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ON THE ABEL-JACOBI MAP FOR NON-COMPACT VARIETIES

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1. Introduction

Let X be a smooth projective variety over $\mathbb{C}$ of dimension n and S be a reduced normal crossing divisor on X. Then the generalized Jacobian $J(X-S)$ is a group $H^{n-1}(X,\omega_X(S))\check{}/H_{2n-1}(X-S,\mathbb{Z})$. When X is a curve, this fits into an exact sequence of algebraic groups:

$$1 \longrightarrow (\mathbb{C}^*)^{\sigma-1} \longrightarrow J(X-S) \longrightarrow J(X) \longrightarrow 0$$

where σ is the number of points in S and $J(X)$ is the usual Jacobian of X. Let $\mathrm{Div}^0(X-S)$ be the set of divisors of degree 0 on X which does not intersect with S. Then integration determines the Abel-Jacobi homomorphism $\alpha : \mathrm{Div}^0(X-S) \to J(X-S)$. We will prove an analogue of Abel's theorem (due to Rosenlicht [8] for curves) that the kernel of α is the following subgroup $\mathrm{Prin}_S(X)$ of S-principal divisors:

$$\mathrm{Prin}_S(X) = \{(f) \in \mathrm{Div}(X-S) | f \in K(X) \text{ and } f = 1 \text{ on } S\}.$$

A proof is a variation of our previous work [1], which involves reinterpretation of the Abel-Jacobi map in the language of mixed Hodge structures and their extensions. As a further application of this technique, we prove a Torelli theorem for a non-compact curve, which states that if X is the complement of at least 2 points in a nonhyperelliptic curve, then it is determined by the graded polarized mixed Hodge structure on $H^1(X,\mathbb{Z})$.

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2. Hodge Structures

DEFINITION 2.1. A (pure) Hodge structure H of weight m consists of a finitely generated abelian group $H_{\mathbb{Z}}$ and a decreasing filtration $F^\bullet$ of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$ such that $H_{\mathbb{C}} = F^p \oplus \overline{F^{m-p+1}}$.

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EXAMPLE 1. The Hodge structure of Tate $\mathbb{Z}(-1)$ is defined to be the Hodge structure of weight 2 with $H_{\mathbb{Z}} = \frac{1}{2\pi\sqrt{-1}}\mathbb{Z} \subset \mathbb{C} = F^1 H_{\mathbb{C}}$.

The most natural example of Hodge structure of weight k is the k-th integral cohomology of a compact Kähler manifold. A differential form lies in F^p if in local coordinate it has at least p "dz's". To extend Hodge theory to any (singular or non-projective) complex algebraic varieties X, Deligne [3] introduced the notion of a mixed Hodge structure. He showed that the cohomology of any variety carries such a structure.

DEFINITION 2.2. A *mixed Hodge structure* (MHS) H consists of a triple $(H_{\mathbb{Z}}, W_\bullet, F^\bullet)$, where

(1) $H_{\mathbb{Z}}$ is a finitely generated abelian group. (In practice $H_{\mathbb{Z}}$ will be free and we will identify it with a lattice in $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes \mathbb{Q}$.)

(2) $W_\bullet$ is an increasing filtration of $H_{\mathbb{Q}}$, called the *weight filtration.*

(3) $F^\bullet$ is a decreasing filtration of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$, called the *Hodge filtration.*

The Hodge filtration $F^\bullet$ is required to induce a (pure) Hodge structure of weight m on each of the graded pieces

$$Gr_m^{W_\bullet} = W_m / W_{m-1}$$

EXAMPLE 2. Let D be a divisor on a smooth projective variety X over $\mathbb{C}$. Set $U = X - D$. By Hironaka, there exists a birational map $\pi : \tilde{X} \to X$, with $\tilde{X}$ non-singular such that $\tilde{D} = \pi^{-1}(D)$ is a normal crossing divisor. Then $H^1(U, \mathbb{Z})$ carries a mixed Hodge structure and the Hodge filtration is given by

$$F^0 = H^1(U, \mathbb{C}), \quad F^1 = H^0(\tilde{X}, \Omega^1(\log \tilde{D})), \quad F^2 = 0.$$

We will denote $H^0(\tilde{X}, \Omega^1(\log \tilde{D}))$ by $H^0(X, \Omega^1(\log D))$. This group does not depend on the choice of $\tilde{X}$.

