{19}$$

REMARK. (i) By the Cauchy-Schwarz inequality, $P(\vec{\psi}) \geq 0$.
(ii) The equalities (17), (18) and (19) are called the L^2-norm, the energy, and the pseudo-conformal conservation laws, respectively.

The proof of Propositions 2.1 is similar to that of the corresponding result for (5)-(6), so we shall omit it (see, for example, [8, 9, 12]).

3. Decay estimates for some norm of the solution

In this section we shall estimate the L^p-norm of the solution $\vec{u}(t)$ of (1)-(2) to prove the main theorem. We use the following transform

$$\begin{aligned} v_j(t) &= \mathcal{F}M(t)U(-t)u_j(t) \\ &= (it)^{n/2}\exp(-it|x|^2/2)u_j(t,tx), \end{aligned}$$

where $\mathcal{F}$ is the Fourier transform in $\mathbf{R}^n$. This transform was introduced by N. Hayashi and T. Ozawa [7]. Then the equations (1) are transformed into the equations

$$i\frac{\partial}{\partial t}v_j = -\frac{1}{2t^2}\Delta v_j + \frac{1}{t^\gamma}f_j(\vec{v}), \quad j = 1, \cdots, N, \tag{20}$$

and if $\vec{\phi} \in \Sigma^{1,m}$, then $\vec{v}(t) \in C((0,\infty);\Sigma^{m,1})$. The relations (17) and (19) are equivalent to

$$\frac{d}{dt}\|v_j(t)\| = 0, \quad j = 1, \cdots, N \tag{21}$$

and

$$t^{-2}\frac{d}{dt}\sum_{j=1}^N \|\nabla v_j(t)\|^2 + t^{-\gamma}\frac{d}{dt}P(\vec{v}(t)) = 0, \tag{22}$$

respectively. Using the relation (22) , we show

Lemma 3.1. *Suppose that $n \geq 2$, $0 < \gamma < \min(4,n)$, and $\vec{\phi} \in \Sigma^{1,1}$. Then, for* $t \geq 1$,

$$\sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq \begin{cases} Ct^{2-\gamma} & \text{if } \gamma \leq \sqrt{2}, \\ C & \text{if } \gamma > \sqrt{2}. \end{cases} \tag{23}$$

Here, the constants C depend on $\|\vec{\phi}\|_{\Sigma^{1,1}}$.

Proof. If $\gamma < 2$,

$$\frac{d}{dt}\left(t^{\gamma-2}\sum_{j=1}^N \|\nabla v_j(t)\|^2 + P(\vec{v}(t))\right) = (\gamma-2)t^{\gamma-3}\sum_{j=1}^N \|\nabla v_j(t)\|^2 \leq 0,$$

and if $\gamma \geq 2$,

$$\frac{d}{dt}\left(\sum_{j=1}^{N} \|\nabla v_j(t)\|^2 + t^{2-\gamma} P(\vec{v}(t))\right) = (2-\gamma)t^{1-\gamma} P(\vec{v}(t)) \leq 0.$$

Hence

$$\sum_{j=1}^{N} \|\nabla v_j(t)\|^2 \leq \begin{cases} Ct^{2-\gamma} & \text{if } \gamma < 2, \\ C & \text{if } \gamma \geq 2. \end{cases} \tag{24}$$

So we shall prove (23) when $\sqrt{2} < \gamma < 2$. We multiply (20) by $\Delta \bar{v}_j$, and integrate the imaginary part over $\mathbf{R}^n$. Then

$$\frac{1}{2}\frac{d}{dt}\|\nabla v_j(t)\|^2 = t^{-\gamma} \mathrm{Im} \int_{\mathbf{R}^n} f_j(\vec{v}) \Delta \bar{v}_j dx.$$

Since $\mathrm{Im} \int_{\mathbf{R}^n} V * |v_k|^2 |\nabla v_j|^2 dx$ and $\mathrm{Im} \sum_{j,k=1}^{N} \int_{\mathbf{R}^n} V * (v_j \bar{v}_k) \nabla v_k \cdot \nabla \bar{v}_j dx$ are equal to zero, we have, by Hölder's inequality and Lemma 2.3,

