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A q-SERIES IDENTITY INVOLVING SCHUR FUNCTIONS AND RELATED TOPICS

Dedicated to Professor Takeshi Hirai on his sixtieth birthday

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1. Introduction

The main purpose of this paper is to prove:

Theorem 1.1. *For a Young diagram* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, $s_\lambda(x) = s_\lambda(x_1, x_2, x_3, \ldots)$ *denotes the corresponding Schur function, and, for each node* v *in the diagram* λ, $h(v)$ *denotes the hook length of* λ *at* v. *Then we have the following identity with a parameter* q:

$$\sum_{\lambda} I_\lambda(q) s_\lambda(x) = \prod_i \prod_{r=0}^{\infty} \frac{1 + x_i q^{r+1}}{1 - x_i q^r} \prod_{i<j} \frac{1}{1 - x_i x_j}, \tag{1.1}$$

where

$$I_\lambda(q) = \prod_{v \in \lambda} \frac{1 + q^{h(v)}}{1 - q^{h(v)}}, \tag{1.2}$$

and the sum on the left of (1.1) *is taken over all Young diagrams* λ.

When $q = 0$, (1.1) reduces to the identity

$$\sum_{\lambda} s_\lambda(x) = \prod_i \frac{1}{1 - x_i} \prod_{i<j} \frac{1}{1 - x_i x_j} \tag{1.3}$$

due to Schur and Littlewood (see [12], I, 5, Ex. 4). On the other hand, when $x_1 = z$ and $x_2 = x_3 = \cdots = 0$, (1.1) reduces to the $t = q$ case of the q-binomial theorem

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(see, e.g., [2]):

$$\sum_{n=0}^{\infty}\left(\prod_{i=1}^{n}\frac{1+tq^{i-1}}{1-q^{i}}\right)z^{n}=\prod_{r=0}^{\infty}\frac{1+tzq^{r}}{1-zq^{r}}. \tag{1.4}$$

Using the Frobenius character formula relating Schur functions with irreducible characters of symmetric groups, we see that Theorem 1.1 is equivalent to:

Theorem 1.2. *For a Young diagram λ with n nodes, χ_λ denotes the corresponding irreducible character of the symmetric group S_n, and $I_\lambda(q)$ as in (1.2). Then we have*

$$I_\lambda(q)=|S_n|^{-1}\sum_{s\in S_n}\chi_\lambda(s^2)\frac{\det(1+q\rho(s))}{\det(1-q\rho(s))}, \tag{1.5}$$

and

$$\sum_{|\lambda|=n}I_\lambda(q)\chi_\lambda(s)=\sum_{\substack{t\in S_n\\ t^2=s}}\frac{\det(1+q\rho(t))}{\det(1-q\rho(t))},\quad s\in S_n, \tag{1.6}$$

where $\rho: S_n\to GL_n(\mathbf{Z})$ is the representation of S_n by permutation matrices.

At $q=0$, the identities (1.5) and (1.6) reduce to well-known ones.

Let $\psi^{(2)}$ be the Adams operator of the second order acting on the space of generalized characters of S_n; $\psi^{(2)}$ is defined by

$$\psi^{(2)}(\chi_\lambda)(s)=\chi_\lambda(s^2),\quad s\in S_n, \tag{1.7}$$

or by

$$\psi^{(2)}(\chi_\lambda)=\chi_{\lambda,s}^{(2)}-\chi_{\lambda,a}^{(2)}, \tag{1.8}$$

where $\chi_{\lambda,s}^{(2)}$ and $\chi_{\lambda,a}^{(2)}$ are the symmetric and anti-symmetric squares of χ_λ respectively (see [15], 2.1). By (1.8), for any pair of Young diagrams λ,μ with n nodes, there exists a unique integer $d_{\lambda\mu}$ such that

$$\psi^{(2)}(\chi_\lambda)=\sum_{\mu}d_{\lambda\mu}\chi_\mu. \tag{1.9}$$

We are interested in the coefficients $d_{\lambda\mu}$. See [16], [14] for some of the known results on this and related problems. See also [5] (p.380, Appendix I.D) from which one can read off the values of $d_{\lambda\mu}$ (and also similar coefficients for the Adams operators of

higher orders) for $n \leq 8$.
Using Theorem 1.2 and a known formula [8], [13], [16], [4] for the sum

$$|S_n|^{-1} \sum_{s \in S_n} \chi_\lambda(s) \frac{\det(1 + q\rho(s))}{\det(1 - q\rho(s))},$$

we get the following.

Theorem 1.3. *For Young diagrams λ and μ with n nodes, let $d_{\lambda\mu}$ be as in (1.9). Then we have*

$$I_\lambda(q) = \sum_{|\mu|=n} d_{\lambda\mu} \prod_{v=v(i,j)\in\mu} \frac{q^{i-1} + q^j}{1 - q^{h(v)}}, \tag{1.10}$$

where $v = v(i,j)$ denotes the node at the intersection of the i-th row and the j-th column of the diagram μ.

Theorem 1.1–Theorem 1.3 will be proved in Section 3 after some preparations in Section 2.
Viewing (1.10) as a set of identities for series in q and comparing coefficients of the corresponding terms on the both hand sides, we get many relations for $d_{\lambda\mu}$'s. The first three of these are :

(1.11) $\quad d_{\lambda(n)} = 1$ (well-known),

(1.12) $\quad d_{\lambda(n)} + d_{\lambda(n-1,1)} = N_1^\lambda,$

(1.13) $\quad 2d_{\lambda(n)} + 3d_{\lambda(n-1,1)} + d_{\lambda(n-2,2)} + d_{\lambda(n-2,1^2)} = (N_1^\lambda)^2 + N_2^\lambda, n \geq 3,$

where

$$N_i^\lambda = |\{v \in \lambda \mid h(v) = i\}|, \tag{1.14}$$

and we understand

$$d_{\lambda(k,l^m)} = 0, \qquad \text{if } k < l.$$

Using these results as well as related techniques, we can determine some of the $d_{\lambda\mu}$'s explicitly. Here are examples:

$$d_{\lambda(n-1,1)} = N_1^\lambda - 1, \tag{1.15}$$

$$d_{\lambda(n-2,2)} = N_1^\lambda(N_1^\lambda - 2) + N_2^\lambda, n \geq 2, \tag{1.16}$$