Given two mixed Hodge structures A and B, we write $B > A$ if there exists m_0 such that $W_m A_{\mathbb{Q}} = A_{\mathbb{Q}}$ for all $m \geq m_0$ and $W_m B_{\mathbb{Q}} = 0$ for all $m < m_0$.

Finally, we define the p-th *Jacobian* of a mixed Hodge structure of H to be the generalized torus

$$J^p H = H_{\mathbb{Z}} \backslash H_{\mathbb{C}} / F^p H_{\mathbb{C}}.$$

The set of mixed Hodge structures forms an abelian category with an internal Hom. Thus one can form the abelian group of extension classes of two objects. Carlson [2] described the structure of this extension group in terms of the Jacobian.

Theorem 2.1 (Carlson). *Let A and B be mixed Hodge structures with $B > A$ and B torsion free. Then there is a natural isomorphism.*

$$\mathrm{Ext}^1_{MHS}(B, A) \cong J^0\mathrm{Hom}(B, A).$$

3. Homologically trivial divisors

Let X be a smooth projective variety over $\mathbb{C}$ of dimension n and S be a reduced normal crossing divisor on X. Let $\mathrm{Div}(X-S)$ be the group of divisors on X which do not intersect S. Moreover, we set

$$\mathrm{Prin}_S(X) = \{(f) \in \mathrm{Div}(X-S) | f \in K(X) \text{ and } f = 1 \text{ on } S\} \tag{1}$$

$$Cl_S(X) = \mathrm{Div}(X-S)/\mathrm{Prin}_S(X). \tag{2}$$

The kernel of the cycle map [5, §19.1]

$$cl : \mathrm{Div}(X-S) \to H_{2n-2}(X-S, \mathbb{Z})$$

will be called the group of homologically trivial divisors and it will be denoted by $\mathrm{Div}^0(X-S)$. Note that $\mathrm{Prin}_S(X) \subset \mathrm{Div}^0(X-S)$.

Let $\mathcal{K}^*$ be the sheaf of invertible rational functions on X and $\mathcal{K}^*(-S)$ be the subsheaf of $\mathcal{K}^*$ consisting of functions which are 1 on S. Similarly, we define $\mathcal{O}^*(-S)$ to be the subsheaf of $\mathcal{O}^*$ consisting of functions which are 1 on S. Consider the following exact sequence

$$1 \longrightarrow \mathcal{O}^*(-S) \longrightarrow \mathcal{K}^*(-S) \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{3}$$

where $\mathcal{Q}$ is the quotient sheaf. Then one can prove that $H^0(X, \mathcal{K}^*(-S)) = \mathrm{Prin}_S(X)$ and $H^0(X, \mathcal{Q}) = \mathrm{Div}(X-S)$ as in [7, II, 6.11]. Let

$$Cl^0_S(X) = \mathrm{Div}^0(X-S)/\mathrm{Prin}_S(X).$$

Consider the following diagram :

$$\begin{array}{ccccc} H^0(X, \mathcal{Q}) & \longrightarrow & H^1(X, \mathcal{O}^*(-S)) & \xrightarrow{\frac{1}{2\pi i} d\log} & H^2(X, j_!\mathbb{Z}) = H^2(X, S) \\ \downarrow & & & & \downarrow \\ \mathrm{Div}(X-S) & & \xrightarrow{\quad cl \quad} & & H_{2n-2}(X-S, \mathbb{Z}) \end{array}$$

