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TOPOLOGY OF EQUILATERAL POLYGON LINKAGES IN THE EUCLIDEAN PLANE MODULO ISOMETRY GROUP

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1. Introduction

We consider the configuration space M_n' of equilateral polygon linkages with n $(n \geq 3)$ vertices, each edge having length 1 in the Euclidean plane $\mathbf{R}^2$ modulo isometry group. More precisely, let $\mathcal{C}_n$ be

(1.1) $$\mathcal{C}_n = \{(u_1, \ldots, u_n) \in (\mathbf{R}^2)^n : |u_{i+1} - u_i| = 1 \ \ (1 \leq i \leq n-1), \ \ |u_1 - u_n| = 1\}.$$

Note that $\mathrm{Iso}(\mathbf{R}^2)$ (= the isometry group of $\mathbf{R}^2$, i.e., a semidirect product of $\mathbf{R}^2$ with $O(2)$), naturally acts on $\mathcal{C}_n$. We define M_n' by

(1.2) $$M_n' = \mathcal{C}_n / \mathrm{Iso}(\mathbf{R}^2).$$

We remark that M_n' has the following description: We set $M_n = \mathcal{C}_n / \mathrm{Iso}^+(\mathbf{R}^2)$, where $\mathrm{Iso}^+(\mathbf{R}^2)$ denotes the orientation preserving isometry group of $\mathbf{R}^2$, i.e., a semidirect product of $\mathbf{R}^2$ with $SO(2)$. Then we can write M_n as

(1.3) $$M_n = \{(u_1, \ldots, u_n) \in \mathcal{C}_n : u_1 = (\frac{1}{2}, 0) \ \text{ and } \ u_2 = (-\frac{1}{2}, 0)\}.$$

M_n admits an involution $\sigma = \mathrm{Iso}(\mathbf{R}^2)/\mathrm{Iso}^+(\mathbf{R}^2)$ such that $M_n' = M_n / \sigma$. Under the identification of (1.3), σ is given by

(1.4) $$\sigma(u_1, \ldots, u_n) = (\bar{u}_1, \ldots, \bar{u}_n),$$

where $\bar{u}_i = (x_i, -y_i)$ when $u_i = (x_i, y_i)$.

Many topological properties of M_n are already known: First we know explicit topological type of M_n $(n \leq 5)$ [3],[4],[8]. Next we have the results on the smoothness of M_n [5],[7],[8]. Finally $H_*(M_n; \mathbf{Z})$ are determined in [6],[7] (cf. Theorem 2.1). In particular, the natural inclusion $i_n : M_n \hookrightarrow (S^1)^{n-1}$ (cf. (1.6)) induces isomorphisms of homology groups up to a certain dimension (cf. Proposition 2.2).

On the other hand, concerning M'_n, what we know already are the following: First we know the following examples.

EXAMPLES 1.5. $M'_3 = \{\text{1-point}\}, M'_4 = S^1$ and $M'_5 = \underset{5}{\sharp}\mathbf{R}P^2$, the five-times connected sum of $\mathbf{R}P^2$.

Next some assertions on the smoothness of M'_n are proved in [5]. However, we have few information on $H_*(M'_n; \mathbf{Z})$, although we know $\chi(M'_n)$, the Euler characteristic of M'_n [5] (cf. Proposition 3.4).

The purposes of this paper are as follows.

(i) We prove assertions on the smoothness of M'_n.

(ii) We determine $H_*(M'_n; \mathbf{Z}_p)$, where p is an odd prime, and $H_*(M'_n; \mathbf{Q})$.

In the following (iii)-(v), we assume n to be odd, and set $n = 2m + 1$. Then by the results of (i) and (ii), M'_{2m+1} is a non-orientable manifold of dimension $2m - 2$.

(iii) Find a space V_{2m} and an inclusion $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$ so that i'_{2m+1} induces isomorphisms of homotopy groups up to a certain dimension.

(iv) As V_{2m} is a natural space, we determine $H_*(V_{2m}; \mathbf{Z})$ completely. Then in particular we know $H_*(M'_{2m+1}; \mathbf{Z})$ up to some dimension by the result of (iii).

As we will see in Remark 1.9, knowing $H_*(V_{2m}; \mathbf{Z})$ is equivalent to knowing $H_*((S^1)^{2m}/\sigma; \mathbf{Z})$.

(v) Finally we determine $H_*(M'_{2m+1}; \mathbf{Z})$ except the possibility of higher two-torsions in $H_{m-1}(M'_{2m+1}; \mathbf{Z})$ when m is even.

