ON HESSIAN STRUCTURES ON THE EUCLIDEAN SPACE
AND THE HYPERBOLIC SPACE

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1. Introduction

Let $M$ be a manifold with a flat affine connection $D$. A Riemannian metric $g$ on $M$ is said to be a Hessian metric if $g$ can be locally written $g = D^2 u$ with a local function $u$. We call such a pair $(D, g)$ a Hessian structure on $M$ and a triple $(M, D, g)$ a Hessian manifold ([5]). Hessian structure appears in affine differential geometry and information geometry ([1], [4]).

If $(D, g)$ is a Hessian structure on $M$, then in terms of an affine coordinate system $(x^i)$ with respect to $D$, $g$ can be expressed by $g = \sum_{ij} (\partial^2 u / \partial x^i \partial x^j) dx^i dx^j$. Since a Kähler metric $h$ on a complex manifold can be locally written $h = \sum_{i,j} (\partial^2 v / \partial z^i \partial \overline{z}^j) dz^i d\overline{z}^j$ with a local real-valued function $v$ in terms of a complex local coordinate system $(z^i)$, a Hessian manifold may be regarded as a real number version of a Kähler manifold. Thus we are interested in similarity between Kähler manifolds and Hessian manifolds.

Given a complex structure on a manifold, the set of Kähler metrics is infinite-dimensional. Similarly, given a flat affine connection, the set of Hessian metrics is infinite-dimensional. We next consider the converse situation. Given a Riemannian metric $g$, the set of almost complex structures $J$ that makes $g$ into a Kähler metric is finite-dimensional because $J$ is parallel with respect to the Riemannian connection. As a Hessian structure version of this, a question arises whether the set of flat affine connections that makes a given Riemannian metric into a Hessian metric is finite-dimensional. In this paper, we shall show that in the cases of the Euclidean space $(\mathbb{R}^n, g_0)$ and the hyperbolic space $(H^n, g_0)$, the set of such connections is infinite-dimensional.

We prepare the terminology and notation. Let $(M, g)$ be a Riemannian manifold of dimension $\geq 2$ and $S^3(M)$ the space of all symmetric covariant tensor fields of degree 3 on $M$. We denote by $R$ and $\nabla$ the curvature tensor and the Riemannian connection, respectively. If $D$ is a flat affine connection of $M$ that makes $g$ into a Hessian metric, then the covariant tensor $T$ corresponding to $\hat{T} = D - \nabla$ by $g$ is an element of $S^3(M)$ satisfying $R_{\nabla + \hat{T}} = 0$ on $M$. Conversely, if the tensor $\hat{T}$ of type $(1, 2)$ corresponding to $T \in S^3(M)$ by $g$ satisfies $R_{\nabla + \hat{T}} = 0$ on $M$, then $D = \nabla + \hat{T}$ defines...
the connection above. By this relation, there is a one-to-one correspondence between the set of flat affine connections of $M$ that makes $g$ into a Hessian metric and the set of $T \in S^3(M)$ satisfying $R^\nabla + \hat{T} = 0$ on $M$. So we say that $T \in S^3(M)$ generates a Hessian structure with $g$ on $M$ if $R^\nabla + \hat{T} = 0$ on $M$ and indicate by $\mathcal{H}(M, g)$ the set of such tensors. To consider a local problem, we also define the set $\mathcal{H}(x, g)$ by the set of symmetric covariant tensors of degree 3 defined on a neighborhood of a point $x \in M$ generating a Hessian structure with $g$ on its domain of definition, where we identify two elements coinciding on a sufficiently small neighborhood of $x$.

Roughly speaking, we shall prove the following:

**Theorem 1.1.** The set $\mathcal{H}(0, g_0)$ at the origin 0 of $\mathbb{R}^2$ has the freedom of three local functions on $\mathbb{R}$.