$$d_{\lambda(n-2,1^2)} = -N_1^\lambda + 1, \tag{1.17}$$

$$d_{\lambda(1^n)} = \begin{cases} (-1)^{\sum \alpha_i}, & \text{if } \lambda = (\alpha_1, \dots, \alpha_p \mid \alpha_1, \dots, \alpha_p) \\ & \qquad \text{in Frobenius notation,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.18}$$

$$d_{\lambda(2,1^{n-2})} = \begin{cases} (-1)^{\sum \alpha_i}, & \text{if } \lambda \neq \lambda' \text{ and } \lambda = \sigma \cup \circ \\ & \qquad \text{for } \sigma = (\alpha_1, \dots, \alpha_p \mid \alpha_1, \dots, \alpha_p), \\ -(-1)^{\sum \alpha_i}, & \text{if } \lambda = (\alpha_1, \dots, \alpha_p \mid \alpha_1, \dots, \alpha_p) \text{ with } \alpha_p \neq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{1.19}$$

where λ' denotes the diagram conjugate to λ, and $\lambda = \sigma \cup \circ$ means λ is obtained by adding just one node to a self-conjugate diagram σ.
We can also give an algorithm for the computation of $d_{\lambda\mu}$ for any diagrams λ and μ. Although our algorithm is not very practical in general, it is rather efficient when μ is of hook-shape. This and (1.15)–(1.19) will be discussed in Section 4.

Our main result (1.1) is a partial generalization of the q-binomial theorem (1.4); a full generalization seems to have the following form.

Conjecture. We have

$$\sum_{\lambda} \left(\prod_{v \in \lambda} \frac{1 + q^{a(v)} t^{l(v)+1}}{1 - q^{a(v)+1} t^{l(v)}} \right) P_\lambda(x; q^2, t^2) = \prod_{i} \prod_{r=0}^{\infty} \frac{1 + t x_i q^r}{1 - x_i q^r} \prod_{i<j} \prod_{r=0}^{\infty} \frac{1 - t^2 x_i x_j q^{2r}}{1 - x_i x_j q^{2r}}, \tag{1.20}$$

where $P_\lambda(x; q^2, t^2)$ denote the Macdonald symmetric functions (see [12], IV), and $a(v)$ and $l(v)$ are the arm-length and the leg-length (see Section 5) of λ at the node v respectively.

For $t = -q$, (1.20) reduces to the Schur-Littlewood identity (1.3), for $t = q$, to our (1.1), and, for $x_1 = z$ and $x_2 = x_3 = \cdots = 0$, to the q-binomial Theorem (1.4). Moreover, for $q = 0$, (1.20) reduces to the following identity, which was essentially proved (using representation theory of general linear groups over finite fields) in [6]:

$$\sum_{\lambda} \prod_{\substack{v \in \lambda \\ a(v)=0}} (1 + t^{l(v)+1}) P_\lambda(x; t^2) = \prod_{i} \frac{1 + t x_i}{1 - x_i} \prod_{i<j} \frac{1 - t^2 x_i x_j}{1 - x_i x_j}, \tag{1.21}$$

where $P_\lambda(x; t^2)$ denote the Hall-Littlewood symmetric functions (see [12]). See Section 5 for the identity (1.21) (and another identity proved in the same way).

2. Preliminaries

2.1. Partitions and diagrams. A partition

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \dots) \tag{2.1}$$

is an infinite sequence of non-negative integers λ_i in non-decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots$$

containing finitely many non-zero terms. In the expression (2.1), zero terms are often omitted. If $m_i(\lambda)$ is the number of times $i(\neq 0)$ occurs as a term of the partition (2.1), we also write

$$\lambda = (\dots, i^{m_i(\lambda)}, \dots, 1^{m_1(\lambda)}).$$

The number of non-zero terms (or *parts*) of λ is denoted by $l(\lambda)$. A partition (2.1) is often identified with the Young diagram with $l(\lambda)$ rows whose i-th row contains exactly λ_i nodes. The number of nodes in the diagram λ is denoted by $|\lambda|$, namely

$$|\lambda| = \sum_i \lambda_i.$$

We define the partition λ' *conjugate* to λ by

$$\lambda' = (\lambda_1', \lambda_2', \dots),$$

where λ_i' is the number of nodes in the i-th column of the diagram λ. If $\lambda = \lambda'$, we say that λ is *self-conjugate*. For the node $v = v(i,j)$ of λ at the intersection of the i-th row and the j-th column, the corresponding *hook length* $h(v)$ is defined by

$$h(v) = \lambda_i + \lambda_j' - i - j + 1.$$

We also need Frobenius notation for partitions. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition. Putting

$$p = \max\{i \mid \lambda_i \geq i\} = \max\{i \mid \lambda_i' \geq i\},$$

and

$$\alpha_i = \lambda_i - i, \quad \beta_i = \lambda_i' - i, \quad 1 \leq i \leq p,$$

we denote the partition λ by

$$\lambda = (\alpha_1, \alpha_2, \dots, \alpha_p \mid \beta_1, \beta_2, \dots, \beta_p).$$

2.2. Symmetrizing operators and Schur functions. Let F_n be the ring of series in n variables $x_1, x_2, \ldots, x_n$. For an element f of F_n, and an element s of the symmetric group S_n, we put

$$f^s(x_1, x_2, \ldots, x_n) = f(x_{s^{-1}(1)}, x_{s^{-1}(2)}, \ldots, x_{s^{-1}(n)}).$$

The *symmetrizing operator* [10]

$$\pi_n : F_n \longrightarrow F_n$$

is defined by

$$\pi_n(f) = \left(\prod_{i<j}(x_i - x_j)\right)^{-1} \sum_{s \in S_n} \operatorname{sgn}(s)(f x^{\delta(n)})^s, \quad f \in F_n, \tag{2.2}$$

where

$$x^{\delta(n)} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

The following properties of π_n are easy to see.

$$\pi_n(f)^s = \pi_n(f), \quad f \in F_n, s \in S_n. \tag{2.3}$$

$$\pi_n(fg) = f\pi_n(g), \tag{2.4}$$

where f, $g \in F_n$ and $f^s = f$ for any $s \in S_n$.

$$\pi_n(f) \in \mathbf{Z}[x_1, x_2, \ldots, x_n], \quad \text{if } f \in \mathbf{Z}[x_1, x_2, \ldots, x_n]. \tag{2.5}$$

$$\pi_n(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) = 0, \tag{2.6}$$

unless $a_i + n - i, 1 \leq i \leq n$, are all distinct.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition with $l(\lambda) \leq n$. Then the symmetric polynomial

$$s_\lambda(x_1, x_2, \ldots, x_n) = \pi_n(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}) \tag{2.7}$$

is called the *Schur polynomial* in n variables corresponding to λ. It is easy to see that

$$s_\lambda(x_1, x_2, \ldots, x_n, x_{n+1})|_{x_{n+1}=0} = s_\lambda(x_1, x_2, \ldots, x_n),$$

which implies that we can define the *Schur function* [12]

$$s_\lambda(x) = s_\lambda(x_1, x_2, \ldots \ldots)$$

in infinite variables $x = (x_1, x_2, \ldots \ldots)$ by letting $n \to \infty$ in (2.7).

