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AN EXTENTION OF MILNOR'S INEQUALITY

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1. Introduction

Milnor showed the following theorem.

Theorem 1.1 (Milnor [4]). *Let $\rho : \pi_1(\Sigma_g) \to SL(2,\mathbb{R})$ be a representation of the surface group and let E be the corresponding flat vector bundle of rank 2 over Σ_g. Then the Euler number $\langle \chi(E), [\Sigma_g] \rangle$ satisfies the following inequality:*

$$|\langle \chi(E), [\Sigma_g] \rangle| \le g - 1.$$

Conversely any integer χ such that $|\chi| \le g-1$ is realized as the Euler number of a flat $SL(2,\mathbb{R})$-vector bundle.

There are various generalizations of this theorem, for which see [5].

In this paper we will prove an extension of this theorem to a representation of surface group to the symplectic group $Sp(2p,\mathbb{R})$. More precisely, let $\rho : \pi_1(\Sigma_g) \to Sp(2p,\mathbb{R})$ be a representation of the surface group and let V be the corresponding flat symplectic vector bundle of rank $2p$ over Σ_g. Since a maximal compact subgroup of $Sp(2p,\mathbb{R})$ is isomorphic to $U(p)$, V admits a positive compatible complex structure with the symplectic structure which always exists and is unique up to homotopy. Hence the first Chern class

$$c_1(V) \in H^2(\Sigma_g;\mathbb{Z})$$

is uniquely defined.

The main theorem of this paper is the following.

Theorem 1.2. *Let g, $p \ge 1$ be integers and Σ_g a closed oriented surface of genus g. For any representation $\rho : \pi_1(\Sigma_g) \to Sp(2p,\mathbb{R})$, let V be the corresponding flat symplectic vector bundle of rank $2p$ over Σ_g. Then the following inequality holds:*

$$|\langle c_1(V), [\Sigma_g] \rangle| \le p(g-1).$$

Conversely any integer c_1 such that $|c_1| \leq p(g-1)$ is realized as the first Chern class of a flat symplectic vector bundle of rank $2p$.

2. The signature of local coefficient systems

In this section we shall review the result due to Lusztig and Atiyah ([1, 3]) which is needed to prove the main theorem.

Let Σ_g be a closed oriented surface of genus $g \geq 1$. Let $E \to \Sigma_g$ be a flat (possibly indefinite) hermitian vector bundle with a hermitian form h. It corresponds to a homomorphism

$$\rho: \pi_1(\Sigma_g) \to U(p,q),$$

where $U(p,q)$ is the unitary group of the hermitian form $\sum_{i=1}^{p} |z_i|^2 - \sum_{i=p+1}^{p+q} |z_i|^2$ on $(z_i) \in \mathbb{C}^{p+q}$. Since a flat vector bundle is identified with a local coefficient system, we have the cohomology group $H^*(\Sigma_g; E)$. Moreover the first cohomology group $H^1(\Sigma_g; E)$ has a skew-hermitian form A induced by the cup product and the hermitian form h. Since $\sqrt{-1}A$ is a hermitian form, we can consider its signature. Let $sign(\Sigma_g, E)$ denote the signature of $\sqrt{-1}A$.

Take a splitting $E = E^+ \oplus E^-$ of E. Here a splitting of the hermitian vector bundle E is defined as a smooth decomposition $E = E^+ \oplus E^-$ of E which is orthogonal relative to the hermitian form and such that the form is positive on E^+ and negative on E^-. Such a splitting is identified with a reduction of the structure group $U(p,q)$ of E to its maximal compact subgroup $U(p) \times U(q)$. It always exists and is unique up to homotopy.

Theorem 2.1 (Lusztig [3], Atiyah [1]). *If E is a flat hermitian vector bundle over the closed surface Σ_g then*

$$\operatorname{sign}(\Sigma_g, E) = 2(c_1(L^+) - c_1(L^-)),$$

where $L^\pm = \det E^\pm$ and $E = E^+ \oplus E^-$ is a splitting of E.

3. Proof of the main theorem

In this section we shall prove the main theorem.

Let $V \to \Sigma_g$ be a flat symplectic vector bundle of rank $2p$ over Σ_g and ω its symplectic form.

Put $E = V \otimes \mathbb{C}$ and extend the symplectic form ω on V to a complex symplectic form on E which is also denoted by ω. The hermitian form h on E is defined by

$$h_x(u,v) = -\sqrt{-1}\omega_x(u, \bar{v})$$

for any $x \in \Sigma_g$ and u, $v \in E_x$.

Let $J: V \to V$ be a positive compatible complex structure with the symplectic form ω, i.e. $J: V \to V$ is a bundle homomorphism which satisfies $J^2 = -1$ and such that $\omega(\cdot, J\cdot)$ is a Euclidean metric of V. Such a complex structure always exists and is unique up to homotopy. The complex vector bundle (V, J) of rank p over $\mathbb{C}$ is denoted by V_J. Extend J to a complex linear map on E, then by $J^2 = -1$, we have the decomposition $E = E' \oplus E''$ of E by the eigenspaces of J, where $J|_{E'} = \sqrt{-1}id_{E'}$ and $J|_{E''} = -\sqrt{-1}id_{E''}$. It is well known that the complex isomorphisms $E' \cong V_J$ and $E'' \cong \bar{E}' \cong \bar{V}_J$ hold, where $^-$ denotes the complex conjugation. Thus we have $E = V_J \oplus \bar{V}_J$. Moreover it is easy to see that this decomposition gives a splitting of E where $E^+ \cong V_J$ and $E^- \cong \bar{V}_J$ (see [6]).