**Corollary 1.2.** The set $\mathcal{H}(\mathbb{R}^n, g_0)$ has at least the freedom of $n$ functions on $\mathbb{R}$. In particular, the set $\mathcal{H}(\mathbb{T}^n, g_0)$ on the $n$-torus $\mathbb{T}^n$ has at least the freedom of $n$ periodic functions on $\mathbb{R}$.

**Theorem 1.3.** The set $\mathcal{H}(\mathbb{H}^n, g_0)$ has at least the freedom of $n - 1$ functions on $\mathbb{R}$.

2. Euclidean case

In this section, we shall show Theorem 1.1 and Corollary 1.2.

**Lemma 2.1.** Let $T$ be an element of $S^3(M)$ with components $T_{ijk}$. Then, $T$ generates a Hessian structure with $g$ on $M$ if and only if

\begin{align}
(2.1) \quad & \nabla_k T_{ijl} = \nabla_l T_{ijk}, \\
(2.2) \quad & R^\nabla_{ijkl} + \sum_s (T_{iks} T^s_{jl} - T_{its} T^s_{jk}) = 0.
\end{align}

**Proof.** By definition, $T$ generates a Hessian structure with $g$ on $M$ if and only if the tensor $\hat{T}$ of type $(1, 2)$ corresponding to $T$ by $g$ satisfies $R^\nabla + \hat{T} = 0$ on $M$. In terms of $T_{ijk}$, $R^\nabla + \hat{T} = 0$ may be expressed by

\[ R^\nabla_{ijkl} + \nabla_k T_{ijl} - \nabla_l T_{ijk} + \sum_s (T_{iks} T^s_{jl} - T_{its} T^s_{jk}) = 0. \]

Subtracting this from the one exchanged $i$ and $j$ in this, we get (2.2) and hence (2.1).

Applying Lemma 2.1 to Euclidean case, we have
Lemma 2.2. Let $U$ be a simply connected neighborhood of the origin $0$ of the Euclidean space $\mathbb{R}^n$ and $T$ an element of $S^3(U)$. Let $T_{ijk}$ be the components of $T$ with respect to the natural coordinate system $x^1, \ldots, x^n$ in $\mathbb{R}^n$. Then, $T$ generates a Hessian structure on $U$ if and only if there exists a function $u$ on $U$ such that

\begin{equation}
T_{ijk} = \frac{\partial^2 u}{\partial x^i \partial x^j \partial x^k},
\end{equation}

\begin{equation}
\sum_{s=1}^{n} \frac{\partial^3 u}{\partial x^i \partial x^k \partial x^s} \frac{\partial^3 u}{\partial x^j \partial x^l \partial x^s} = \sum_{s=1}^{n} \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^s} \frac{\partial^3 u}{\partial x^k \partial x^l \partial x^s}.
\end{equation}

Proof. We obtain $\partial T_{ijk}/\partial x^k = \partial T_{ijl}/\partial x^l$ on $U$ from (2.1). Thus by Poincaré’s lemma, there exists a function $u_{ij}$ on $U$ such that $T_{ijk} = \partial u_{ij}/\partial x^k$. Moreover, because $\partial u_{ij}/\partial x^k = \partial u_{ik}/\partial x^j$ from the symmetry of $T$, again by Poincaré’s lemma, there exists a function $u_i$ on $U$ such that $T_{ijk} = \partial^2 u_i/\partial x^j \partial x^k$. Once again by using the symmetry of $T$ and Poincaré’s lemma, finally we get $T_{ijk} = \partial^3 u/\partial x^i \partial x^j \partial x^k$. Substituting this to (2.2), we have (2.4). \qed

By Lemma 2.2, we see that, up to the quadratic functions of $x^1, \ldots, x^n$, there is a one-to-one correspondence between the solutions $u$ of (2.4) on a neighborhood of $0 \in \mathbb{R}^n$ and $\mathcal{H}(0,g_0)$ at $0 \in \mathbb{R}^n$ by $u \mapsto (\partial^3 u/\partial x^i \partial x^j \partial x^k)$. So we investigate equation (2.4) in a neighborhood of $0 \in \mathbb{R}^n$.