Lemma 2.1 ([10]). *Let π_n and $s_\lambda(x_1, x_2, \ldots, x_n)$ be as above. Then:*

(i) *Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition with $l(\lambda) \leq n-1$, and m a non-negative integer. Then we have*

$$\pi_n(s_\lambda(x_1, x_2, \ldots, x_{n-1})x_n^m) = 0$$

if $m = \lambda_i + n - i$ for some $1 \leq i \leq n-1$, and

$$\pi_n(s_\lambda(x_1, x_2, \ldots, x_{n-1})x_n^m) = \mathrm{sgn}(w)s_\mu(x_1, x_2, \ldots, x_n)$$

otherwise,where the element $w = w(\lambda, m)$ of S_n and the partition $\mu = \mu(\lambda, m)$ are uniquely determined by the conditions:

$$l(\mu) \leq n$$

and

$$\begin{aligned}&(\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_{n-1} + 1, m)\\ &= w(\mu_1 + n - 1, \mu_2 + n - 2, \ldots, \mu_n).\end{aligned}$$

(ii) *We have*

$$\pi_n\left(\prod_{i=1}^{n-1}(1 - x_i x_n)\right) = \begin{cases} 1 \quad , & \text{if } n \text{ is odd;} \\ 1 - x_1 x_2 \cdots x_n, & \text{if } n \text{ is even.}\end{cases}$$

Proof. (i) Let $\{w_1, \ldots, w_n\} \subset S_n$ be a set of representatives of the coset $S_{n-1}\backslash S_n$. Then, by (2.2)–(2.4) and (2.7),

$$\begin{aligned}
&\pi_n(s_\lambda(x_1, \ldots, x_{n-1})x_n^m)\prod_{i<j}(x_i - x_j) \\
&= \sum_{k=1}^{n}\sum_{s\in S_{n-1}} \mathrm{sgn}(sw_k)(s_\lambda(x_1, \ldots, x_{n-1})x^{\delta(n)}x_n^m)^{sw_k} \\
&= \sum_{k=1}^{n}\mathrm{sgn}(w_k)\left\{s_\lambda(x_1, \ldots, x_{n-1})(x_1\cdots x_{n-1})x_n^m \sum_{s\in S_{n-1}}\mathrm{sgn}(s)(x^{\delta(n-1)})^s\right\}^{w_k} \\
&= \sum_{k=1}^{n}\mathrm{sgn}(w_k)\left\{s_\lambda(x_1, \ldots, x_{n-1})(x_1\cdots x_{n-1})x_n^m \prod_{i<j<n}(x_i - x_j)\right\}^{w_k} \\
&= \sum_{k=1}^{n}\mathrm{sgn}(w_k)\left\{\sum_{s\in S_{n-1}}\mathrm{sgn}(s)(x_1^{\lambda_1}\cdots x_{n-1}^{\lambda_{n-1}}x^{\delta(n-1)})^s(x_1\cdots x_{n-1})x_n^m\right\}^{w_k} \\
&= \pi_n(x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_{n-1}^{\lambda_{n-1}}x_n^m)\prod_{i<j}(x_i - x_j).
\end{aligned}$$

We have shown

$$\pi_n(s_\lambda(x_1,\ldots,x_{n-1})x_n^m) = \pi_n(x_1^{\lambda_1}\cdots x_{n-1}^{\lambda_{n-1}}x_n^m).$$

Hence Part (i) follows from (2.6) and (2.7).

(ii) We have

$$\prod_{i=1}^{n-1}(1-x_ix_n) = \sum_{k=0}^{n}(-1)^k\Big\{\sum_{i_1<i_2<\cdots<i_k<n} x_{i_1}x_{i_2}\cdots x_{i_k}\Big\}x_n^k.$$

On the other hand, by (2.6), we have

$$\pi_n(x_{i_1}x_{i_2}\cdots x_{i_k}x_n^k) = 0, \quad k>0, i_1<i_2<\cdots<i_k<n$$

unless n is even, $k=n/2$, and $(i_1,i_2,\ldots,i_k) = (1,2,\ldots,n/2)$. Moreover, by (2.2) and (2.4), we have

$$\begin{aligned}\pi_n(x_1x_2\cdots x_{n/2}x_n^{n/2}) &= (-1)^{n/2-1}\pi_n(x_1x_2\cdots x_n)\\ &= (-1)^{n/2-1}x_1x_2\cdots x_n\end{aligned}$$

if n is even. Now Part (ii) follows. □

For a positive integer k, we put

$$p_k(x) = \sum_i x_i^k,$$

and, for an element s of the symmetric group S_n, we put

$$p_s(x) = p_{\mu_1}(x)p_{\mu_2}(x)\cdots p_{\mu_n}(x),$$

where $(\mu_1,\mu_2,\ldots\mu_n)$ $(\sum_i \mu_i = n)$ is the cycle-type of s. For a partition λ with$|\lambda| = n$, χ_λ denotes the corresponding irreducible character of S_n. This means

$$s_\lambda(x) = |S_n|^{-1}\sum_{s\in S_n}\chi_\lambda(s)p_s(x) \tag{2.8}$$

(Frobenius character formula).

2.3. Adams operator. The following lemma relates the infinite product

$$\prod_i\prod_{r=0}^{\infty}\frac{1+tx_iq^r}{1-x_iq^r}\prod_{i<j}\prod_{r=0}^{\infty}\frac{1-t^2x_ix_jq^{2r}}{1-x_ix_jq^{2r}} \tag{2.9}$$

appearing on the right hand side of (1.20) to the Adams operator $\psi^{(2)}$ (see (1.7) and (1.8)).