Now we state our results. Concerning (i), we have the following:

Theorem A. (a) *M'_{2m+1} is a manifold of dimension $2m - 2$.*

(b) *M'_{2m} is a manifold of dimension $2m - 3$ with singular points. $(u_1, \dots, u_{2m}) \in M'_{2m}$ is a singular point iff all of u_i lie on the x-axis, i.e., the line determined by u_1 and u_2* (cf. (1.3)). *Moreover every singular point of M'_{2m} is a cone-like singularity and has a neighborhood as $C(S^{m-2} \times_{\mathbf{Z}_2} S^{m-2})$, where C denotes cone and action of $\mathbf{Z}_2$ on both factors is generated by the antipodal map.*

Concerning (ii), first we prove the following:

Theorem B. *$H_*(M'_n; \mathbf{Z})$ are odd-torsion free.*

Thus in order to know $H_*(M'_n; \mathbf{Z}_p)$, we need to know $H_*(M'_n; \mathbf{Q})$, which is given by the following:

Theorem C. *The Poincaré polynomials* $PS_{\mathbf{Q}}(M'_n) = \Sigma_\lambda \dim_{\mathbf{Q}} H_\lambda(M'_n; \mathbf{Q})t^\lambda$ *are given by*

$$\text{(a) } PS_{\mathbf{Q}}(M'_{2m+1}) = \sum_{0\le 2a\le m-2} \binom{2m}{2a} t^{2a} + \binom{2m}{m-1} t^{m-1} + \sum_{m\le 2b+1\le 2m-3} \binom{2m}{2b+3} t^{2b+1},$$

$$\text{(b) } PS_{\mathbf{Q}}(M'_{4l}) = \sum_{0\le 2a\le 2l-2} \binom{4l-1}{2a} t^{2a} + \binom{4l-1}{2l-2} t^{2l-1} + \sum_{2l+1\le 2b+1\le 4l-3} \binom{4l-1}{2b+3} t^{2b+1},$$

$$PS_{\mathbf{Q}}(M'_{4l+2}) = \sum_{0\le 2a\le 2l-2} \binom{4l+1}{2a} t^{2a} + \binom{4l+1}{2l+1} t^{2l} + \sum_{2l+1\le 2b+1\le 4l-1} \binom{4l+1}{2b+3} t^{2b+1},$$

where $\binom{a}{b}$ *denotes the binomial coefficient.*

Next we go to (iii). By setting $z_i = u_{i+2} - u_{i+1}$ $(1 \le i \le n-2)$, $z_{n-1} = u_1 - u_n$, and identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can write M_n $(n \ge 3)$ as

$$(1.6) \qquad M_n \cong \{(z_1, \dots, z_{n-1}) \in (S^1)^{n-1} : z_1 + \cdots + z_{n-1} - 1 = 0\}.$$

Let $i_n : M_n \hookrightarrow (S^1)^{n-1}$ be the inclusion.

As we have mentioned, $(S^1)^{n-1}$ approximates the topology of M_n up to some dimension (cf. Proposition 2.2). However, for an odd $n = 2m+1$, our low-dimensional computations lead us to give up the hope that $(S^1)^{2m}/\sigma$ might approximate $M'_{2m+1} = M_{2m+1}/\sigma$, where σ acts on $(S^1)^{2m}$ in the same way as in (1.4). The essential reason for this is that the action of σ on $(S^1)^{2m}$ is not free, although on M_{2m+1} is.

Thus we define V_{2m} by

$$(1.7) \qquad V_{2m} = \left\{(S^1)^{2m} - \Sigma_{2m}\right\}/\sigma,$$

where we set

$$\Sigma_{2m} = \{(z_1, \dots, z_{2m}) \in (S^1)^{2m} : z_i = \pm 1 \ (1 \le i \le 2m)\}.$$

Let $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$ be the inclusion. Then we have the following map of covering spaces:

$$
(1.8)\qquad
\begin{array}{ccc}
\mathbf{Z}_2 & = & \mathbf{Z}_2 \\
\downarrow & & \downarrow \\
M_{2m+1} & \xrightarrow{i_{2m+1}} & (S^1)^{2m} - \Sigma_{2m} \\
\downarrow & & \downarrow \\
M'_{2m+1} & \xrightarrow{i'_{2m+1}} & V_{2m}.
\end{array}
$$

Note that $(S^1)^{2m} - \Sigma_{2m}$ is a maximal subspace of $(S^1)^{2m}$ on which σ acts freely. Thus it is natural to consider the topology of V_{2m}.

Now concerning the relation between M'_{2m+1} and V_{2m}, we have the following theorem.

Theorem D. *$(i'_{2m+1})_* : \pi_q(M'_{2m+1}) \to \pi_q(V_{2m})$ are isomorphisms for $q \leq m-2$, and an epimorphism for $q = m-1$.*

Concerning (iv), we have the following:

Theorem E. *$H_*(V_{2m};\mathbf{Z})$ is given by*

$$
H_q(V_{2m};\mathbf{Z}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Z} & q: \textit{ even } \leq 2m-2 \\ \bigoplus_{\sum_{i\leq q}\binom{2m}{i}} \mathbf{Z}_2 & q: \textit{ odd } \leq 2m-3 \\ \bigoplus_{2^{2m}-1} \mathbf{Z} & q = 2m-1 \\ 0 & \textit{otherwise,} \end{cases}
$$

where $\bigoplus_{\binom{2m}{q}} \mathbf{Z}$ denotes the $\binom{2m}{q}$-times direct sum of $\mathbf{Z}$.

Note that Theorems D and E give $H_q(M'_{2m+1};\mathbf{Z})$ for $q \leq m-2$.

REMARK 1.9. By the Poincaré-Lefschetz duality $H^q((S^1)^{2m}/\sigma, \Sigma_{2m}; \mathbf{Z}) \cong H_{2m-q}(V_{2m};\mathbf{Z})$, knowing $H_*(V_{2m};\mathbf{Z})$ is equivalent to knowing $H_*((S^1)^{2m}/\sigma;\mathbf{Z})$.