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\sum_{j=1}^{N} \|\nabla v_j(t)\|^2 \\ = \; & t^{-\gamma} \mathrm{Im} \sum_{j,k=1}^{N} \left[\int_{\mathbf{R}^n} v_j \nabla (V * |v_k|^2) \cdot \nabla \bar{v}_j dx + \int_{\mathbf{R}^n} v_k \nabla (V * (v_j \bar{v}_k)) \cdot \nabla \bar{v}_j dx\right] \\ \leq \; & Ct^{-\gamma} \|\vec{v}(t)\|_\rho^2 \sum_{j=1}^{N} \|\nabla v_j(t)\|^2, \end{aligned}$$

where $\rho = 2n/(n-\gamma)$. By Lemma 2.1 and (24), we have

$$\begin{aligned} \|v_j(t)\|_\rho & \leq C\|v_j\|^{1-\gamma/2} \|\nabla v_j\|^{\gamma/2} \\ & \leq Ct^{(2\gamma-\gamma^2)/4}. \end{aligned}$$

Therefore,

$$\frac{d}{dt}\sum_{j=1}^{N} \|\nabla v_j(t)\|^2 \leq Ct^{-\gamma^2/2} \sum_{j=1}^{N} \|\nabla v_j(t)\|^2. \tag{25}$$

Since $\gamma^2/2 > 1$ if $\gamma > \sqrt{2}$, (25) and Gronwall's inequality yield (23).

Lemma 3.1 immediately implies

Proposition 3.1. *Suppose that* $\sqrt{2} < \gamma < \min(4, n)$, *and* $\vec{\phi} \in \Sigma^{1,1}$. *Then for the number* p *satisfying* $0 \leq \delta(p) \leq 1$ *if* $n \geq 3$ *and* $0 \leq \delta(p) < 1$ *if* $n = 2$, *the solution of* (1)-(2) *has the estimate*

$$\|\vec{u}(t)\|_p \leq C(1 + |t|)^{-\delta(p)}. \tag{26}$$

Proof. Since $\|\vec{u}(t)\|_p = t^{-\delta(p)}\|\vec{v}(t)\|_p$, Lemma 2.1 and Lemma 3.1 yield (26).

Now we show the L^p decay estimate of the solution in case $1 < \gamma \leq \sqrt{2}$.

Lemma 3.2. *Suppose that* $1 < \gamma \leq \sqrt{2}$ *and* $\vec{\phi} \in \Sigma^{1,2}$. *Then we have for* $t \geq 1$,

$$\sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \leq \begin{cases} Ct^{(\gamma^2-8\gamma+10)/(2-\gamma)} & \text{if } n \geq 3, \\ Ct^{(\gamma^2-8\gamma+10)/(2-\gamma)+\varepsilon} & \text{if } n = 2. \end{cases} \tag{27}$$

Here ε *is a positive number which can be chosen arbitrarily small, and the constant* C *depends on* $\|\vec{\phi}\|_{\Sigma^{1,2}}$, *and* ε (*the case* $n = 2$).

Proof. We apply Δ to the both side of (20) and obtain

$$i\frac{\partial}{\partial t}\Delta v_j = -\frac{1}{2t^2}\Delta^2 v_j + \frac{1}{t^\gamma}\Delta f_j(\vec{v}), \quad j = 1, \cdots, N. \tag{28}$$

Multiplying (28) by $\Delta\bar{v}_j$, integrating the imaginary part over $\mathbf{R}^n$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta v_j(t)\|^2 = \frac{1}{t^\gamma}\mathrm{Im}\int_{\mathbf{R}^n} \Delta f_j(\vec{v})\Delta\bar{v}_j dx.$$