The map $1/2\pi i d\log$ is the connecting homomorphism associated to the exponential sequence:

$$0 \longrightarrow j_!\mathbb{Z} \longrightarrow \mathcal{O}(-S) \overset{\exp(2\pi i)}{\longrightarrow} \mathcal{O}^*(-S) \longrightarrow 1 \tag{4}$$

where j is the natural inclusion from $X - S$ to X. By Lefschetz duality [9, Theorem 6.2.19], the right vertical arrow is an isomorphism. Moreover, the diagram is commutative since the cycle map is compatible with Chern class map. Therefore, $Cl_S^0(X)$ is isomorphic to a subgroup of the kernel of the connecting homomorphism

$$H^1(X, \mathcal{O}^*(-S)) \xrightarrow{\frac{1}{2\pi i} d\log} H^2(X, j_!\mathbb{Z}) = H^2(X, S).$$

So, $Cl_S^0(X)$ is isomorphic to a subgroup of $H^1(X, \mathcal{O}(-S))/H^1(X, S; \mathbb{Z})$. By duality, we can identify $H^1(X, \mathcal{O}(-S))/H^1(X, S; \mathbb{Z})$ with $H^{n-1}(X, \omega_X(S))\check{}\,/H_{2n-1}(X - S, \mathbb{Z})$ where $\omega_X(S) = \wedge^n \Omega_X^1 \otimes \mathcal{O}_X(S)$. Thus we obtain an injection

$$\beta : Cl_S^0(X) \to H^{n-1}(X, \omega_X(S))\check{}\,/H_{2n-1}(X - S, \mathbb{Z}), \tag{5}$$

which will be identified with the Abel-Jacobi map later.

Lemma 3.1. *In the diagram,*

$$\begin{array}{ccc} \mathrm{Div}(X - S) & \xrightarrow{\quad cl_{X-S} \quad} & H_{2n-2}(X - S) \\ & {\scriptstyle cl_X}\searrow \qquad \swarrow & \\ & H_{2n-2}(X) & \end{array}$$

the kernel of the map cl_{X-S} is equal to the kernel of the map cl_X.

Proof. Clearly, we have $\ker cl_{X-S} \subset \ker cl_X$. We will prove the converse in dual form. Let $D \in \ker cl_X$. Consider a long exact sequence of Mixed Hodge structures, so called a "Thom-Gysin" sequence, associated to a triple $S \subset X - |D| \subset X$;

$$0 \longrightarrow H^1(X, S) \longrightarrow H^1(X - |D|, S) \longrightarrow H^2(X, X - |D|) \overset{Gysin}{\longrightarrow} H^2(X, S) \tag{6}$$

Note that $H^2(X, X - |D|) \cong H_D^2(X)$. By Fujiki [4], we have

$$H^2(X, X - |D|) = \mathrm{Hom}(H^{2n-2}(D), \mathbb{Z}(-n))$$

as a mixed Hodge structure. Also observe that $H^{2n-2}(D) = \bigoplus \mathbb{Z}(-n + 1)$ where the sum is over all irreducible components of D. Thus $H^2(X, X - |D|)$ has a pure Hodge structure of weight 2. On the other hand, it follows from the long exact sequence of cohomologies associated to the pair (X, S) that $Gr_2^{W_\bullet} H^2(X, S)$ injects into $H^2(X)$. Hence if the class of D in $H^2(X)$ vanishes, then so does the class of D in $H^2(X, S)$. □

4. Extensions of MHS

Let $D \in \mathrm{Div}^0(X - S)$ be a homologically trivial divisor. From the sequence (6), we get an extension of mixed Hodge structures

$$0 \longrightarrow H^1(X,S) \longrightarrow H^1(X - |D|, S) \longrightarrow K \longrightarrow 0 \tag{7}$$

where $K = \ker[H^2(X, X - |D|) \overset{Gysin}{\longrightarrow} H^2(X,S)]$. Let

$$\phi_D : \mathbb{Z}(-1) \longrightarrow \bigoplus \mathbb{Z}(-1) = H^2(X, X - |D|)$$

be a morphism of Hodge structures defined by $\phi_D(1/2\pi\sqrt{-1}) = \sum D_i$ where D_i are irreducible components of D. Since D is homologically trivial, ϕ_D factors through K. By pulling back the extension (7) along ϕ_D, we get a new extension of mixed Hodge structures:

$$0 \longrightarrow H^1(X,S) \longrightarrow E_D \longrightarrow \mathbb{Z}(-1) \longrightarrow 0. \tag{8}$$