Concerning (v), we have the following:

Theorem F. (a) *For an odd m, we have*

$$H_q(M'_{2m+1};\mathbf{Z}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Z} & q:\ \textit{even}\ \le m-1 \\ \bigoplus_{\sum_{i\le q}\binom{2m}{i}} \mathbf{Z}_2 & q:\ \textit{odd}\ \le m-2 \\ \bigoplus_{\binom{2m}{q+2}} \mathbf{Z} \oplus \bigoplus_{\sum_{i\ge q+3}\binom{2m}{i}} \mathbf{Z}_2 & q:\ \textit{odd}\ \ge m \\ 0 & \textit{otherwise.} \end{cases}$$

(b) *For an even m, we have*

$$H_q(M'_{2m+1};\mathbf{Z}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Z} & q:\ \textit{even}\ \le m-2 \\ \bigoplus_{\sum_{i\le q}\binom{2m}{i}} \mathbf{Z}_2 & q:\ \textit{odd}\ \le m-3 \\ \bigoplus_{\binom{2m}{m-1}} \mathbf{Z} \oplus Tor_{m-1} & q = m-1 \\ \bigoplus_{\binom{2m}{q+2}} \mathbf{Z} \oplus \bigoplus_{\sum_{i\ge q+3}\binom{2m}{i}} \mathbf{Z}_2 & q:\ \textit{odd}\ \ge m+1 \\ 0 & \textit{otherwise,} \end{cases}$$

where Tor_{m-1}, the torsion submodule of $H_{m-1}(M'_{2m+1};\mathbf{Z})$, satisfies that $\dim_{\mathbf{Z}_2} Tor_{m-1} \otimes \mathbf{Z}_2 = \sum_{i\le m-2}\binom{2m}{i}$.

Thus, in particular, $H_{even}(M'_{2m+1};\mathbf{Z})$ are torsion free for all m.

REMARK 1.10. (a) By Theorems D, E and F, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1};\mathbf{Z}) \to H_{m-1}(V_{2m};\mathbf{Z})$ is an isomorphism when m is odd, but not an isomorphism when m is even.

(b) In order to prove Theorem F, we first determine $H_*(M'_{2m+1};\mathbf{Z}_2)$, which is given in Proposition 5.1. In particular, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1};\mathbf{Z}_2) \to H_{m-1}(V_{2m};\mathbf{Z}_2)$ is an isomorphism for all m (cf. Remark 5.2).

This paper is organized as follows. In §2 we recall the results of [7], then prove Theorems A, B and D. In §3 we prove Theorem C. In §4 we prove Theorem E, and in §5 we prove Theorem F.

2. Proofs of Theorems A, B and D

In [7], the following theorem is proved.

Theorem 2.1. *$H_*(M_n;\mathbf{Z})$ are free $\mathbf{Z}$-modules and the Poincaré polynomials $PS(M_n)=\sum_\lambda \mathrm{rank} H_\lambda(M_n;\mathbf{Z})t^\lambda$ are given by*

$$PS(M_{2m+1})=\sum_{\lambda=0}^{m-2}\binom{2m}{\lambda}t^\lambda+2\binom{2m}{m-1}t^{m-1}+\sum_{\lambda=m}^{2m-2}\binom{2m}{\lambda+2}t^\lambda,$$

$$PS(M_{2m})=\sum_{\lambda=0}^{m-2}\binom{2m-1}{\lambda}t^\lambda+\binom{2m}{m-1}t^{m-1}+\sum_{\lambda=m}^{2m-3}\binom{2m-1}{\lambda+2}t^\lambda.$$

The essential facts to prove Theorem 2.1 are the following three propositions.

Proposition 2.2. (i) *$(i_{2m+1})_*:\pi_q(M_{2m+1})\to\pi_q((S^1)^{2m})$ are isomorphisms for $q\le m-2$, and an epimorphism for $q=m-1$.*

(ii) *$(i_{2m})_*:H_q(M_{2m};\mathbf{Z})\to H_q((S^1)^{2m-1};\mathbf{Z})$ are isomorphisms for $q\le m-2$, and an epimorphism for $q=m-1$.*

Proposition 2.3. (i) *M_{2m+1} is an orientable manifold of dimension $2m-2$. Thus the Poincaré duality homomorphisms $\cap[M_{2m+1}]:H^q(M_{2m+1};\mathbf{Z})\to H_{2m-2-q}(M_{2m+1};\mathbf{Z})$ are isomorphisms for all q, where $[M_{2m+1}]\in H_{2m-2}(M_{2m+1};\mathbf{Z})$ is a fundamental class.*

(ii) *M_{2m} is a manifold of dimension $2m-3$ with singular points. $(u_1,\dots,u_{2m})\in M_{2m}$ is a singular point iff all of u_i lie on the x-axis. Moreover every singular point of M_{2m} is a cone-like singularity and has a neighborhood as $C(S^{m-2}\times S^{m-2})$. Thus the Poincaré duality homomorphisms $\cap[M_{2m}]:H^q(M_{2m};\mathbf{Z})\to H_{2m-3-q}(M_{2m};\mathbf{Z})$ are isomorphisms for $q\le m-3$ or $q\ge m$, an epimorphism for $q=m-1$, and a monomorphism for $q=m-2$.*

Proposition 2.4. (i) $\chi(M_{2m+1})=(-1)^{m+1}\binom{2m}{m}$.