By Theorem 2.1, we have

$$\operatorname{sign}(\Sigma_g, E) = 2(c_1(V_J) - c_1(\bar{V}_J)) = 4c_1(V_J).$$

Note that, for any complex vector bundle F, the equalities $c_1(F) = c_1(\det F)$ and $c_1(\bar{F}) = -c_1(F)$ hold.

On the other hand, since E is a flat vector bundle, we have the following de Rham complex with local coefficients in E:

$$d_E^\bullet : 0 \to \Omega^0(\Sigma_g; E) \xrightarrow{d_E} \Omega^1(\Sigma_g; E) \xrightarrow{d_E} \Omega^2(\Sigma_g; E) \to 0.$$

By the Atiyah-Singer index theorem ([2]), we obtain

$$\operatorname{ind}(d_E^\bullet) = \operatorname{rank}_{\mathbb{C}} E \cdot \chi(\Sigma_g) = 4p(1-g).$$

We shall prove the inequality in the main theorem in the following two cases.

In the first case we suppose that $H^0(\Sigma_g; E) = 0$. We then have $H^2(\Sigma_g; E) = 0$ by the duality after identifications $E \cong \bar{E} \cong E^*$. Thus we have $\dim_{\mathbb{C}} H^1(\Sigma_g; E) = 4p(g-1)$, and hence obtain

$$|\langle c_1(V_J), [\Sigma_g] \rangle| = \frac{1}{4} |\operatorname{sign}(\Sigma_g, E)| \leq p(g-1).$$

In the second case we suppose that $H^0(\Sigma_g; E) \neq 0$ and hence $H^0(\Sigma_g; V)$ is also nontrivial. Thus there exists a nonzero element s of $H^0(\Sigma_g; V)$ which does not vanish at any point of Σ_g. $W := \mathbb{R}s$ defines an $\mathbb{R}$-line subbundle of V. Let $W^{\perp_\omega} \subset V$ be the subbundle of V orthogonal to W with respect to the symplectic form ω, then W is also a subbundle of $W^{\perp_\omega}$. The quotient bundle $W^{\perp_\omega}/W$ has the symplectic form $\underline{\omega}$ induced from ω. From the construction, it is clear that this bundle is a flat symplectic vector bundle of rank $2(p-1)$.

Put $W_J = \mathbb{R}s \oplus J\mathbb{R}s$, then it is a trivial complex line subbundle of V_J. Let $W_J^{\perp_{h'}}$ be the subbundle of V_J orthogonal to W_J with respect to the hermitian form $h'(\cdot,\cdot) := \omega(\cdot, J\cdot) - \sqrt{-1}\omega(\cdot,\cdot)$, which is also a symplectic vector bundle with a positive compatible complex structure. It is easy to see that the inclusion $W_J^{\perp_{h'}} \hookrightarrow W^{\perp_\omega}$

induces the isomorphism

$$(W_J^{\perp_{h'}}, \omega|_{W_J^{\perp_{h'}}}) \cong (W^{\perp_\omega}/W, \underline{\omega})$$

as smooth symplectic bundles. So a positive compatible complex structure with $\underline{\omega}$ on $W^{\perp_\omega}/W$ is induced from the one on $W_J^{\perp_{h'}}$ via the isomorphism. Thus we have

$$c_1(V_J) = c_1(W_J^{\perp_{h'}} \oplus W_J) = c_1(W_J^{\perp_{h'}}) = c_1(W^{\perp_\omega}/W).$$

In particular if $p = 1$, then we obtain $c_1(V_J) = 0$. So we complete the proof of the inequality in Theorem 1.2 for $p = 1$.

For $p > 1$, we may suppose that $c_1(V) \neq 0$. If necessary, repeating the above argument, we can construct a flat symplectic vector bundle Z over Σ_g of rank $q < p$ such that $H^0(\Sigma_g; Z \otimes \mathbb{C}) = 0$ and $c_1(V) = c_1(Z)$. From the first case, we have

$$|\langle c_1(V), [\Sigma_g]\rangle| = |\langle c_1(Z), [\Sigma_g]\rangle| \leq q(g-1) \leq p(g-1).$$

This completes the proof of the inequality.

Conversely for any integer c_1 such that $|c_1| \leq p(g-1)$, since $Sp(2,\mathbb{R}) = SL(2,\mathbb{R})$, the existence of a flat symplectic vector bundle V over Σ_g of rank $2p$ with $\langle c_1(V), [\Sigma_g]\rangle = c_1$ is true for $p = 1$ by Theorem 1.1.

For $p > 1$, suppose that $c_1 = p(g-1)$ since the other cases can easily be obtained by the same method. Let V_1 be a flat symplectic vector bundle over Σ_g of rank 2 with $c_1(V_1) = g - 1$. The direct sum $V = \overset{p}{\oplus} V_1$ of p copies of V_1 is a flat symplectic vector bundle of rank $2p$ with

$$\langle c_1(V), [\Sigma_g]\rangle = p\langle c_1(V_1), [\Sigma_g]\rangle = p(g-1).$$

This vector bundle V is a required one.

This finishes the proof of the main theorem.

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