Since $\mathrm{Im}\int_{\mathbf{R}^n} V * |v_k|^2 |\Delta v_j|^2 dx$ and $\mathrm{Im}\sum_{j,k=1}^{N}\int_{\mathbf{R}^n} V * (v_j\bar{v}_k)\Delta v_k \Delta\bar{v}_j dx$ are equal to zero,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{N}\|\Delta v_j(t)\|^2 \\ = \; & t^{-\gamma}\mathrm{Im}\sum_{j,k=1}^{N}\left[\int_{\mathbf{R}^n}\Delta(V * |v_k|^2)v_j\Delta\bar{v}_j dx + 2\int_{\mathbf{R}^n}\nabla(V * |v_k|^2)\cdot\nabla v_j\Delta\bar{v}_j dx \right. \\ & \left. + \int_{\mathbf{R}^n}\Delta(V * (v_j\bar{v}_k))v_k\Delta\bar{v}_j dx + 2\int_{\mathbf{R}^n}\nabla(V * (v_j\bar{v}_k))\cdot\nabla v_k\Delta\bar{v}_j dx\right]. \end{aligned} \tag{29}$$

(i) Case $n \geq 3$. Hölder's inequality, Lemma 2.1 and Lemma 2.3 imply that the first term in the brackets of the right of (29) is dominated by

$$\begin{aligned} & C\int_{\mathbf{R}^n} |x|^{-\gamma-1} * (|\nabla v_k|\ |v_k|)\ |v_j|\ |\Delta v_j| dx \\ \leq\ & C\|\nabla v_k\|\ \|v_k\|_{2n/(n-2\gamma)}\|v_j\|_{2n/(n-2)}\|\Delta v_j\| \\ \leq\ & C(\sum_{j=1}^N \|\nabla v_j\|)^{4-\gamma}(\sum_{j=1}^N \|\Delta v_j\|^2)^{\gamma/2}. \end{aligned}$$

The other terms are estimated similarly. Therefore, it follows from (23) that for $t \geq 1$,

$$\begin{aligned} \frac{d}{dt}\sum_{j=1}^N \|\Delta v_j(t)\|^2 \ &\leq\ Ct^{-\gamma}\left(\sum_{j=1}^N \|\nabla v_j\|\right)^{4-\gamma}\left(\sum_{j=1}^N \|\Delta v_j(t)\|^2\right)^{\gamma/2} \\ &\leq\ Ct^{(8-8\gamma+\gamma^2)/2}\left(\sum_{j=1}^N \|\Delta v_j(t)\|^2\right)^{\gamma/2}. \end{aligned} \tag{30}$$

Integrating this differential inequality, we have

$$\left(\sum_{j=1}^N \|\Delta v_j(t)\|^2\right)^{1-\gamma/2} \leq Ct^{(10-8\gamma+\gamma^2)/2} + \left(\sum_{j=1}^N \|\Delta v_j(1)\|^2\right)^{1-\gamma/2}, \tag{31}$$

which implies (27). Since $\|\Delta v_j(1)\| = \|\ |x|^2 U(-1)u(1)\| \leq C\|\vec{\phi}\|_{\Sigma^{1,2}}$, the constant C in (23) depends on $\|\vec{\phi}\|_{\Sigma^{1,2}}$.

(ii) Case $n = 2$. Since

$$V* = \frac{2^{n-\gamma}\pi^{n/2}\Gamma(\frac{n-\gamma}{2})}{\Gamma(\frac{\gamma}{2})}(-\Delta)^{(\gamma-n)/2}, \quad 0 < \gamma < n,$$

we have for $n = 2$, $-\Delta V* = C(-\Delta)^{\gamma/2}$. Hence, by using Hölder's inequality, Lemma 2.1 and Lemma 2.2, we can estimate the first term in the brackets of the right of (29) by

$$\begin{aligned} & C\|(-\Delta)^{\gamma/2}|v_k|^2\|\ \|v_j\|_\infty\|\Delta v_j\| \\ \leq\ & C\|(-\Delta)^{\gamma/2}v_k\|\ \|\vec{v}\|_\infty^2\|\Delta v_j\| \\ \leq\ & C\|\vec{v}\|_\infty^2\left(\sum_{j=1}^N \|\nabla v_j\|\right)^{2-\gamma}\left(\sum_{j=1}^N \|\Delta v_j\|^2\right)^{\gamma/2}. \end{aligned} \tag{32}$$