(ii) $\chi(M_{2m})=(-1)^{m+1}\binom{2m-1}{m}$.

REMARK 2.5. In order to prove Theorem 2.1, the homological assertion is sufficient for Proposition 2.2 (i). But actually we can prove the homotopical assertion.

Proof of Theorem A. Since σ acts freely on M_{2m+1}, and M_{2m}^σ (=the fixed point set of the involution) equals to the set of singular points in M_{2m}, all of the assertions except the type of the singular points of M'_{2m} are deduced from Proposition 2.3.

Let $(z_1,\dots,z_{2m-1})$ be a singular point of M_{2m} in the identification of (1.6). By Proposition 2.3, we must have $z_i=\pm1$ $(1\le i\le 2m-1)$. As the symmetric group on $(2m-1)$-letters acts on M_{2m}, we can assume that $z_i=1$ $(1\le i\le m)$ and $z_i=-1$ $(m+1\le i\le 2m-1)$. A neighborhood of $(z_1,\dots,z_{2m-1})$ in $(S^1)^{2m-1}$ is written by

$$\left\{\left(\begin{pmatrix}\sqrt{1-y_1^2}\\ y_1\end{pmatrix},\dots,\begin{pmatrix}\sqrt{1-y_m^2}\\ y_m\end{pmatrix},\begin{pmatrix}-\sqrt{1-y_{m+1}^2}\\ y_{m+1}\end{pmatrix},\dots,\begin{pmatrix}-\sqrt{1-y_{2m-1}^2}\\ y_{2m-1}\end{pmatrix}\right): \\ -\epsilon\le y_i\le\epsilon\ (1\le i\le 2m-1)\right\},$$

where $\epsilon > 0$ is a fixed small number. As ϵ is small, it is easy to see that we can write this neighborhood as

$$\left\{\left(\begin{pmatrix}1-\frac{1}{2}y_1^2\\ y_1\end{pmatrix},\dots,\begin{pmatrix}1-\frac{1}{2}y_m^2\\ y_m\end{pmatrix},\begin{pmatrix}-1+\frac{1}{2}y_{m+1}^2\\ y_{m+1}\end{pmatrix},\dots,\begin{pmatrix}-1+\frac{1}{2}y_{2m-1}^2\\ y_{2m-1}\end{pmatrix}\right): \\ -\epsilon\le y_i\le\epsilon\ (1\le i\le 2m-1)\right\}.$$

Thus a neighborhood of a singular point in M_{2m} is written as a subspace of $\mathbf{R}^{2m-1}$ defined by two equations

$$\text{(2.6)} \qquad \begin{cases} y_1^2+\cdots+y_m^2-y_{m+1}^2-\cdots-y_{2m-1}^2=0\\ y_1+\cdots+y_m+y_{m+1}+\cdots+y_{2m-1}=0.\end{cases}$$

By a linear transformation of parameters, we can write the quadratic form of (2.6), i.e.,

$$y_1^2+\cdots+y_{m-1}^2+(y_1+\cdots+y_{m-1}+y_{m+1}+\cdots+y_{2m-1})^2-y_{m+1}^2-\cdots-y_{2m-1}^2,$$

as

$$w_1^2+\cdots+w_{m-1}^2-w_m^2-\cdots-w_{2m-2}^2.$$

Thus a singular point of M_{2m} has a neighborhood $C\{(w_1,\dots,w_{m-1},w_m,\dots,w_{2m-2}):w_1^2+\cdots+w_{m-1}^2=1,w_m^2+\cdots+w_{2m-2}^2=1\}$, which is homeomorphic to $C(S^{m-2}\times S^{m-2})$.

Now it is clear that $\sigma w_i=-w_i$. Hence a singular point of M'_{2m} has a neighborhood $C(S^{m-2}\times_{\mathbf{Z}_2}S^{m-2})$, where $\sigma(\zeta_1,\zeta_2)=(-\zeta_1,-\zeta_2)$ $(\zeta_1,\zeta_2\in S^{m-2})$. □

Proof of Theorem B. For $F=\mathbf{Z}_p$ (p : an odd prime) or $\mathbf{Q}$, we have that $H_*(M'_n;F)\cong H_*(M_n;F)^\sigma$ (= the fixed point set of $H_*(M_n;F)$ under the σ-action) (see for example [2]). As $H_*(M_n;\mathbf{Z})$ are free modules by Theorem 2.1, we have that $\dim_{\mathbf{Z}_p}H_q(M'_n;\mathbf{Z}_p)=\dim_{\mathbf{Q}}H_q(M'_n;\mathbf{Q})$. Hence Theorem B follows. □

Proof of Theorem D. Let $j_{2m} : (S^1)^{2m} - \Sigma_{2m} \hookrightarrow (S^1)^{2m}$ be the inclusion. Since Σ_{2m} is a discrete set, the general position argument shows that $(j_{2m})_* : \pi_q((S^1)^{2m} - \Sigma_{2m}) \to \pi_q((S^1)^{2m})$ are isomorphisms for $q \leq 2m-2$. Then Proposition 2.2 (i) shows that $(i_{2m+1})_* : \pi_q(M_{2m+1}) \to \pi_q((S^1)^{2m} - \Sigma_{2m})$ are isomorphisms for $q \leq m-2$ and an epimorphism for $q = m-1$, where $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m}$ is the inclusion.