In case $n = 2$, (2.4) is reduced to the only one equation:

\begin{equation}
\bbox[black, 2pt, 3pt]{u_{xxx}u_{yyy} + u_{yyy}u_{yxx} = u_{yxx}^2 + u_{xyy}^2,}
\end{equation}

where $x = x^1$, $y = x^2$. Then

\begin{align*}
0 &= (u_{xxx} - u_{xyy})u_{xyy} + (u_{yyy} - u_{yxx})u_{yxx} \\
&= (u_{xx} - u_{yy})_x (u_{xy})_y - (u_{xx} - u_{yy})_y (u_{xy})_x \\
&= \begin{vmatrix}
(u_{xx} - u_{yy})_x & (u_{xx} - u_{yy})_y \\
(u_{xy})_x & (u_{xy})_y \\
\end{vmatrix}.
\end{align*}

This is equivalent to having a functional relation

\begin{equation}
F(u_{xx} - u_{yy}, u_{xy}) = 0
\end{equation}

on a neighborhood of $0 \in \mathbb{R}^2$, where $F = F(s,t)$ is an arbitrary function satisfying $F_s^2 + F_t^2 \neq 0$. Furthermore, this can be written

\begin{equation}
u_{xx} - u_{yy} = f(u_{xy}) \quad \text{if } F_s \neq 0
\end{equation}
and

\[ u_{xy} = \hat{f}(u_{xx} - u_{yy}) \quad \text{if } F_t \neq 0. \]

Since (2.6) is reduced to the type of (2.5): \( u_{\xi \xi} - u_{\eta \eta} = \hat{f}(4u_{\xi \eta}) \) by the change of variables \( \xi = x + y, \eta = x - y \), we study (2.5).

We know by the following theorem that (2.5) has a unique solution \( u(x, y) \) for any given initial data \( (u(0, y), u_x(0, y)) \):

**Fact** ([12]). Let \( u_0(y), u_1(y) \) and \( A(x, y, u, p, q, s, t) \) are smooth functions. Then, Cauchy problem

\[
\begin{align*}
&u_{xx} = A(x, y, u, u_x, u_y, u_{xy}, u_{yy}) \\
&u(0, y) = u_0(y), \; u_x(0, y) = u_1(y)
\end{align*}
\]

has a unique solution \( u(x, y) \) on a neighborhood of \( x = 0 \) if its linearized equation

\[
\begin{align*}
u_{xx} - au_{xy} - bu_{yy} - \text{(the terms of lower order)} = 0
\end{align*}
\]

with coefficients

\[
\begin{align*}
a(x, y) &= A_s(x, y, U, U_x, U_y, U_{xy}, U_{yy}), \\
b(x, y) &= A_t(x, y, U, U_x, U_y, U_{xy}, U_{yy}),
\end{align*}
\]

where \( U(x, y) = u_0(y) + xu_1(y) \), is hyperbolic.

We check that the linearized equation of (2.5) is hyperbolic for any functions \( u_0(y), u_1(y) \). We need to verify that its characteristic equation \( \lambda^2 - a\lambda - b\xi^2 = 0 \) has two different real roots \( \lambda_1, \lambda_2 \), i.e., its discriminant is positive for any real number \( \xi \neq 0 \). We get \( a(x, y) = f'(u_1(y)) \) and \( b(x, y) = 1 \). Thus the characteristic equation is written \( \lambda^2 - f'(u_1(y))\xi\lambda - \xi^2 = 0 \). Then because the discriminant is computed as \( (f'(u_1(y)))^2 + 4\xi^2 = \xi^2(f'(u_1(y))^2 + 4) \), it is positive for any \( \xi \neq 0 \).