Since Lemma 2.1 implies

$$\begin{aligned} \|v_k\|_\infty\ &\leq\ C\|\Delta v_k\|^{2/(\theta+2)}\|v_k\|_\theta^{\theta/(\theta+2)} \\ &\leq\ C\|v_k\|^{2/(\theta+2)}\|\nabla v_k\|^{(\theta-2)/(\theta+2)}\|\Delta v_k\|^{2/(\theta+2)}, \end{aligned}$$

where $2 \leq \theta < \infty$, the right of (32) is dominated by

$$C\|v\|^a \left(\sum_{j=1}^{N} \|\nabla v_j\| \right)^{4-\gamma-2a} \left(\sum_{j=1}^{N} \|\Delta v_j\|^2 \right)^{(\gamma+a)/2}$$

with $a = 2/(\theta + 2)$. The second term in the brackets of the right of (29) is estimated by

$$\begin{aligned} & \|V * (|\nabla v_k||v_k|)\|_{n/(\gamma-1)} \|\nabla v_j\|_{2n/(n-\gamma+1)} \|\Delta v_j\| \\ \leq \ & C\|\vec{v}\|_\infty \left(\sum_{j=1}^{N} \|\nabla v_j\| \right)^{3-\gamma} \left(\sum_{j=1}^{N} \|\Delta v_j\|^2 \right)^{\gamma/2} \\ \leq \ & C\|\vec{v}\|^a \left(\sum_{j=1}^{N} \|\nabla v_j\| \right)^{4-\gamma-2a} \left(\sum_{j=1}^{N} \|\Delta v_j\|^2 \right)^{(\gamma+a)/2} \end{aligned}$$

The other terms are estimated similarly. Therefore, we have

$$\frac{d}{dt} \sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \leq Ct^{(8-8\gamma+\gamma^2)/2} \left(\sum_{j=1}^{N} \|\Delta v_j(t)\|^2 \right)^{(\gamma+a)/2} \tag{33}$$

Since the number a can be chosen arbitrarily small, this differential equation implies (27).

Lemma 3.3. *Suppose that $n \geq 2$, $1 < \gamma \leq \sqrt{2}$ and $\vec{\phi} \in \Sigma^{1,2}$. Then we have for $t \geq 1$,*

$$\|\vec{v}(t)\|_p \leq C. \tag{34}$$

Here, p satisfies $0 < \delta(p) < (\gamma - 1)(2 - \gamma)/(6 - 4\gamma)$, and the constant C depends on $\|\vec{\phi}\|_{\Sigma^{1,2}}$.

Proof. For simplicity, we prove the lemma in case $n \geq 3$. We put $\|\vec{v}\|_{p,*} = [\int_{\mathbf{R}^n} (\sum_{l=1}^{N} |v_l|^2)^{p/2}\, dx]^{1/p}$, which is equivalent to the norm $\|\vec{v}\|_p = \sum_{l=1}^N \|v_l\|_p$. We multiply the equation (20) by $(\sum_{l=1}^N |v_l|^2)^{(p-2)/2}\bar{v}_j$, integrate their imaginary part over $\mathbf{R}^n$, and add them. Then we have

$$\frac{1}{p}\frac{d}{dt}\|\vec{v}(t)\|_{p,*}^p = -\frac{1}{2t^2}\mathrm{Im} \sum_{j=1}^{N} \int_{\mathbf{R}^n} \Delta v_j \left(\sum_{l=1}^{N} |v_l|^2 \right)^{(p-2)/2} \bar{v}_j dx, \tag{35}$$

since $\mathrm{Im}\int_{\mathbf{R}^n} V*|v_k|^2\left(\sum_{l=1}^{N}|v_l|^2\right)^{(p-2)/2}|v_j|^2dx$ and $\mathrm{Im}\sum_{j,k=1}^{N}\int_{\mathbf{R}^n} V*(v_j\bar{v}_k)v_k\bar{v}_j\left(\sum_{l=1}^{N}|v_l|^2\right)^{(p-2)/2}dx$ are equal to zero. By the integration by parts and Hölder's inequality,