Thus this corresponds to an element in $\mathrm{Ext}^1(\mathbb{Z}(-1), H^1(X,S))$. By a theorem of Carlson, $\mathrm{Ext}^1(\mathbb{Z}(-1), H^1(X,S))$ is isomorphic to

$$J^0\mathrm{Hom}(\mathbb{Z}(-1), H^1(X,S)) = H^1(X,S;\mathbb{C})/H^1(X,S;\mathbb{Z}) + F^1H^1(X,S;\mathbb{C}),$$

which will be denoted by $J(X-S)$. Note that $J(X-S)$ is independent of the choice of a compactification of $X - S$.

Lemma 4.1. *The Jacobian $J(X - S)$ is naturally isomorphic to*

$$H^1(X, \mathcal{O}(-S))/H^1(X,S;\mathbb{Z}).$$

Proof. Consider an exact sequence of cohomologies on X.

$$\ldots \longrightarrow H^0(X, \mathbb{C}_S) \longrightarrow H^1(X, j_!\mathbb{C}) \longrightarrow H^1(X,\mathbb{C}) \longrightarrow \ldots$$

where j is the natural inclusion from $X - S$ to X. Since this is an exact sequence of mixed Hodge structures and the Hodge filtrations are strictly preserved by the maps, this induces an exact sequence:

$$\ldots \longrightarrow Gr^0_{F^\bullet}H^0(X, \mathbb{C}_S) \longrightarrow Gr^0_{F^\bullet}H^1(X, j_!\mathbb{C}) \longrightarrow Gr^0_{F^\bullet}H^1(X,\mathbb{C}) \longrightarrow \ldots$$

Now consider the following diagram of cohomologies on X:

$$\begin{array}{ccccccccc}
Gr^0_{F^\bullet}H^0(\mathbb{C}_X) & \to & Gr^0_{F^\bullet}H^0(\mathbb{C}_S) & \to & Gr^0_{F^\bullet}H^1(j_!\mathbb{C}) & \to & Gr^0_{F^\bullet}H^1(\mathbb{C}_X) & \to & Gr^0_{F^\bullet}H^1(\mathbb{C}_S) \\
\downarrow \alpha_0 & & \downarrow \beta_0 & & \downarrow \gamma_1 & & \downarrow \alpha_1 & & \downarrow \beta_1 \\
H^0(\mathcal{O}_X) & \longrightarrow & H^0(\mathcal{O}_S) & \longrightarrow & H^1(\mathcal{O}(-S)) & \longrightarrow & H^1(\mathcal{O}_X) & \longrightarrow & H^1(\mathcal{O}_S)
\end{array}$$

The vertical arrows β_0 and β_1 are isomorphisms because the spectral sequence associated to the Hodge filtration on $H^*(S, \mathbb{Z}_S)$ degenerates at E_1 [10, (1.5)] [3, (8.2.1), (8.1.12), (8.1.9)]. The vertical arrows α_0 and α_1 are isomorphisms by the E_1-degeneration of the usual Hodge to DeRham spectral sequence. Hence by 5-lemma, the map γ_1 is an isomorphism. It follows that the sequence

$$0 \longrightarrow H^0(d\mathcal{O}(-S)) \longrightarrow H^1(j_!\mathbb{C}) \longrightarrow H^1(\mathcal{O}(-S)) \longrightarrow 0 \tag{9}$$

is exact. Note that $F^1H^1(j_!\mathbb{C}) = H^0(d\mathcal{O}(-S))$. This completes the proof. □

Thus for a homologically trivial divisor $D \in \mathrm{Div}^0(X - S)$, we can associate an element in the Jacobian $H^1(X, \mathcal{O}(-S))/H^1(X, S; \mathbb{Z})$. By duality, the Jacobian can be identified with

$$H^{n-1}(X, \omega_X(S))\check{}/H_{2n-1}(X - S; \mathbb{Z}).$$

The map

$$\alpha : \mathrm{Div}^0(X - S) \to H^{n-1}(X, \omega_X(S))\check{}/H_{2n-1}(X - S; \mathbb{Z})$$

obtained in this way will be called the *Abel-Jacobi* map. We will show that the Abel-Jacobi map α can be realized in the following way ;

Theorem 4.2. *Given a cohomology class in $H^{n-1}(X, \omega_X(S))$, choose a $(n, n-1)$-form ω representing this cohomology class. Then $\alpha(D)$ is given by*

$$\omega \mapsto \int_{\Gamma_D} \omega$$

where Γ_D is a $(2n-1)$-chain in $X - S$ whose boundary is D.