Consequently we have a bijection from the solutions \( u \) of \( F(u_{xx} - u_{yy}, u_{xy}) = 0 \) with \( F_s \neq 0 \) into the triples of local functions on \( \mathbb{R} \) by \( u \mapsto (f, u(0, y), u_x(0, y)) \).

Therefore we obtain

**Theorem 1.1.** The set \( \mathcal{H}(0, g_0) \) at the origin 0 of \( \mathbb{R}^2 \) can be expressed by the union of two sets each of which is in one-to-one correspondence with the set of triples of local functions on \( \mathbb{R} \) up to finite-dimensional factor.

Now setting \( \hat{f} = 0 \) at (2.6), we get \( u_{xy} = 0 \) and, from this, \( u = \varphi_1(x) + \varphi_2(y) \) with arbitrary functions \( \varphi_1, \varphi_2 \). If they are global functions on \( \mathbb{R} \), this is a global solution of (2.4) on \( \mathbb{R}^2 \). Especially, if they are periodic, this is one on 2-torus \( T^2 \).
Hence we have an injection from the pairs of functions on $\mathbb{R}$ into $\mathcal{H}(\mathbb{R}^2, g_0)$ by $(\varphi_1, \varphi_2) \mapsto (\partial^3(\varphi_1(x^1) + \varphi_2(x^2))/\partial x^i \partial x^j \partial x^k)$. Restricting it on the periodic ones, we also obtain a mapping into $\mathcal{H}(T^2, g_0)$.

We prove the following lemma to generalize this:

**Lemma 2.3.** If $u$ is a solution of (2.4) on $\mathbb{R}^n$ and $v$ is an arbitrary function on $\mathbb{R}$, then $u(x_1, \ldots, x_n) + v(x_{n+1})$ is a solution of (2.4) on $\mathbb{R}^{n+1}$.

**Proof.** We have to establish

$$\sum_{s=1}^{n+1} \partial_i \partial_k \partial_s (u + v) \partial_j \partial_t \partial_s (u + v) = \sum_{s=1}^{n+1} \partial_i \partial_t \partial_s (u + v) \partial_j \partial_k \partial_s (u + v),$$

where we write $\partial_i$ for $\partial/\partial x^i$. We may assume $i < j$, $k < l$ at (2.7) from symmetry. Then $i, k \neq n + 1$ and

the left-hand side of (2.7)

$$= \sum_{s=1}^{n+1} (\partial_i \partial_k \partial_s u \partial_j \partial_t \partial_s u + \partial_i \partial_k \partial_s u \partial_j \partial_t \partial_s v + \partial_i \partial_k \partial_s v \partial_j \partial_t \partial_s u + \partial_i \partial_k \partial_s v \partial_j \partial_t \partial_s v)

= \left( \sum_{s=1}^{n} \partial_i \partial_k \partial_s u \partial_j \partial_t \partial_s u \right) + \partial_i \partial_k \partial_t v' \partial_j \partial_t v'

= \sum_{s=1}^{n} \partial_i \partial_k \partial_s u \partial_j \partial_t \partial_s u.$$

Similarly

the right-hand side of (2.7) $= \sum_{s=1}^{n} \partial_i \partial_t \partial_s u \partial_j \partial_k \partial_s u.$

Since $u$ is a solution of (2.4) on $\mathbb{R}^n$ by the assumption, both sides are equal to one another.

Combining the result of 2-dimensional case and Lemma 2.3, we obtain

**Corollary 1.2.** The mapping $\Phi : (\varphi_1, \ldots, \varphi_n) \mapsto (\partial^3(\varphi_1(x^1) + \cdots + \varphi_n(x^n))/\partial x^i \partial x^j \partial x^k)$ gives an injection from the set of $n$-tuples of functions on $\mathbb{R}$ into the set $\mathcal{H}(\mathbb{R}^n, g_0)$ up to finite-dimensional factor. Particularly, $\Phi$ restricted on the set of periodic ones gives a mapping into the set $\mathcal{H}(T^n, g_0)$ on $n$-torus $T^n$. 
3. Hyperbolic case

In this section, we shall show Theorem 1.3.