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\vec{v}(t)\|_{p,*}^p &= \frac{1}{2t^2}\mathrm{Im}\sum_{j=1}^{N}\int_{\mathbf{R}^n}\nabla v_j\cdot\nabla\left(\left(\sum_{l=1}^{N}|v_l|^2\right)^{(p-2)/2}\bar{v}_j\right)dx\\ &\leq Ct^{-2}\sum_{j=1}^{N}\int_{\mathbf{R}^n}|\nabla v_j|^2\left(\sum_{l=1}^{N}|v_l|^2\right)^{(p-2)/2}dx\\ &\leq Ct^{-2}\sum_{j=1}^{N}\|\nabla v_j\|_p^2\|\vec{v}(t)\|_{p,*}^{(p-2)}.\end{aligned}$$

We note that when $1<\gamma\leq\sqrt{2}$, we have $0<(\gamma-1)(2-\gamma)/(6-4\gamma)<1$, and so $2<p<2n/(n-2)$. Then, Lemma 2.1, Lemma 3.1 and Lemma 3.2 yield

$$\begin{aligned}\|\nabla v_j\|_p &\leq C\|\nabla v_j\|^{1-\delta(p)}\|\Delta v_j\|^{\delta(p)}\\ &\leq Ct^{\eta}.\end{aligned}$$

Here

$$\eta = 2-\gamma+\frac{6-4\gamma}{2-\gamma}\delta(p),$$

and the constant C depends on $\|\vec{\phi}\|_{\Sigma^{1,2}}$. Therefore,

$$\frac{d}{dt}\|\vec{v}(t)\|_{p,*}^p \leq Ct^{-2+\eta}\|\vec{v}(t)\|_{p,*}^{(p-2)}. \tag{36}$$

Since $\eta<1$ for p satisfying $0<\delta(p)<(\gamma-1)(2-\gamma)/(6-4\gamma)$, the estimate (34) follows by integrating the differential inequality (36).

By this lemma, we have

Proposition 3.2. *Suppose that $n\geq 2$, $1<\gamma\leq\sqrt{2}$ and $\vec{\phi}\in\Sigma^{1,2}$. Then the solution of* (1)-(2) *has the following estimate*

$$\|\vec{u}(t)\|_p \leq C(1+|t|)^{-\delta(p)}, \tag{37}$$

where p satisfies $0<\delta(p)<(\gamma-1)(2-\gamma)/(6-4\gamma)$.

4. Proof of the main theorem

In this section, we shall prove Theorem 1.1. Since we can prove part (i) of the Theorem similar to the Theorem [A] for Hartree type equation, we omit the proof. Throughout this section, we put $q = 4n/(2n-\gamma)$ and $r = 8/\gamma$. Then the pair (q, r) is admissible. To prove part (ii), we introduce the following Banach space:

$$X^{l,m}(I) = \left\{u \in C(I; H^l); \|u\|_{X^{l,m}(I)} < \infty\right\},$$

where

$$\|u\|_{X^{l,m}(I)} = \sum_{|\alpha| \leq l} \left(\|\nabla^\alpha u\|_{2,\infty,I} + \|\nabla^\alpha u\|_{q,r,I}\right) + \sum_{|\beta| \leq m} \left(\|J^\beta u\|_{2,\infty,I} + \|J^\beta u\|_{q,r,I}\right).$$

Let $I = [T, \infty)$, where T will be defined later. Using Hölder's inequality, Lemma 2.1 and Lemma 2.3, we have

$$\sum_{|\alpha|=l} \|\nabla^\alpha f_j(\vec{u})\|_{q'} \leq C\|\vec{u}\|_q^2 \sum_{k=1}^{N} \sum_{|\alpha|=l} \|\nabla^\alpha u_k\|_q \tag{38}$$

and

$$\sum_{|\beta|=m} \|J^\beta f_j(\vec{u})\|_{q'} \leq C\|\vec{u}\|_q^2 \sum_{k=1}^{N} \sum_{|\beta|=m} \|J^\beta u_k\|_q. \tag{39}$$