Lemma 2.2. *Let $A(x;q,t)$ be the infinite product (2.9). For a partition λ, let $a_\lambda(q,t)$ be a function in q and t defined by*

$$A(x;q,t)=\sum_\lambda a_\lambda(q,t)s_\lambda(x). \tag{2.10}$$

Then we have

$$a_\lambda(q,t)=|S_n|^{-1}\sum_{s\in S_n}\psi^{(2)}(\chi_\lambda)(s)\frac{\det(1+t\rho_n(s))}{\det(1-q\rho_n(s))}, \tag{2.11}$$

where $n=|\lambda|$, and $\rho_n:S_n\to GL_n(\mathbf{Z})$ is the representation of S_n by permutation matrices.

Proof. We calculate, as in [12], p.120, Ex.11,

$$\begin{aligned}
\log A(x;q,t) &= \sum_i\sum_r\{\log(1+tx_iq^r)-\log(1-x_iq^r)\}\\
&\quad+\sum_{i<j}\sum_r\{\log(1-t^2x_ix_jq^{2r})-\log(1-x_ix_jq^{2r})\\
&=\sum_i\sum_r\sum_{k=1}^{\infty}\left\{-\frac{(-tx_iq^r)^k}{k}+\frac{(x_iq^r)^k}{k}\right\}\\
&\quad+\sum_{i<j}\sum_r\sum_{k=1}^{\infty}\left\{-\frac{(t^2x_ix_jq^{2r})^k}{k}+\frac{(x_ix_jq^{2r})^k}{k}\right\}\\
&=\sum_i\sum_k\left\{-\frac{1}{1-q^k}\frac{(-tx_i)^k}{k}+\frac{1}{1-q^k}\frac{x_i^k}{k}\right\}\\
&\quad+\sum_{i<j}\sum_k\left\{-\frac{1}{1-q^{2k}}\frac{(t^2x_ix_j)^k}{k}+\frac{1}{1-q^{2k}}\frac{(x_ix_j)^k}{k}\right\}\\
&=\sum_i\sum_k\frac{1-(-t)^k}{1-q^k}\frac{x_i^k}{k}+\sum_{i<j}\sum_k\frac{1-t^{2k}}{1-q^{2k}}\frac{(x_ix_j)^k}{k}\\
&=\sum_k\frac{1-(-t)^k}{1-q^k}\frac{p_k(x)}{k}+\sum_k\frac{1-t^{2k}}{1-q^{2k}}\frac{p_k(x)^2-p_{2k}(x)}{2k}\\
&=\sum_{k\ \mathrm{odd}}\frac{1+t^k}{1-q^k}\frac{p_k(x)}{k}+\sum_k\frac{1-t^{2k}}{1-q^{2k}}\frac{p_k(x)^2}{2k}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
A(x;q,t) &= \prod_{k\ \mathrm{odd}} \exp\left(\frac{1+t^k}{1-q^k}\frac{p_k(x)}{k}\right)\prod_k \exp\left(\frac{1-t^{2k}}{1-q^{2k}}\frac{p_k(x)^2}{2k}\right) \\
&= \prod_{k\ \mathrm{odd}}\left\{\sum_m \frac{1}{m!}\left(\frac{1+t^k}{1-q^k}\frac{p_k(x)}{k}\right)^m\right\}\prod_k\left\{\sum_l \frac{1}{l!}\left(\frac{1-t^{2k}}{1-q^{2k}}\frac{p_k(x)^2}{2k}\right)^l\right\} \\
&= \sum_{n=0}^{\infty}\sum_{m_k,l_j}\prod_{\substack{k,j\\ k\ \mathrm{odd}}}\frac{1}{m_k!k^{m_k}l_j!(2j)^{l_j}} \\
&\qquad\times\left(\frac{1+t^k}{1-q^k}\right)^{m_k}\left(\frac{1-t^{2j}}{1-q^{2j}}\right)^{l_j} p_k(x)^{m_k}p_j(x)^{2l_j},
\end{aligned}$$

where the sum on m_k, l_j is taken over the set of sequences $(m_1, m_3, m_5, \ldots; l_1, l_2, \ldots)$ of non-negative integers m_k, l_j such that

$$\sum_{k\ \mathrm{odd}} km_k + \sum_j j(2l_j) = n.$$

Let u be an element of S_n with cycle-type $(\mu_1, \mu_2, \ldots, \mu_n)$. If we put

$$m_k = |\{j \mid \mu_j = k\}|, \quad 1 \le k \le n,$$

then

$$\det(1 - q\rho_n(u)) = \prod_k (1 - q^k)^{m_k},$$

and the order of the centralizer of u in S_n is equal to

$$\prod_k m_k!k^{m_k}.$$

Hence we have

$$\begin{aligned}
A(x;q,t) &= \sum_{n=0}^{\infty}|S_n|^{-1}\sum_{u\in S_n}\frac{\det(1+t\rho_n(u))}{\det(1-q\rho_n(u))}p_{u^2}(x) \\
&= \sum_n |S_n|^{-1}\sum_{s\in S_n}\left\{\sum_{\substack{u\in S_n\\ u^2=s}}\frac{\det(1+t\rho_n(u))}{\det(1-q\rho_n(u))}\right\}p_s(x).
\end{aligned}$$

The lemma now follows from Frobenius character formula (2.8). □

3. Proofs of Theorem 1.1–Theorem 1.3

3.1. Proof of Theorem 1.1. It is enough to prove this for a finite set of variables $x_1, x_2, \ldots, x_n$, i.e. in the case when $x_{n+1} = x_{n+2} = \cdots = 0$. Then (1.1) takes the following form:

$$\sum_{\lambda} I_\lambda(q) s_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} \prod_{r=0}^{\infty} \frac{1 + x_i q^{r+1}}{1 - x_i q^r} \prod_{\substack{i,j=1 \\ i<j}}^{n} \frac{1}{1 - x_i x_j}, \tag{3.1}$$

where the sum on the left is over all partitions λ with $l(\lambda) \leq n$. As noted in Section 1, this is true for $n = 1$. Let $F(n)$ be the right hand side of (3.1). Then we have

$$F(n-1) \prod_{r=0}^{\infty} \frac{1 + x_n q^{r+1}}{1 - x_n q^r} = F(n) \prod_{i=1}^{n-1} (1 - x_i x_n).$$