We set

\[ H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n | x^n > 0\} \quad \text{and} \quad g_0 = \frac{1}{(x^n)^2}\{(dx^1)^2 + \cdots + (dx^n)^2\}. \]

It is known that there exists an element \( T_0 = ((T_0)_{ijk}) \in S^3(H^n) \) generating a Hessian structure with \( g_0 \) on \( H^n \), which is given for \( 1 \leq i \leq j \leq k \leq n \) as follows ([3]):

\[
(T_0)_{ijk} = \begin{cases} 
\frac{1}{(x^n)^3} & 1 \leq i = j \leq n - 1, \quad k = n \\
\frac{2}{(x^n)^3} & i = j = k = n \\
0 & \text{otherwise.}
\end{cases}
\]

We consider the case \( n = 2 \) for a while.

An element \( X \) of \( S^3(M) \) is called an infinitesimal deformation of \( \Gamma \in H(M,g) \) if \( (d/dt)_{t=0}R^\nabla + tX = 0 \).

Lemma 3.1. An infinitesimal deformation \( X = (X_{ijk}) \in S^3(H^2) \) of \( T_0 \in H(H^2,g_0) \) is given by

\[ X_{111} = \frac{f''(x) + g'(x)}{8}, \]

\[ X_{112} = \frac{f'(x)}{2} + \frac{g(x)}{y}, \]

\[ X_{122} = f(x), \]

\[ X_{222} = 0, \]

where \( x = x^1, \quad y = x^2, \quad \text{and} \quad f, \ g, \ h \) are arbitrary functions.

Proof. In general, by differentiating each of ones substituted \( T + tX \) for \( T \) in (2.1) and (2.2), we obtain equations for an infinitesimal deformation \( X \) of \( T \in H(M,g) \) as follows:

\[ \nabla_k X_{ijkl} - \nabla_l X_{ijk} = 0, \]

\[ \sum_s (X_{iks} T^s_{jl} + T_{iks} X^s_{jl} - X_{ils} T^s_{jk} - T_{ils} X^s_{jk}) = 0. \]

In case \( (M,g) = (H^2,g_0) \) and \( T = T_0 \), this is reduced to
(3.5) \( (X_{111})_y - (X_{112})_x + \frac{2}{y}(X_{111} + X_{122}) = 0 \)

(3.6) \( (X_{112})_y - (X_{122})_x + \frac{1}{y}X_{112} = 0, \)

(3.7) \( (X_{122})_y = 0, \)

(3.8) \( X_{222} = 0. \)

First from (3.7), we get (3.3). Then equation (3.6) is written as

\[ (X_{112})_y + \frac{1}{y}X_{112} = f'(x). \]

Solving this, we have (3.2). Finally by (3.2) and (3.3), equation (3.5) is written as

\[ (X_{111})_y + \frac{2}{y}X_{111} = \frac{f''(x)y}{2} + \frac{g'(x)}{y} - 2\frac{f(x)}{y}. \]

Solving this, we obtain (3.1).

We find out the elements of \( \mathcal{H}(H^2, g_0) \) that has the form of \( T_0 + X \). Since both of \( T_0 \) and \( X \) satisfy (2.1), \( T_0 + X \) satisfies it. Thereby \( T_0 + X \) belongs to \( \mathcal{H}(H^2, g_0) \) if and only if it satisfies (2.2) in \( H^2 \). In the present case, it is reduced to the only one equation:

\[ X_{111}X_{122} + X_{222}X_{112} - X_{112}^2 - X_{122}^2 = 0. \]