So we have, by Lemma 2.1 and Lemma 2.5,

$$\sum_{|\alpha| \leq l} \|\nabla^\alpha u_j\|_{2,\infty,I} \leq \sum_{|\alpha| \leq l} \|\nabla^\alpha U(-T)u_j(T)\| + C \sum_{|\alpha| \leq l} \|\nabla^\alpha f_j(\vec{u})\|_{q',r',I}. \tag{40}$$

Under the assumption of the theorem, Proposition 3.1 or Proposition 3.2 implies $\|\vec{u}(t)\|_q \leq Ct^{-\gamma/4}$. Therefore, by using (38) and Hölder's inequality, the second term in the right of (40) is dominated by

$$\begin{aligned} & C\sum_{k=1}^{N} \sum_{|\alpha| \leq l} \left[\int_T^\infty \left(\|\vec{u}(\tau)\|_q^2 \|\nabla^\alpha u_k(\tau)\|_q\right)^{r'} d\tau\right]^{1/r'} \\ \leq\ & C\sum_{k=1}^{N} \sum_{|\alpha| \leq l} \left[\int_T^\infty \left(\tau^{-\gamma/2} \|\nabla^\alpha u_k(\tau)\|_q\right)^{r'} d\tau\right]^{1/r'} \\ \leq\ & C\left(\int_T^\infty \tau^{-2\gamma/(4-\gamma)} d\tau\right)^{(4-\gamma)/4} \sum_{k=1}^{N} \sum_{|\alpha| \leq l} \|\nabla^\alpha u_k\|_{q,r,I}. \end{aligned} \tag{41}$$

If $\gamma > 4/3$, the integral in the right of (41) converges. Hence,

$$\sum_{|\alpha|\le l} \|\nabla^\alpha u_j\|_{2,\infty,I} \le \|U(-T)u_j(T)\|_{\Sigma^{l,m}} + CT^{(4-3\gamma)/4}\|\vec{u}\|_{X^{l,m}(I)}. \tag{42}$$

We can estimate

$$\sum_{|\alpha|\le l} \|\nabla^\alpha u_j\|_{q,r,I}, \ \sum_{|\beta|\le m} \|J^\beta u_j\|_{2,\infty,I}, \text{and} \sum_{|\beta|\le m} \|J^\beta u_j\|_{q,r,I}$$

similarly. Therefore,

$$\|\vec{u}\|_{X^{l,m}(I)} \le C\|U(-T)\vec{u}(T)\|_{\Sigma^{l,m}} + CT^{(4-3\gamma)/4}\|\vec{u}\|_{X^{l,m}(I)}. \tag{43}$$

If we choose T sufficiently large so that $CT^{(4-3\gamma)/4} \le 1/2$, (43) implies

$$\|\vec{u}\|_{X^{l,m}(I)} \le C\|U(-T)\vec{u}(T)\|_{\Sigma^{l,m}}.$$

Therefore, $\|\vec{u}\|_{X^{l,m}(\mathbf{R})}$ is finite. Once this has been proved, by the similar argument, for $t > s > 0$, we have

$$\begin{aligned} \|U(-t)\vec{u}(t) - U(-s)\vec{u}(s)\|_{\Sigma^{l,m}} &\le C\left(\int_s^t \tau^{-2\gamma/(4-\gamma)}d\tau\right)^{(4-\gamma)/4} \|\vec{u}\|_{X^{l,m}(\mathbf{R})} \\ &\le C\left(t^{(4-3\gamma)/4} - s^{(4-3\gamma)/4}\right). \end{aligned} \tag{44}$$

The right of (44) tends to zero as s, t tend to infinity. Thus the theorem has been proved.

Corollary 4.1. *Suppose that* $4/3 < \gamma < \min(4, n)$, *and* $l, m \ge 1 + [n/2]$. *Then for any* $\vec{\phi} \in \Sigma^{l,m}$, *the solution* $\vec{u}(t)$ *of* (1)-(2) *satisfies*

$$\|\vec{u}(t)\|_\infty \le C(1 + |t|)^{-n/2}. \tag{45}$$

Proof. By the relation $J^\beta \vec{u}(t) = M(t)x^\beta M(-t)\vec{u}(t)$ and Lemma 2.1.

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Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka560, Japan