Hence, by induction assumption, we have

$$\Big(\sum_{\lambda} I_\lambda(q) s_\lambda(x_1, \ldots, x_{n-1}) \Big) \Big(\sum_{m=0}^{\infty} \prod_{i=1}^{m} \frac{1 + q^i}{1 - q^i} x_n^m \Big) = F(n) \prod_{i<n} (1 - x_i x_n), \tag{3.2}$$

where the sum on λ is over all partitions with $l(\lambda) \leq n - 1$. By applying the symmetrizing operator π_n (see Section 2.2) on the both hand sides of (3.2), we get

$$\begin{aligned} &\sum_{\lambda} \sum_{m=0}^{\infty} \Big(I_\lambda(q) \prod_{i=1}^{m} \frac{1 + q^i}{1 - q^i} \Big) \pi_n(s_\lambda(x_1, \ldots, x_{n-1}) x_n^m) \\ &\quad = F(n) \pi_n \Big(\prod_{i<n} (1 - x_i x_n) \Big). \end{aligned} \tag{3.3}$$

Now let $F(n)^*$ be the left hand side of (3.1). In view of (3.3), for a proof of (3.1), it is enough to show:

$$\begin{aligned} &\sum_{\lambda} \sum_{m=0}^{\infty} \Big(I_\lambda(q) \prod_{i=1}^{m} \frac{1 + q^i}{1 - q^i} \Big) \pi_n(s_\lambda(x_1, \ldots, x_{n-1}) x_n^m) \\ &\quad = F(n)^* \pi_n \Big(\prod_{i<n} (1 - x_i x_n) \Big). \end{aligned} \tag{3.4}$$

Since the both hand sides of (3.4) are symmetric polynomials in $x_1, \ldots, x_n$, they can be written as linear combinations of Schur polynomials $s_\mu(x_1, \ldots, x_n), l(\mu) \leq n$. If $l(\mu) \leq n - 1$, then the coefficients of $s_\mu(x_1, \ldots, x_n)$ on the left and right hand sides of (3.4) are both equal to $I_\mu(q)$ by Lemma 2.1(i)(ii) and the multiplication rule ([12], p.73, (5.17)) between Schur functions and elementary symmetric functions. If

$\mu = (\mu_1, \dots, \mu_n)$ is such that $l(\mu) = n$, then, by Part (i) of Lemma 2.1, the coefficient of $s_\mu(x_1, \dots, x_n)$ of the left hand side of (3.4) is equal to

$$\sum_{j=1}^{n} (-1)^{n-j} I_{\mu(j)}(q) \prod_{k=1}^{\mu_j+n-j} \frac{1+q^k}{1-q^k},$$

where $\mu(j) = (\mu(j)_1, \mu(j)_2, \dots)$ is a partition with $l(\mu(j)) \leq n-1$ defined by:

$$\begin{gathered} \mu(j)_1 = \mu_1, \mu(j)_2 = \mu_2, \dots, \mu(j)_{j-1} = \mu_{j-1}, \\ \mu(j)_j = \mu_{j+1} - 1, \mu(j)_{j+1} = \mu_{j+2} - 1, \dots, \mu(j)_{n-1} = \mu_n - 1, \end{gathered}$$

and, by Part (ii) of Lemma 2.1, the coefficient of $s_\mu(x_1, \dots, x_n)$ on the right hand side of (3.4) is equal to $I_\mu(q)$ or $I_\mu(q) - I_{\mu-(1^n)}(q)$ according as n is odd or even, where $\mu - (1^n) = (\mu_1 - 1, \dots, \mu_n - 1)$. Thus (3.4) is equivalent to:

Lemma 3.1. *Let μ be a partition with $l(\mu) = n$. In the above notation, we have*

$$I_\mu(q) = \sum_{j=1}^{n} (-1)^{n-j} I_{\mu(j)}(q) \prod_{m=1}^{\mu_j+n-j} \frac{1+q^m}{1-q^m}, \tag{3.5}$$

if n is odd, and

$$I_\mu(q) = \sum_{j=1}^{n} (-1)^{n-j} I_{\mu(j)}(q) \prod_{m=1}^{\mu_j+n-j} \frac{1+q^m}{1-q^m} + I_{\mu-(1^n)}(q), \tag{3.6}$$

if n is even.

Proof. We put $\nu_j = \mu_j + n - j$ for $1 \leq j \leq n$. Multiplying the both hand sides of (3.5) and (3.6) by

$$\left(\prod_{j=1}^{n} \prod_{m=1}^{\nu_j} \frac{1+q^m}{1-q^m} \right)^{-1},$$

we see that (3.5) and (3.6) are equivalent to:

$$\prod_{i<j} \frac{1-q^{\nu_i-\nu_j}}{1+q^{\nu_i-\nu_j}} = \sum_{k=1}^{n} (-1)^{n-k} \prod_{\substack{i<j \\ i\neq k \\ j\neq k}} \frac{1-q^{\nu_i-\nu_j}}{1+q^{\nu_i-\nu_j}} \prod_{i\neq k} \frac{1-q^{\nu_i}}{1+q^{\nu_i}} \tag{3.7}$$

and

$$\prod_{i<j} \frac{1-q^{\nu_i-\nu_j}}{1+q^{\nu_i-\nu_j}} = \sum_{k=1}^{n} (-1)^{n-k} \prod_{\substack{i<j\\ i\neq k\\ j\neq k}} \frac{1-q^{\nu_i-\nu_j}}{1+q^{\nu_i-\nu_j}} \prod_{i\neq k} \frac{1-q^{\nu_i}}{1+q^{\nu_i}} + \prod_{i<j} \frac{1-q^{\nu_i-\nu_j}}{1+q^{\nu_i-\nu_j}} \prod_{i=1}^{n} \frac{1-q^{\nu_i}}{1+q^{\nu_i}} \tag{3.8}$$

respectively. Putting $A_i = q^{\nu_i}$, we can rewrite (3.7) and (3.8) as

$$\prod_{i<j} \frac{A_j - A_i}{A_j + A_i} = \sum_{k=1}^{n} (-1)^{n-k} \prod_{\substack{i<j\\ i\neq k\\ j\neq k}} \frac{A_j - A_i}{A_j + A_i} \prod_{i\neq k} \frac{1-A_i}{1+A_i}$$

and

$$\prod_{i<j} \frac{A_j - A_i}{A_j + A_i} = \sum_{k=1}^{n} (-1)^{n-k} \prod_{\substack{i<j\\ i\neq k\\ j\neq k}} \frac{A_j - A_i}{A_j + A_i} \prod_{i\neq k} \frac{1-A_i}{1+A_i} + \prod_{i<j} \frac{A_j - A_i}{A_j + A_i} \prod_{i} \frac{1-A_i}{1+A_i}$$

respectively. We can further rewrite these equalities as

$$\prod_{i=1}^{n} \frac{1+A_i}{1-A_i} = \sum_{k=1}^{n} \prod_{i\neq k} \frac{A_k + A_i}{A_k - A_i} \frac{1+A_k}{1-A_k} \tag{3.9}$$