Substituting (3.1) ~ (3.4), we get

\[
0 = \left( \frac{f''(x)y^2}{8} + \frac{g'(x)}{2} - f(x) + \frac{h(x)}{y^2} \right) f(x) - \left( \frac{f'(x)y}{2} + \frac{g(x)}{y} \right)^2 - f(x)^2 \\
= \left( \frac{f(x)f''(x)}{8} - \frac{f'(x)^2}{4} \right) y^2 + \frac{f(x)g'(x)}{2} \\
- f'(x)g(x) - 2f(x)^2 + (f(x)h(x) - g(x)^2) \frac{1}{y^2}.
\]

Hence \( T_0 + X \) belongs to \( \mathcal{H}(H^2, g_0) \) if and only if

\begin{align*}
(3.9) \quad & f f'' - 2f'^2 = 0, \\
(3.10) \quad & fg' - 2f'g - 4f^2 = 0, \\
(3.11) \quad & fh - g^2 = 0.
\end{align*}

We find the global solutions of this:

A. The case \( f = 0 \).
From (3.11), we have \( g = 0 \). So the solution is

\[
\begin{cases}
  f = 0 \\
  g = 0 \\
  h: \text{an arbitrary function}
\end{cases}
\]

B. The case \( f \neq 0 \).

By supposing \( f' \neq 0 \), (3.9) can be written

\[
\frac{f''}{f'} = 2 \frac{f'}{f}.
\]

From this, we obtain \( f = 1/(Ax + B) \) with arbitrary constants \( A, B \). Then \( f \) is a global solution if and only if \( A = 0 \) and \( B \neq 0 \). But this contradicts with \( f' \neq 0 \). Thus \( f' = 0 \), i.e., \( f \) is a constant. Setting \( f = C_1(\neq 0) \), from (3.10) and (3.11), we get \( g = 4C_1x + C_2 \) and \( h = g^2/C_1 \). So the solution is

\[
\begin{cases}
  f = C_1 \\
  g = 4C_1x + C_2 \\
  h = \frac{g^2}{C_1}.
\end{cases}
\]

Therefore we have

**Proposition 3.2.** For an infinitesimal deformation \( X = (X_{ijk}) \in S^3(H^2) \) of \( T_0 \in \mathcal{H}(H^2,g_0) \), \( T_0 + X \) belongs to \( \mathcal{H}(H^2,g_0) \) if and only if \( X \) is given as follows:

\[
\begin{cases}
  X_{111} = \frac{h(x)}{y^2} \\
  X_{112} = 0 \\
  X_{122} = 0 \\
  X_{222} = 0
\end{cases}
\]

(3.12)

or

\[
\begin{cases}
  X_{111} = C_1 + \frac{(4C_1x + C_2)^2}{C_1y^2} \\
  X_{112} = \frac{4C_1x + C_2}{y} \\
  X_{122} = C_1 \\
  X_{222} = 0,
\end{cases}
\]

(3.13)

where \( h \) is an arbitrary function and \( C_1 \neq 0 \) and \( C_2 \) are arbitrary constants.
We go back to the general case. On the analogy of (3.12), we obtain

**Theorem 1.3.** Let \( \tilde{X} = (\tilde{X}_{ijk}) \in S^3(H^n) \) be given by

\[
\tilde{X}_{ijk} = \begin{cases} 
\frac{f_i(x^i)}{(x^n)^2} & 1 \leq i = j = k \leq n - 1 \\
0 & \text{otherwise},
\end{cases}
\]

where \( f_i \) are arbitrary functions. Then, \( T_0 + \tilde{X} \) belongs to \( \mathcal{H}(H^n, g_0) \).