Proof. After a birational change of X, we may assume that the support of D is a reduced normal crossing divisor. To each homologically trivial divisor $D \in \mathrm{Div}^0(X - S)$, one can associate a form $\eta_D \in H^0(X, \Omega^1_X(\log|D|)) = F^1H^1(X - |D|, \mathbb{C})$ with $\mathrm{Res}\eta_D = D$ since the map in the sequence (8) strictly preserves the Hodge filtration $F^\bullet$ and $F^1H^1(X - |D|, S; \mathbb{C}) \subset F^1H^1(X - |D|, \mathbb{C})$. To construct a retraction $r : E_D \to H^1(X, S; \mathbb{Z})$, choose a set $\{\xi_1, \cdots, \xi_m\}$ of differential $(2n-1)$-forms on $X - S$ representing a basis of $H^{2n-1}(X - S, \mathbb{Z})$ such that ξ_i vanishes in a neighborhood $N(D)$ of $|D|$. This is possible since we have a surjection $H^{2n-1}(X - S, D) \to H^{2n-1}(X - S)$. Let $\{\xi^1, \cdots, \xi^m\}$ be the dual basis of $H^1(X, S; \mathbb{Z})$. We now set

$$r(\eta) = \sum_i \int_X (\eta \wedge \xi_i)\xi^i.$$

Let $B(D)$ be a small tubular neighborhood of $|D|$ in $X - S$ such that the closure of $B(D)$ is contained in $N(D)$. We can write

$$\omega = \sum_{i=1}^{m} c_i \xi_i + d\phi$$

where ϕ is a $C^\infty(2n-2)$-form on $X - S$. Set $\eta = \eta_D$. Via the isomorphism given in Theorem 2.1 (cf. [1, Theorem 6.2]), $\alpha(D)$ is given by sending ω to

$$\begin{aligned}
&\int_X r(\eta) \wedge \omega \\
&= \int_X r(\eta) \wedge \sum_i c_i \xi_i = \int_X \left(\sum_i \left(\int_X \eta \wedge \xi_i \right) \xi^i \right) \wedge \left(\sum_j c_j \xi_j \right) \\
&= \int_X \eta \wedge \sum_i c_i \xi_i = \int_{X - B(D)} \eta \wedge (\omega - d\phi) \\
&= \int_{X - B(D)} -\eta \wedge d\phi \\
&\quad \text{(since } \eta \wedge \omega = 0 \text{ on } X - B(D)) \\
&= \int_{\partial B(D)} \eta \wedge \phi \quad \text{(by Stokes' Theorem.)} \\
&= \int_{\partial B(D)} \eta \wedge \int \omega \quad (\textstyle\int \omega \text{ is a primitive of } \omega \text{ on } B(D)) \\
&= \int_D \int \omega
\end{aligned}$$

Since D is also algebraically equivalent to zero [6, p. 462] it is enough to consider the following case: Let T be a non-singular curve and $\mathcal{D}$ be an irreducible divisor on $X \times T$, flat over T. D is given by $p_{2*}(p_1^*(0) - p_1^*(1))$ for some points $0, 1 \in T$.

$$\begin{array}{ccc}
\mathcal{D} \subset X \times T & \xrightarrow{p_2} & X \\
\Big\downarrow p_1 & & \\
T & &
\end{array}$$