(which is to be proved for odd n) and

$$\prod_{i=1}^{n} \frac{1+A_i}{1-A_i} = \sum_{k=1}^{n} \prod_{i\neq k} \frac{A_k + A_i}{A_k - A_i} \frac{1+A_k}{1-A_k} + 1 \tag{3.10}$$

(which is to be proved for even n). Now, by the partial fraction expansion

$$\prod_{i=1}^{n} \frac{z-tA_i}{z-A_i} = \sum_{k=1}^{n} \frac{(1-t)A_k}{z-A_k} \prod_{i\neq k} \frac{A_k - tA_i}{A_k - A_i} + 1, \tag{3.11}$$

we see that

$$\prod_{i=1}^{n} \frac{1+A_i}{1-A_i} = \sum_{k=1}^{n} \frac{2A_k}{1-A_k} \prod_{i\neq k} \frac{A_k + A_i}{A_k - A_i} + 1. \tag{3.12}$$

We also have

$$\frac{1-(-1)^n}{2} = \sum_{k=1}^{n} \prod_{i\neq k} \frac{A_k + A_i}{A_k - A_i} \tag{3.13}$$

(put $z = 0$ and $t = -1$ in (3.11)). Adding (3.12) and (3.13), we get (3.9) and (3.10). This proves Lemma 3.1. □

The proof of Theorem 1.1 is now complete.

3.2. Proofs of Theorem 1.2 and Theorem 1.3. By putting $t = q$ in Lemma 2.2, and using Theorem 1.1, we get (1.5). The formula (1.6) follows from (1.5) via the orthogonality relations for χ_λ. This proves Theorem 1.2.
For a proof of Theorem 1.3, we need the following formula (see [8], [13], [16], [4]):

$$(3.14) \qquad |S_n|^{-1} \sum_{s \in S_n} \chi_\mu(s) \frac{\det(1 + t\rho_n(s))}{\det(1 - q\rho_n(s))} = \prod_{v=v(i,j)\in\mu} \frac{q^{i-1} + tq^{j-1}}{1 - q^{h(v)}},$$

where μ is a partition with $|\mu| = n$, and $v = v(i,j)$ is as in (1.10). This formula, together with (1.9) and (2.11), implies

$$(3.15) \qquad a_\lambda(q,t) = \sum_\mu d_{\lambda\mu} \prod_{v=v(i,j)\in\mu} \frac{q^{i-1} + tq^{j-1}}{1 - q^{h(v)}},$$

where the sum on the right is over all partitions μ with $|\mu| = |\lambda|$. Combining (3.15) for $t = q$ with (2.10) and (1.1), we get Theorem 1.3.

REMARK. It would be interesting to generarize Theorem 1.2 (and Theorem 1.3) to finite reflection groups. The formula (3.14) has been generalized to the case of Weyl groups by Gyoja, Nishiyama and Shimura [4]. See also [8], [13].

4. The coefficients $d_{\lambda\mu}$

Comparing the coefficients of q^0, q^1 and q^2 on the both hand sides of (1.10), we get (1.11)–(1.13). These formula imply:

$$(4.1) \qquad d_{\lambda(n)} = 1 \quad \text{(well-known)},$$

$$(4.2) \qquad d_{\lambda(n-1,1)} = N_1^\lambda - 1,$$

$$(4.3) \qquad d_{\lambda(n-2,2)} + d_{\lambda(n-2,1^2)} = (N_1^\lambda)^2 - 3N_1^\lambda + N_2^\lambda + 1$$

Thus (1.10) is not sufficient in determining $d_{\lambda(n-2,2)}$ and $d_{\lambda(n-2,1^2)}$ for all n and λ. To determine these and some other coefficients $d_{\lambda\mu}$, we shall use:

Theorem 4.1. *For a partition λ, we put*

$$a_\lambda(t) = a_\lambda(0,t)$$

in the notation of Lemma 2.2. *Then*:

(i) *We have*

$$\left(\sum_{\mu} s_{\mu}(x)\right)\left(\sum_{\sigma}(-1)^{\sum \alpha_i} s_{\sigma}(x)t^{|\sigma|}\right) = \sum_{\lambda} a_{\lambda}(t)s_{\lambda}(x) \tag{4.4}$$

and

$$\left(\sum_{\mu} s_{\mu}(x)t^{|\mu|}\right)\left(\sum_{\sigma}(-1)^{\sum \alpha_i} s_{\sigma}(x)\right) = \sum_{\lambda} t^{|\lambda|} a_{\lambda}(t^{-1})s_{\lambda}(x), \tag{4.5}$$

where the sums on μ and λ are over all partitions, and the sums on σ are over all self-conjugate partitions $\sigma = (\alpha_1, \dots, \alpha_p \mid \alpha_1, \dots, \alpha_p)$ (in Frobenius notation; see Section 2.1).

(ii) *We have*

$$a_{\lambda}(t) = \sum_{r=0}^{n} (d_{\lambda(n-r+1,1^{r-1})} + d_{\lambda(n-r,1^r)})t^r, \quad n = |\lambda|, \tag{4.6}$$

where $d_{\lambda(n+1,1^{-1})}$ and $d_{\lambda(0,1^n)}$ are understood to be 0.