By comparing the homotopy exact sequences of two covering spaces of (1.8), we see that $(i'_{2m+1})_* : \pi_q(M'_{2m+1}) \to \pi_q(V_{2m})$ are isomorphisms for $q \leq m-2$ and an epimorphism for $q = m-1$, where $i'_{2m+1} : M'_{2m+1} \hookrightarrow V_{2m}$ is the map induced from the σ-equivariant inclusion $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m} - \Sigma_{2m}$.

This completes the proof of Theorem D. □

3. Proof of Theorem C

Let $i_n : M_n \hookrightarrow (S^1)^{n-1}$ be the inclusion. Note that i_n is a σ-equivariant map. Hence $(i_n)_* : H_*(M_n; \mathbf{Q}) \to H_*((S^1)^{n-1}; \mathbf{Q})$ is also a σ-equivariant homomorphism. Since $H_*(M'_n; \mathbf{Q}) = H_*(M_n; \mathbf{Q})^\sigma$, Proposition 2.2 tells us the following:

Proposition 3.1. (i) *For $q \leq m-2$, we have*

$$H_q(M'_{2m+1}; \mathbf{Q}) = \begin{cases} \bigoplus_{\binom{2m}{q}} \mathbf{Q} & q : \textit{even} \\ 0 & q : \textit{odd}. \end{cases}$$

(ii) *For $q \leq m-2$, we have*

$$H_q(M'_{2m}; \mathbf{Q}) = \begin{cases} \bigoplus_{\binom{2m-1}{q}} \mathbf{Q} & q : \textit{even} \\ 0 & q : \textit{odd}. \end{cases}$$

We assume the truth of the following Lemma for the moment. Let $[M_n] \in H_{n-3}(M_n; \mathbf{Q})$ be the fundamental class.

Lemma 3.2. $\sigma_*[M_n] = (-1)^n [M_n]$.

Then we have the following:

Proposition 3.3. (i) *For $q \geq m$, we have*

$$H_q(M'_{2m+1}; \mathbf{Q}) = \begin{cases} 0 & q : \textit{even} \\ \bigoplus_{\binom{2m}{q+2}} \mathbf{Q} & q : \textit{odd}. \end{cases}$$

(ii) *For* $q \geq m$, *we have*

$$H_q(M'_{2m};\mathbf{Q}) = \begin{cases} 0 & q:\ \textit{even} \\ \bigoplus_{\binom{2m-1}{q+2}} \mathbf{Q} & q:\ \textit{odd.} \end{cases}$$

Proof of Proposition 3.3. Take an element $\alpha \in H_q(M_{2m+1};\mathbf{Q})$ $(q \geq m)$. By Proposition 2.3, there is an element $f \in H^{2m-2-q}(M_{2m+1};\mathbf{Q})$ such that $\alpha = f \cap [M_{2m+1}]$. As $\sigma_*(f \cap [M_{2m+1}]) = \sigma^* f \cap \sigma_*[M_{2m+1}] = -\sigma^* f \cap [M_{2m+1}]$, we have that

$$H_q(M_{2m+1};\mathbf{Q})^\sigma = \left\{ f \in H^{2m-2-q}(M_{2m+1};\mathbf{Q}) : \sigma^* f = -f \right\}.$$

Now (i) follows from Proposition 3.1.

(ii) can be proved similarly. □

Now in order to determine $H_*(M'_n;\mathbf{Q})$, we need to know only $H_{m-1}(M'_{2m+1};\mathbf{Q})$ and $H_{m-1}(M'_{2m};\mathbf{Q})$, which are determined if we know $\chi(M'_n)$.

Proposition 3.4 ([5]). (i) $\chi(M'_{2m+1}) = (-1)^{m+1}\binom{2m-1}{m}$.

(ii) $\chi(M'_{2m}) = \begin{cases} 0 & m:\ \textit{even} \\ \binom{2m-1}{m} & m:\ \textit{odd.} \end{cases}$

Proof. By a general formula of an involution (see for example [1]), we have $\chi(M_n) + \chi(M_n^\sigma) = 2\chi(M'_n)$. Then the result follows from Proposition 2.4. □

Proof of Lemma 3.2. First we treat the case of $n = 2m+1$. We define a volume element ω of M_{2m+1} as follows. Fix $(z_1,\ldots,z_{2m}) \in M_{2m+1}$ in the identification of (1.6). It is easy to see that the tangent space $T_{(z_1,\ldots,z_{2m})}M_{2m+1}$ is given by

$$T_{(z_1,\ldots,z_{2m})}M_{2m+1} \cong \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \mathbf{R}^{2m} : \xi_1 z_1 + \cdots + \xi_{2m} z_{2m} = 0 \right\}. \tag{3.5}$$

Write z_i as (x_i, y_i). Then for $\eta_1,\ldots,\eta_{2m-2} \in T_{(z_1,\ldots,z_{2m})}M_{2m+1}$, we set

$$\omega(\eta_1,\ldots,\eta_{2m-2}) = \det\left(\eta_1,\ldots,\eta_{2m-2}, \begin{pmatrix} x_1 \\ \vdots \\ x_{2m} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{2m} \end{pmatrix} \right). \tag{3.6}$$

It is easy to see that ω is nowhere zero on M_{2m+1}.