**Proof.** We prove that \( T_0 + \tilde{X} \) satisfies (2.1) and (2.2). We first verify to satisfy (2.1). Because \( T_0 \) satisfies it, we need only verify that \( \tilde{X} \) satisfies it, that is,

\[
(3.14) \quad \partial_k \tilde{X}_{ijl} - \partial_l \tilde{X}_{ijk} + \sum_s (\Gamma^s_{i} \tilde{X}_{sjk} + \Gamma^s_{j} \tilde{X}_{isk} - \Gamma^s_{k} \tilde{X}_{sjl} - \Gamma^s_{k} \tilde{X}_{isl}) = 0,
\]

where the Christoffel symbols \( \Gamma^i_{jk} \) of \( \nabla \) is given by

\[
\Gamma^i_{jk} = \begin{cases} 
\frac{1}{x^n} & i = n, 1 \leq j = k \leq n - 1 \\
\frac{-1}{x^n} & 1 \leq i = j \leq n - 1, k = n; \text{ or } i = j = k = n \\
0 & \text{otherwise}.
\end{cases}
\]

It suffices to consider (3.14) for \( i < j, k < l \) by symmetry.

A. The case \( i = j \).

Then the left-hand side of (3.14) = \( \partial_k \tilde{X}_{iil} - \partial_l \tilde{X}_{iik} + 2 \sum_s (\Gamma^s_{i} \tilde{X}_{sik} - \Gamma^s_{k} \tilde{X}_{sii}) \)

= \( \partial_k \tilde{X}_{iil} - \partial_l \tilde{X}_{iik} + 2 \sum_s \Gamma^s_{i} \tilde{X}_{sii} \).

If \( i = k \), then we get

\[
\partial_k \tilde{X}_{iil} - \partial_l \tilde{X}_{iik} + 2 \sum_s \Gamma^s_{i} \tilde{X}_{sik} = -\partial_l \tilde{X}_{iili} + 2 \sum_s \Gamma^s_{i} \tilde{X}_{sii}
\]

= \( -\frac{f_i(x^i)}{(x^n)^2} + 2\Gamma^i_{i} \tilde{X}_{iii} \)

= \begin{cases} 
2\Gamma^i_{i} \tilde{X}_{iii} = 0 & l < n \\
2\frac{f_i(x^i)}{(x^n)^3} + 2\frac{-1}{x^n} \frac{f_i(x^i)}{(x^n)^2} = 0 & l = n.
\end{cases}
If $i \neq k$, then we have

$$\partial_k \tilde{X}_{iil} - \partial_l \tilde{X}_{iik} + 2 \sum_s \Gamma_{l}^{s} \tilde{X}_{sik} = \partial_k \tilde{X}_{iil} = \delta_{il} \partial_k \frac{f_i(x^i)}{(x^n)^2} = 0,$$

where $\delta_{ij}$ is Kronecker's delta.

B. The case $i < j$.

Then

the left-hand side of (3.14) = \sum_s (\Gamma_{l}^{s} \tilde{X}_{sjk} + \Gamma_{l}^{s} \tilde{X}_{isk} - \Gamma_{k}^{s} \tilde{X}_{sij} - \Gamma_{k}^{s} \tilde{X}_{isl})

= \sum_s (\Gamma_{l}^{s} \tilde{X}_{sjk} + \Gamma_{l}^{s} \tilde{X}_{isk} - \Gamma_{k}^{s} \tilde{X}_{isl}).

Since (3.14) is equal to the one exchanged a pair $(i, j)$ and a pair $(k, l)$, we need only check the following three cases:

If $i = k, j = l$, then we obtain

$$\sum_s (\Gamma_{l}^{s} \tilde{X}_{sjk} + \Gamma_{l}^{s} \tilde{X}_{isk} - \Gamma_{k}^{s} \tilde{X}_{sij} - \Gamma_{k}^{s} \tilde{X}_{isl}) = \sum_s (\Gamma_{l}^{s} \tilde{X}_{sji} + \Gamma_{l}^{s} \tilde{X}_{isi} - \Gamma_{k}^{s} \tilde{X}_{isl})