Proof. Putting $q = 0$ in (2.10) and (2.11), we have

$$\prod_i \frac{1+tx_i}{1-x_i} \prod_{i<j} \frac{1-t^2x_ix_j}{1-x_ix_j} = \sum_{\lambda} a_{\lambda}(t)s_{\lambda}(x) \tag{4.7}$$

and

$$a_{\lambda}(t) = |S_n|^{-1} \sum_{s \in S_n} \psi^{(2)}(\chi_{\lambda}) \det(1 + t\rho_n(s)), \quad n = |\lambda|. \tag{4.8}$$

Since

$$\prod_i (1-x_i) \prod_{i<j} (1-x_ix_j) = \sum_{\sigma} (-1)^{p+\sum \alpha_i} s_{\sigma}(x)$$

(see [12], I, 5, Ex.9(c)), we have

$$\prod_i (1+tx_i) \prod_{i<j} (1-t^2x_ix_j) = \sum_{\sigma} (-1)^{\sum \alpha_i} s_{\sigma}(x)t^{|\sigma|}. \tag{4.9}$$

By (4.7), (4.9) and (1.3), we get (4.4). The identity (4.5) follows from (4.4) by just replacing, in the latter, t with t^{-1} and then x_i with tx_i. This proves Part(i). Part (ii) follows from (4.8) and

$$\det(1 + t\rho_n(s)) = \sum_{r=0}^{n} (\chi_{(n-r+1,1^{r-1})} + \chi_{(n-r,1^r)})(s)t^r,$$

which is well-known and is equivalent to the $q = 0$ case of (3.14).

We now show how we can derive formulas like (1.16)–(1.19) from Theorem 4.1 and Theorem 1.1. Comparing the coefficients of $t^r (r = 0, 1, 2, \dots)$ on the both hand sides of (4.4) and (4.5) and taking (4.6) into consideration, we have

$$\left(\sum_{\mu} s_\mu(x)\right)\left(\sum_{\substack{\sigma \\ |\sigma|=r}} (-1)^{\sum \alpha_i} s_\sigma(x)\right) = \sum_{\lambda} (d_{\lambda(n-r+1,1^{r-1})} + d_{\lambda(n-r,1^r)}) s_\lambda(x) \tag{4.10}$$

and

$$\left(\sum_{\substack{\mu \\ |\mu|=r}} s_\mu(x)\right)\left(\sum_{\sigma} (-1)^{\sum \alpha_i} s_\sigma(x)\right) = \sum_{\lambda} (d_{\lambda(r,1^{n-r})} + d_{\lambda(r+1,1^{n-r-1})}) s_\lambda(x) \tag{4.11}$$

respectively, where $n = |\lambda|$ and the sums on σ are taken over all self-conjugate partitions $\sigma = (\alpha_1, \dots, \alpha_p \mid \alpha_1, \dots, \alpha_p)$. For three partitions μ, ν and λ, let $c^\lambda_{\mu\nu}$ be the Littlewood-Richarson coefficient in the expansion

$$s_\mu(x) s_\nu(x) = \sum_{\lambda} c^\lambda_{\mu\nu} s_\lambda(x). \tag{4.12}$$

As is well known, there exists a nice combinatorial rule (the Littlewood-Richardson rule) for computing $c^\lambda_{\mu\nu}$. See, e.g., [12], I, 9. By (4.10), (4.11) and (4.12), we have

$$\sum_{\substack{|\sigma|=r \\ |\mu|=n-r}} (-1)^{\sum \alpha_i} c^\lambda_{\mu\sigma} = d_{\lambda(n-r+1,1^{r-1})} + d_{\lambda(n-r,1^r)} \tag{4.13}$$

and

$$\sum_{\substack{|\mu|=r \\ |\sigma|=n-r}} (-1)^{\sum \alpha_i} c^\lambda_{\mu\sigma} = d_{\lambda(r,1^{n-r})} + d_{\lambda(r+1,1^{n-r-1})} \tag{4.14}$$

for any partition λ and any integer r with $0 \leq r \leq n = |\lambda|$, where the sums on σ are as in (4.10) and (4.11). By (4.13) and/or (4.14), we have an algorithm for the computation of $d_{\lambda\nu}$ for any partition $\nu = (s, 1^{n-s})$ of 'hook-shape'. (Note that the individual values of the Littlewood-Richardson coefficients $c^\lambda_{\mu\sigma}$ are not needed here; it is enough to know the sum $\sum_{|\mu|=|\lambda|-|\sigma|} c^\lambda_{\mu\sigma}$ for each pair (λ, σ) with $\sigma = \sigma'$.) For example, if we put $r = 0$ in (4.13) and (4.14), then we get (4.1) (again) and (1.18). If we put $r = 1$ in (4.13) and (4.14), then we get (1.12) (again) and

$$d_{\lambda(1^n)} + d_{\lambda(2,1^{n-2})} = \begin{cases} (-1)^{\sum \alpha_i}, & \text{if } \lambda = \sigma \cup \circ \text{for } \sigma = (\alpha_1, \dots, \alpha_p \mid \alpha_1, \dots, \alpha_p), \\ 0 \quad , & \text{otherwise,} \end{cases}$$

where $\lambda = \sigma \cup \circ$ means that the diagram λ is obtained by adding just one node to a self-conjugate diagram σ. This, together with (1.18), implies (1.19). If we put $r = 2$ in (4.13), then we get

$$d_{\lambda(n-1,1)} + d_{\lambda(n-2,1^2)} = 0.$$

This and (4.2) imply (1.17), and (1.17) and (4.3) imply (1.16). □

EXAMPLES.

(i) $d_{(7,1)(7,1)} = 1, d_{(5,2,1)(7,1)} = 2, d_{(4,4)(7,1)} = 0, d_{(4,2,1,1)(7,1)} = 2.$

(ii) $d_{(7,1)(6,2)} = 1, d_{(5,2,1)(6,2)} = 4, d_{(4,4)(6,2)} = 1, d_{(4,2,1,1)(6,2)} = 5.$

(iii) $d_{(7,1)(6,1,1)} = -1, d_{(5,2,1)(6,1,1)} = -2, d_{(4,4)(6,1,1)} = 0, d_{(4,2,1,1)(6,1,1)} = -2.$

(iv) $d_{\lambda(1^8)} = \begin{cases} -1, & \text{if } \lambda = (4,2,1,1) \text{ or } (3,3,2), \\ 0, & \text{otherwise.} \end{cases}$

(v) $d_{\lambda(2,1^6)} = \begin{cases} -1, & \text{if } \lambda = (5,1^3) \text{ or } (4,1^4), \\ 1, & \text{if } \lambda = (3,3,2), \\ 0, & \text{otherwise.} \end{cases}$

REMARK.