For $\eta = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in T_{(z_1,\ldots,z_{2m})}M_{2m+1}$, we see that

$$di_{2m+1}(\eta) = \xi_1(\sqrt{-1}z_1) + \cdots + \xi_{2m}(\sqrt{-1}z_{2m}),$$

where $i_{2m+1} : M_{2m+1} \hookrightarrow (S^1)^{2m}$ denotes the inclusion. Hence we see that $d\sigma : T_{(z_1,\ldots,z_{2m})}M_{2m+1} \to T_{(\bar{z}_1,\ldots,\bar{z}_{2m})}M_{2m+1}$ is given by

$$d\sigma(\eta) = -\eta. \tag{3.7}$$

Now the formulae $d\sigma(\eta_i) = -\eta_i$ and $\sigma(x_i, y_i) = (x_i, -y_i)$ tell us that $(\sigma^*\omega)(\eta_1, \ldots, \eta_{2m-2}) = -\omega(\eta_1, \ldots, \eta_{2m-2})$. Hence $\sigma^*\omega = -\omega$ and the result follows.

Next we treat the case of $n = 2m$. Let $\bar{M}_{2m}$ be $M_{2m} - \{\text{singular points}\}$. By the same argument as in the case of $n = 2m+1$, we see that $\sigma : \bar{M}_{2m} \to \bar{M}_{2m}$ preserves orientation. As $H_c^{2m-3}(\bar{M}_{2m}; \mathbf{Q}) \cong H^{2m-3}(M_{2m}; \mathbf{Q})$ (H_c= cohomology with compact supports), the result follows. □

4. Proof of Theorem E

First we determine $H_{2m-1}(V_{2m}; \mathbf{Z})$. The Poincaré-Lefschetz duality tells us that $H_{2m-1}(V_{2m}; \mathbf{Z}) \cong H^1((S^1)^{2m}/\sigma, \Sigma_{2m}; \mathbf{Z})$. As $H^1((S^1)^{2m}/\sigma; \mathbf{Z}) = 0$, we have $H_{2m-1}(V_{2m}; \mathbf{Z}) \cong \bigoplus_{2^{2m}-1} \mathbf{Z}$.

As V_{2m} is a non-compact manifold of dimension $2m$, we have $H_q(V_{2m}; \mathbf{Z}) = 0$ ($q \geq 2m$). Hence in order to complete the proof of Theorem E, we need to determine $H_q(V_{2m}; \mathbf{Z})$ ($q \leq 2m-2$).

Recall that we have a fibration $(S^1)^{2m} - \Sigma_{2m} \to V_{2m} \to \mathbf{R}P^\infty$. Set $F_{2m} = (S^1)^{2m} - \Sigma_{2m}$. The local systems of this fibration of dimensions less than or equal to $2m-2$ are easy to describe: We write the generator of $\pi_1(\mathbf{R}P^\infty)$ by σ. Then as a σ-module, we have

$$H_q(F_{2m}; \mathbf{Z}) \cong H_q((S^1)^{2m}; \mathbf{Z}) \ (q \leq 2m-2). \tag{4.1}$$

Let $\{E^r_{s,t}\}$ be the $\mathbf{Z}$-coefficient homology Serre spectral sequence of the above fibration. It is elementary to describe $E^2_{s,t}$ ($t \neq 2m-1$) by using the following fact: We define a σ-module $\mathcal{S}$ to be the free abelian group of rank 1 on which σ acts by -1. Then have that

$$H_q(\mathbf{R}P^\infty; \mathcal{S}) = \begin{cases} \mathbf{Z}_2 & q: \ \text{even} \\ 0 & q: \ \text{odd}. \end{cases} \tag{4.2}$$

REMARK 4.3. For our reference, we give $E^2_{s,2m-1}$. Let $\mathcal{T}$ be the free abelian group of rank 2 on which σ acts by $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$. And let σ act on $\mathbf{Z}$ trivially. Then we can prove that

$$H_{2m-1}(F_{2m};\mathbf{Z}) \cong \bigoplus_{2m} \mathcal{T} \oplus \bigoplus_{2^{2m}-2m-1} \mathbf{Z}.$$

As

$$H_q(\mathbf{R}P^\infty;\mathcal{T}) = \begin{cases} \mathbf{Z} & q=0 \\ 0 & q>0, \end{cases}$$

we can determine $E^2_{s,2m-1}$.

We return to $E^2_{s,t}$ $(t \neq 2m-1)$. By the dimensional reason, we have the following:

Proposition 4.4. *For $s+t \leq 2m-2$, we have that $E^2_{s,t} \cong E^\infty_{s,t}$.*

Hence in order to complete the proof of Theorem E, it suffices to determine the extensions of $E^\infty_{s,t}$, where $s+t$ are odd $\leq 2m-3$. To do so, it is convenient to study $H_*(V_{2m};\mathbf{Z}_2)$.