= \sum_s \Gamma_{j}^{i} \tilde{X}_{isi}

= \Gamma_{j}^{i} \tilde{X}_{iii}

= 0.$$

If $i = k, j < l$, then we get

$$\sum_s (\Gamma_{l}^{s} \tilde{X}_{sjk} + \Gamma_{l}^{s} \tilde{X}_{isk} - \Gamma_{k}^{s} \tilde{X}_{sij} - \Gamma_{k}^{s} \tilde{X}_{isl}) = \sum_s (\Gamma_{l}^{s} \tilde{X}_{sji} + \Gamma_{l}^{s} \tilde{X}_{isi} - \Gamma_{k}^{s} \tilde{X}_{isl})

= \sum_s \Gamma_{l}^{i} \tilde{X}_{isi}

= \Gamma_{l}^{i} \tilde{X}_{iii}

= 0.$$

If $i < k$, then we have

$$\sum_s (\Gamma_{l}^{s} \tilde{X}_{sjk} + \Gamma_{l}^{s} \tilde{X}_{isk} - \Gamma_{k}^{s} \tilde{X}_{isl}) = \sum_s \Gamma_{l}^{s} \tilde{X}_{sjk}

= \begin{cases} 
\Gamma_{n}^{i} \tilde{X}_{njk} = 0 & l < n \\
\Gamma_{n}^{i} \tilde{X}_{ijk} = 0 & l = n.
\end{cases}$$
We next establish that $\bar{T} = T_0 + \bar{X}$ satisfies (2.2), i.e.,

$$\sum_s (\bar{T}_{iks} \bar{T}_{jls} - \bar{T}_{ils} \bar{T}_{jks}) = \frac{1}{(x^n)^6} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}),$$

where $\bar{T} = (\bar{T}_{ijk})$ is given by

$$\bar{T}_{ijk} = \begin{cases} \frac{1}{(x^n)^3} & 1 \leq i = j \leq n - 1, \ k = n \\ \frac{f_i(x^j)}{(x^n)^2} & 1 \leq i = j = k \leq n - 1 \\ \frac{2}{(x^n)^3} & i = j = k = n \\ 0 & \text{otherwise}. \end{cases}$$

It suffices to consider (3.15) in the case $i = k, j = l$, in the case $i = k, j < l$ and in the case $i < k$ under $1 \leq i < j \leq n, 1 \leq k < l \leq n$ from symmetry.

A. The case $i = k, j = l$.

Equality (3.15) is written as

$$\sum_s (\bar{T}_{iis} \bar{T}_{jjs} - \bar{T}_{ijs}^2) = \frac{1}{(x^n)^6}.$$ 

Then

the left-hand side of (3.16) = $\bar{T}_{iis} \bar{T}_{jji} + \bar{T}_{iin} \bar{T}_{jjn} - \bar{T}_{ijs}^2$.

\[= \bar{T}_{iin} \bar{T}_{jjn} - \bar{T}_{iij}^2 = \begin{cases} \frac{1}{(x^n)^3} (x^n)^3 = \frac{1}{(x^n)^6} & j < n \\ \frac{1}{(x^n)^3} \frac{2}{(x^n)^3} - \left(\frac{1}{(x^n)^3}\right)^2 = \frac{1}{(x^n)^6} & j = n. \end{cases} \]

B. The case $i = k, j < l$.

Equality (3.15) is simplified as

$$0 = \sum_s (\bar{T}_{iis} \bar{T}_{jls} - \bar{T}_{ils} \bar{T}_{jks}) = \sum_s \bar{T}_{iis} \bar{T}_{jls}.$$ 

Then

the right-hand side of (3.17) = $\bar{T}_{iis} \bar{T}_{jls} + \bar{T}_{iin} \bar{T}_{jln} = 0$. 
C. The case \( i < k \).

Equality (3.15) is simplified as

\[
(3.18) \quad 0 = \sum_i (\tilde{T}_{iks} \tilde{T}_{jls} - \tilde{T}_{ils} \tilde{T}_{jks}) = -\sum_i \tilde{T}_{ils} \tilde{T}_{jks}.
\]

Then

the right-hand side of (3.18) = \(-\tilde{T}_{ili} \tilde{T}_{jki} = 0\).

References


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