(i) By (4.7) and (2.10), we have

$$\prod_{r=0}^{\infty} \left(\sum_{\mu} a_\mu(t) q^{r|\mu|} s_\mu(x) \right) = \sum_{\lambda} a_\lambda(q,t) s_\lambda(x).$$

Hence

$$a_\lambda(q,t) = \sum_{n=0}^{\infty} \left\{ \sum_{\substack{\sum r|\mu_r| = n \\ \sum |\mu_r| = |\lambda|}} c^{\lambda}_{\mu_1 \mu_2 \dots \mu_k \dots} \prod_{r=0}^{\infty} a_{\mu_r}(t) \right\} q^n, \tag{4.15}$$

where, for partitions $\lambda, \mu_1, \mu_2, \dots$, we define $c^{\lambda}_{\mu_1 \mu_2 \dots \mu_k \dots}$ by

$$s_{\mu_1}(x) s_{\mu_2}(x) \cdots s_{\mu_k}(x) \cdots = \sum_{\lambda} c^{\lambda}_{\mu_1 \mu_2 \dots \mu_k \dots} s_\lambda(x).$$

By (3.15), the knowledge of the function $a_\lambda(q,t)$ is sufficient for the determination of $d_{\lambda\mu}$ for any μ with $|\mu| = |\lambda|$. Thus (4.4), (4.5) and (4.15), together with the Littlewood-Richardson rule, give an algorithm for the computation of $d_{\lambda\mu}$. Theorem 1.1, which gives an explicit formula for $a_\lambda(q,q)$, is sometimes helpful to shorten the computation.

(ii) For a positive integer r, the Adams operator $\psi^{(r)}$ of the r-th order is defined by

$$\psi^{(r)}(\chi_\lambda)(s) = \chi_\lambda(s^r), \quad s \in S_n.$$

Since

$$p_r(x) = \sum_{t=0}^{r-1}(-1)^t s_{(r-t,1^t)}(x)$$

(see [12], I, 3, Ex.11), we have the following generalization of (1.8):

$$\psi^{(r)}(\chi_\lambda) = \sum_{t=0}^{r-1}(-1)^t \chi_{\lambda(r-t,1^t)}, \tag{4.16}$$

where $\chi_{\lambda(r-t,1^t)}$ is the symmetrization of χ_λ by $\chi_{(r-t,1^t)}$ (in the terminology of [5], 5.2). For any pair of partitions (λ,μ) with $l(\lambda) = l(\mu) = n$, let $d^{(r)}_{\lambda\mu}$ be an integer defined by

$$\psi^{(r)}(\chi_\lambda) = \sum_\mu d^{(r)}_{\lambda\mu}\chi_\mu.$$

By (4.16), one can read off the values of $d^{(r)}_{\lambda\mu}$ for $r \le 5$ and $n \le 8$ from the tables in [5], Appendix I, D. We observe that the absolute values of these numbers are relatively 'small'; perhaps this suggests the existence of a nice theory for the coefficients $d^{(r)}_{\lambda\mu}$.

5. Symmetric spaces over finite fields

It is known ([3], [7], [6]) that the permutation representation of the general linear group $\mathrm{GL}_n(\mathbf{F}_{q^2})$ over a finite field $\mathbf{F}_{q^2}$ of q^2 elements on the 'symmetric space' $\mathrm{GL}_n(\mathbf{F}_{q^2})/\mathrm{GL}_n(\mathbf{F}_q)$ is multiplicity-free. As noted in [6], Theorem 3.2.6 (ii), this fact can be expressed as a set of identities for Green polynomials. By the relation ([12], III, 7) between Hall-Littlewood symmetric functions $P_\lambda(x,;t)$ and Green polynomials, the latter is equivalent to:

$$\sum_{\substack{\lambda \\ |\lambda|=n}} \frac{b_\lambda(t^2)}{b_\lambda(t)} P_\lambda(x;t^2) = |S_n|^{-1} \sum_{s\in S_n} \prod_j \frac{(1-t^{2j})^{m_j(s^2)}}{(1-t^j)^{m_j(s)}} p_{s^2}(x), \tag{5.1}$$

where $m_j(s) = m_j(\nu_s)$ denotes the number of times j occurs as a part of the cycle-type ν_s of $s \in S_n$, and $b_\lambda(t)$ is a polynomial in t defined by

$$b_\lambda(t) = \prod_j (1-t)(1-t^2)\cdots(1-t^{m_j(\lambda)})$$

(see [6], Remark 3.2.7(ii)). It is easy to see that

$$\prod_j \frac{(1-t^{2j})^{m_j(s^2)}}{(1-t^j)^{m_j(s)}} = \det(1 + t\rho_n(s)),$$

and that

$$\frac{b_\lambda(t^2)}{b_\lambda(t)} = \prod_{\substack{v\in\lambda \\ a(v)=0}} (1 + t^{l(v)+1}),$$

where, for a node $v = v(i,j)$ of the diagram λ, we define the *arm-length* $a(v) = a_\lambda(v)$ and the *leg-length* $l(v) = l_\lambda(v)$ of λ at v by

$$a(v) = \lambda_i - j, \quad l(v) = \lambda'_j - i.$$

Moreover, by the proof of Lemma 2.2 with $q = 0$, we have

$$\prod_i \frac{1+tx_i}{1-x_i} \prod_{i<j} \frac{1-t^2x_ix_j}{1-x_ix_j} = \sum_{n=0}^{\infty} |S_n|^{-1} \sum_{s\in S_n} \det(1+t\rho_n(s))p_{s^2}(x).$$

Hence, (1.21) follows from (5.1).

Another well-known multiplicity-free permutation representation of a finite general linear group comes from the action of $\mathrm{GL}_{2n}(\mathbf{F}_q)$ on the symmetric space $\mathrm{GL}_{2n}(\mathbf{F}_q)/\mathrm{Sp}_{2n}(\mathbf{F}_q)$. See [9], [1]. It is easy to see that an exact analogue of [6],Theorem 3.2.6 also holds in this case. (See [11] for a much more general result.) Using this result and results [17] on unipotent conjugacy classes of symplectic groups over finite fields, we can prove the following identity. (Since the argument is very similar to the one shown above, we omit the details.)

$$(5.2) \qquad \sum_\lambda t^{o(\lambda)/2} \prod_{\substack{v\in\lambda a(v)=0 \\ l(v)\ \text{even}}} (1 - t^{l(v)+1})P_\lambda(x;t) = \prod_{i\le j} \frac{1-tx_ix_j}{1-x_ix_j},$$

where the sum on the left is taken over all partitions λ such that $m_i(\lambda)$ is even for odd i, and

$$o(\lambda) = \sum_{i\ \text{odd}} m_i(\lambda).$$

Note that, when $t=0$, (5.2) reduces to the identity given in [12], I, Ex.5(a).

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