Proposition 4.5. *For $q \leq 2m-2$, we have*

$$H_q(V_{2m};\mathbf{Z}_2) = \bigoplus_{\sum_{i\leq q}\binom{2m}{i}} \mathbf{Z}_2.$$

From Proposition 4.5, we see that the extensions of $E^\infty_{s,t}$ $(s+t \leq 2m-2)$ are trivial. Hence Theorem E follows.

Thus in order to complete the proof of Theorem E, we need to prove Proposition 4.5, which we prove for the rest of this section.

Let $\{E_r^{s,t}\}$ be the $\mathbf{Z}_2$-coefficient cohomology Serre spectral sequence of the fibration $F_{2m} \to V_{2m} \to \mathbf{R}P^\infty$. We prove the following:

Lemma 4.6. $d_2 : E_2^{0,1} \to E_2^{2,0}$ *equals to* 0.

Lemma 4.6 tells us that elements of $E_2^{s,t}$ $(t \leq 2m-2)$ are permanent cycles. Hence Proposition 4.5 follows.

Proof of Lemma 4.6. Suppose that Lemma 4.6 fails. Then we have $H^1(V_{2m};\mathbf{Z}_2) = \bigoplus_{2m}\mathbf{Z}_2$. By Theorem D and the $\mathbf{Z}_2$-coefficient Poincaré duality of M'_{2m+1}, we have $H_{2m-3}(M'_{2m+1};\mathbf{Z}_2) = \bigoplus_{2m}\mathbf{Z}_2$. Since $H_{2m-3}(M'_{2m+1};\mathbf{Q}) = \bigoplus_{2m}\mathbf{Q}$ by Theorem C (a), we have

$$H_{2m-3}(M'_{2m+1};\mathbf{Z}) = \bigoplus_{2m}\mathbf{Z}. \tag{4.7}$$

By Theorem C (a), we have $H_{2m-2}(M'_{2m+1};\mathbf{Q}) = 0$. Hence by Theorem A (a), M'_{2m+1} is a non-orientable manifold of dimension $2m-2$. Thus we have $H_{2m-2}(M'_{2m+1};\mathbf{Z}) = 0$. Then by (4.7), we have $H_{2m-2}(M'_{2m+1};\mathbf{Z}_2) = 0$. This contradicts the fact that $H_{2m-2}(M'_{2m+1};\mathbf{Z}_2) = \mathbf{Z}_2$, i.e., M'_{2m+1} is a compact manifold of dimension $2m-2$.

This completes the proof of Lemma 4.6, and hence also that of Theorem E. □

5. Proof of Theorem F

In order to calculate $H_*(M'_{2m+1};\mathbf{Z})$, first we need to determine $H_*(M'_{2m+1};\mathbf{Z}_2)$. By the Poincaré duality, it suffices to determine $H_q(M'_{2m+1};\mathbf{Z}_2)$ $(q \leq m-1)$, which are given by the following:

Proposition 5.1. *For $q \leq m-1$, we have*

$$H_q(M'_{2m+1};\mathbf{Z}_2) = \bigoplus_{\sum_{i\leq q}\binom{2m}{i}} \mathbf{Z}_2.$$

Proof. First, $H_q(M'_{2m+1};\mathbf{Z}_2)$ $(q \leq m-2)$ are determined by Theorems D and E together with the universal coefficient theorem. Then $H_{m-1}(M'_{2m+1};\mathbf{Z}_2)$ is determined by Proposition 3.4. □

REMARK 5.2. From Theorems D, E and Proposition 5.1, we see that $(i'_{2m+1})_* : H_{m-1}(M'_{2m+1};\mathbf{Z}_2) \to H_{m-1}(V_{2m};\mathbf{Z}_2)$ is an isomorphism for all m.

Now we begin to determine $H_*(M'_{2m+1};\mathbf{Z})$.

(I) $H_{even}(M'_{2m+1};\mathbf{Z})$.

These modules are determined from Theorem C and the following:

Proposition 5.3. *$H_{even}(M'_{2m+1};\mathbf{Z})$ are torsion free.*

Proof. We can inductively prove this proposition from Theorem C and Proposition 5.1 together with the universal coefficient theorem. □

(II) $H_{odd}(M'_{2m+1};\mathbf{Z})$.

In order to determine these modules from Theorem C and Proposition 5.1, we need to prove the non-existence of higher two-torsions, i.e., elements of order 2^i $(i \geq 2)$.

Let $p : M_{2m+1}\underset{\sigma}{\times}\mathbf{R} \to M'_{2m+1}$ be the real line bundle associated to the covering space $M_{2m+1} \to M'_{2m+1}$. And let $O(M_{2m+1}\underset{\sigma}{\times}\mathbf{R})$ denote the local system of the above vector bundle. Finally, let $O(TM'_{2m+1})$ denote the local system of TM'_{2m+1}, the tangent bundle of M'_{2m+1}.