Let $\widetilde{\mathcal{D}}$ be a desingularization of $\mathcal{D}$ and p_1' be the composition of $\widetilde{\mathcal{D}} \to \mathcal{D}$ and p_1. Let

$$
\begin{array}{ccc}
\widetilde{\mathcal{D}} & \xrightarrow{f} & X' \\
 & & \Big\downarrow g \\
 & & X
\end{array}
$$

be the Stein factorization of $\widetilde{\mathcal{D}} \to X$. So f has connected fibers and g is a finite surjective map. Now choose a path γ from 0 to 1 in T such that p_1' is smooth over $\gamma - \partial\gamma$. Let $\widetilde{\Gamma_D} = {p_1'}^{-1}(\gamma)$, $\Gamma_D' = f_*\widetilde{\Gamma_D}$ and $\Gamma_D = g_*\Gamma_D'$. Take a division $\gamma = \Sigma_i\gamma_i$ of γ so that p_1' is trivial over γ_i. Set $(\widetilde{\Gamma_D})_i = {p_1'}^{-1}(\gamma_i)$, $(\Gamma_D')_i = f_*(\widetilde{\Gamma_D})_i$. Then each $(\widetilde{\Gamma_D})_i$ shrinks to a fiber, hence $H^{2n-1}((\widetilde{\Gamma_D})_i, \mathbb{C}) = 0$ and so $H^{2n-1}((\Gamma_D')_i, \mathbb{C}) = 0$. Therefore we have

$$
\begin{aligned}
&\int_D \int \omega \\
&= g_*\left(\int_{D'} g^*\left(\int \omega\right)\right) \quad \text{where } D' = f_*({p_1'}^*(0) - {p_1'}^*(1)) \\
&= g_*\left(\Sigma_i \int_{\partial(\Gamma_D')_i} g^*\left(\int \omega\right)\right) \\
&= g_*\left(\Sigma_i \int_{(\Gamma_D')_i} g^*\omega\right) \quad \text{by Stokes' theorem} \\
&= g_*\left(\int_{\Gamma_D'} g^*\omega\right) = \int_{\Gamma_D} \omega
\end{aligned}
$$

Note that $g^*(\int \omega)$ is extendable to Γ_D' since $H^{2n-1}(\Gamma_D', \mathbb{C}) = 0$. □

Note that when $S = \emptyset$, our Abel-Jacobi map agrees with the classical Abel-Jacobi map. The initial step of the proof contains a useful method for calculating α. Under the original definition

$$J(X - S) = H^1(X, S; \mathbb{C})/H^1(X, S; \mathbb{Z}) + F^1H^1(X, S; \mathbb{C})$$

$\alpha(D)$ is represented by $r(\eta_D)$, where $\eta_D \in F^1H^1(X - D, S; \mathbb{C})$ is given by a logarithmic 1-form with $\text{Res}\eta_D = D$ and r an integral retraction onto $H^1(X, S; \mathbb{C})$. Note that a form in $H^0(\Omega_X^1(\log|D|))$ defines a class in $H^1(X - D, S)$ if and only if it vanishes on S.

5. Abel's Theorem

We will establish Abel's Theorem by showing that the two definitions of the Abel-Jacobi map α and β (5) agree up to sign.

Theorem 5.1. *α and β coincide up to sign.*

Proof. First, we will give an explicit description of the map β in (5). By construction β is a composite of the injection

$$\beta' : Cl^0_S(X) \longrightarrow H^1(O(-S))/H^1(X,S;\mathbb{Z})$$

and the duality map

$$H^1(O(-S))/H^1(X,S;\mathbb{Z}) \cong H^{n-1}(X,\omega_X(S))\check{}/H_{2n-1}(X-S,\mathbb{Z}).$$

Let D be a homologically trivial divisor on $X - S$. Choose a finite open covering $\{U_i\}_{i=0,\cdots,m}$ of X such that D is defined by $f_i = 0$ on U_i and $U_0 = X - |D|$. As S and D are disjoint, there is no loss in assuming that $f_i = 1$ on S. Then the cohomology class $[D]$ of D in $H^1(\mathcal{O}^*(-S))$ can be represented by $\{f_i/f_j\}$. Since D is homologically trivial, there is a cocycle $\phi_{ij} \in Z^1(\mathcal{O}(-S))$ such that $exp(2\pi i\phi_{ij}) = f_i/f_j$. Then $\beta'(D)$ is represented by ϕ_{ij}.