Concerning these local systems, we have the following:

Lemma 5.4. *As local systems on M'_{2m+1}, we have $O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R}) \cong O(TM'_{2m+1})$.*

Proof. Let $\mathbf{R}^2 \to \nu \to M_{2m+1}$ denote the normal bundle of M_{2m+1} in $(S^1)^{2m}$ (cf. (1.6)). As $TM_{2m+1} \oplus \nu \cong T((S^1)^{2m})|M_{2m+1}$, we have

$$TM'_{2m+1} \oplus \nu/\sigma \cong TV_{2m}|M'_{2m+1}, \tag{5.5}$$

where $\nu/\sigma \to M'_{2m+1}$ denotes the vector bundle obtained from $\nu \to M_{2m+1}$ by the action of σ.

We study ν/σ. Recall that TM_{2m+1} is given by (3.5). Similarly, for $(z_1, \ldots, z_{2m}) \in M_{2m+1}$, we have

$$T_{(z_1,\ldots,z_{2m})}((S^1)^{2m}) \cong \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \mathbf{R}^{2m} \right\}.$$

Hence by assigning $\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2m} \end{pmatrix} \in \nu_{(z_1,\ldots,z_{2m})}$ to $\xi_1 z_1 + \cdots + \xi_{2m} z_{2m}$, we have

$$\nu \cong M_{2m+1} \times \mathbf{R}^2. \tag{5.6}$$

Under this identification, the bundle homomorphism $d\sigma : \nu \to \nu$ is given by

$$d\sigma((z_1, \ldots, z_{2m}); \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = ((\bar{z}_1, \ldots, \bar{z}_{2m}); \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}), \tag{5.7}$$

(cf. (3.7)).

Then (5.6)-(5.7) tell us that

$$\nu/\sigma \cong M_{2m+1} \underset{\sigma}{\times} \mathbf{R} \oplus M'_{2m+1} \times \mathbf{R}. \tag{5.8}$$

Now, as V_{2m} is orientable, we see from (5.5) and (5.8) that

$$O(TM'_{2m+1}) \otimes O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R}) \cong \mathbf{Z}, \tag{5.9}$$

where $\mathbf{Z}$ denotes the simple local system on M'_{2m+1}. By taking a tensor $\otimes O(TM'_{2m+1})$ on both sides of (5.9), the result follows. □

Let us denote the local systems $O(M_{2m+1} \underset{\sigma}{\times} \mathbf{R}) \cong O(TM'_{2m+1})$ (cf. Lemma 5.4) by $\mathcal{Z}$.

(A) *The case of an odd* m.

We can determine $H_q(M'_{2m+1};\mathbf{Z})$ $(q$: odd $\leq m-2)$ by Theorems D and E. Thus we need to determine $H_q(M'_{2m+1};\mathbf{Z})$ $(q$: odd $\geq m)$. By the Poincaré duality: $H_q(M'_{2m+1};\mathbf{Z}) \cong H^{2m-2-q}(M'_{2m+1};\mathcal{Z})$, it suffices to determine $H^r(M'_{2m+1};\mathcal{Z})$ $(r$: odd $\leq m-2)$.

Consider the Gysin sequence of $p : M_{2m+1} \underset{\sigma}{\times} \mathbf{R} \to M'_{2m+1}$:

$$\cdots \xrightarrow{\psi} H^{r-1}(M'_{2m+1};\mathcal{Z}) \xrightarrow{\mu} H^r(M'_{2m+1};\mathbf{Z}) \xrightarrow{p^*} H^r(M_{2m+1};\mathbf{Z})$$
$$\xrightarrow{\psi} H^r(M'_{2m+1};\mathcal{Z}) \xrightarrow{\mu} \cdots .$$

Lemma 5.10. *For an odd* $r \leq m-2$, *we have*

(i) $H^r(M'_{2m+1};\mathbf{Z}) = 0$.

(ii) $H^r(M_{2m+1};\mathbf{Z})$ *is a free module.*

(iii) *The order of a torsion element of* $H^{r+1}(M'_{2m+1};\mathbf{Z})$ *is exactly* 2, *i.e.,* $H^{r+1}(M'_{2m+1};\mathbf{Z})$ *does not contain higher two-torsions.*

Proof. This lemma is an easy consequence of Theorems D, E, 2.1 and Proposition 5.3. □

Now suppose that $H^r(M'_{2m+1};\mathcal{Z})$ contains a higher two-torsion. Then by Lemma 5.10 (iii), Ker $[\mu : H^r(M'_{2m+1};\mathcal{Z}) \to H^{r+1}(M'_{2m+1};\mathbf{Z})]$ contains a torsion element.

But by Lemma 5.10 (i)-(ii), Im $[\psi : H^r(M_{2m+1};\mathbf{Z}) \to H^r(M'_{2m+1};\mathcal{Z})]$ is a free module. This is a contradiction. Thus $H^r(M'_{2m+1};\mathcal{Z})$ $(r$: odd $\leq m-2)$ does not contain higher two-torsions.

This completes the proof of Theorem F (a).

(B) *The case of an even* m.

As in (A), it suffices to determine $H^r(M'_{2m+1};\mathcal{Z})$ $(r$: odd $\leq m-1)$. For an odd $r \leq m-3$, Lemma 5.10 applies and, by the same argument as in (A), we see that $H^r(M'_{2m+1};\mathcal{Z})$ does not contain higher two-torsions.

But Lemma 5.10 fails when $r = m-1$. Thus our argument cannot apply in this case.

This completes the proof of Theorem F (b). □

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