Next we calculate $\alpha(D)$. We make use of the identification

$$J(X-S) \cong H^1(O(-S))/H^1(X,S;\mathbb{Z})$$

to view $\alpha(D)$ as an element of the latter group. By degeneration of the Hodge to De Rham spectral sequence, there exists $\psi_i \in H^0(\Omega^1_X(U_i))$ such that $d\phi_{ij} = \psi_j - \psi_i$. Therefore $\eta = 1/2\pi i d\log f_i + \psi_i$ is a globally defined logarithmic 1-form satisfying $\text{Res}\eta = D$ which also vanishes on S. As explained in the remarks at the end of the last section, $\alpha(D)$ is represented by $r(\eta)$, and in fact we are free to modify η by adding an element of $F^1H^1(X,S)$. Note that $H^1(X-D,S;\mathbb{C})$ is isomorphic to the first hypercohomology group of $\Omega^\bullet_X(\log D + S)(-S)$, and this can be described using Cech methods. In particular $(\phi_{ij}, d\log f_i)$ is a cocycle defining a class $\Phi \in H^1(X-D,S;\mathbb{C})$. We claim that Φ can be decomposed as a sum $\Phi_1 + \Phi_2$ where the first term lies in $L = H^1(X-D,S;\mathbb{Z})$ and the second in $F^1H^1(X-D,S)$. To see this, first observe that the quotient $H^1(X,O(-S))$ by the image of L is isomorphic to the quotient of $J(X-S)$ by the subgroup of homologically trivial divisors with support in $|D|$ (under β'). Therefore as the image of Φ in $J(X-S)$ is D, it follows that modulo L, Φ can be represented by a form in $F^1H^0(X-D,S)$. As Φ has integral residues, it follows that after subtracting off an addition element of L, the difference lies in $F^1H^1(X,S)$. In other words, we have obtained the desired decomposition of Φ. Now set $\eta' = \eta - \Phi_2$. Let $E_D \subset H^1(X-D,S)$ be the extension defined in (8). Consider the unique retraction $\eta : E_D \to H^1(X,S;\mathbb{Z})$ with kernel $\mathbb{Z}\Phi_1$. Then $\eta' = a\Phi_1 + r(\eta')$, and by matching residues, we see that $a = 1$. Therefore $r(\eta') = \eta - \Phi = -(\phi_{ij}, \psi_i)$ represents $\alpha(D)$. But of course $\alpha(D)$ is the image of this class in $J(X-S)$ and this is represented by $-\phi_{ij}$, or $-\beta'(D)$. □

6. Hodge Theoretic Proof of Abel's Theorem

We give an alternative proof of Abel's theorem based on Carlson's theorem.

Theorem 6.1. *A homologically trivial divisor $D \in \mathrm{Div}^0(X - S)$ is S-principal if and only if there exists $\eta \in H^0(X, \Omega^1_X(\log|D|)$ such that*

(1) $\mathrm{Res}\eta = D$

(2) *η has integral periods for any closed loop in $X - |D|$.*

(3) *$\int_\gamma \eta \in \mathbb{Z}$ where γ is a path in $X - S$ connecting points of S.*

By the way, the statement (2) is included in (3).

Proof. Given η as above, set

$$f(z) = \exp\left(2\pi\sqrt{-1}\int_{z_0}^{z} \eta\right).$$

Conversely, if $D = (f) \in \mathrm{Prin}_S(X)$, let

$$\eta = \frac{1}{2\pi\sqrt{-1}}\frac{df}{f}.$$

□

Corollary 6.2. *$\alpha(D) = 0$ if and only if D is S-principal.*

Proof. $\alpha(D) = 0$ if and only if the extension (8) splits in the category of Mixed Hodge Structures. Hence $\alpha(D) = 0$ if and only if η_D represents an integral class in $H^1(X - |D|, S)$. Thus $\alpha(D) = 0$ if and only if η_D satisfies the conditions in Theorem 6.1. □

7. Non-compact Curves

When X is a curve, $J(X - S)$ is an extension of the classical Jacobian $J(X)$ by the complex multiplicative group.

Lemma 7.1. *Let X be a smooth projective curve and S be a set of distinct points. Then we have an exact sequence of algebraic groups:*

$$1 \longrightarrow (\mathbb{C}^*)^{\sigma-1} \longrightarrow J(X - S) \longrightarrow J(X) \longrightarrow 0$$

where σ is the number of points in S and $J(X)$ is the usual Jacobian of X.

Proof. Consider an exact sequence of cohomologies on X:

$$0 \to H^0(\Omega^1_X) \to H^0(\Omega^1_X(\log S)) \to H^0(\mathcal{O}_S) \to \mathcal{H}^\infty(\otimes^\infty_{\mathcal{X}}) \to \mathcal{H}^\infty(\otimes^\infty_{\mathcal{X}}(\log \mathcal{S})) \to \prime$$

By Serre duality, $H^1(X,\Omega^1_X(\log S)) = H^0(X,\mathcal{O}(-S)) = {}'$ and $h^1(X,\Omega^1_X) = 1$. This sequence fits into the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(X,X-S)/H_2(X,\mathbb{Z}) & \longrightarrow & H_1(X-S,\mathbb{Z}) & \longrightarrow & H_1(X,\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^{\sigma-1} & \longrightarrow & H^0(X,\Omega^1_X(\log S))\check{} & \longrightarrow & H^0(X,\Omega^1_X)\check{} & \longrightarrow & 0
\end{array}$$

The cokernels of the vertical arrows will give the desired sequence. The cokernel of the leftmost arrow is identified with the multiplicative group $(\mathbb{C}^*)^{\sigma-1}$ via the exponential map $\exp(2\pi i(\ \ ))$. □

EXAMPLE 3. Let $X = \mathbb{P}^1$ and $S = \{0,\infty\}$. Then $H^0(X,\omega_X(S))$ is generated by dz/z. By the above Lemma, we have $J(X-S) = \mathbb{C}^*$. By Theorem 4.2, the Abel-Jacobi map $\alpha : \mathrm{Div}^0(X-S) \to \mathbb{C}^*$ is the natural linear extension of

$$\alpha(x-1) = \exp\int_1^x \frac{dz}{z} = x, \quad \text{for} x \in X-S$$

if we choose $1 \in X-S$ as a base point. Thus $\ker\alpha = \{\sum n_p p - (\sum n_p)\cdot 1 \in \mathrm{Div}^0(X-S) | \Pi p^{n_p} = 1\}$ On the other hand, a rational function f on X is in $\mathrm{Prin}_S(X)$ iff

$$f(z) = \frac{\Pi_{i=1}^n (z-a_i)}{\Pi_{i=1}^n (z-b_i)}$$

with $\Pi a_i = \Pi b_i \neq 0,\infty$. As expected by our Abel-Jacobi theorem, $\ker\alpha = \mathrm{Prin}_S(X)$.

As an application, we give a version of Torelli theorem for noncompact curves. A similar result for complete singular curves was obtained by Carlson [2]. Let X be a smooth non-compact curve and $\bar{X}$ its unique smooth compactification. Then the mixed Hodge structure on $H^1(X,\mathbb{Z})$ carries a natural graded polarization given as follows: The polarization on $Gr_1^{W_\bullet}H^1(X,\mathbb{Z})$ is induced by the polarized Hodge structure on $H^1(\bar{X},\mathbb{Z})$, which is determined by the intersection product of one-cycles on $\bar{X}$. For $Gr_2^{W_\bullet}H^1(X,\mathbb{Z})$, choose the unique symmetric bilinear form on $\bigoplus_{i=1}^n \mathbb{Z}(-1)$ so that $\{e_j\}$ forms an orthonormal basis. Then restrict this polarization to $Gr_2^{W_\bullet}H^1(X,\mathbb{Z})$.

Theorem 7.2. *Let X be a smooth non-compact curve and $\bar{X}$ its unique smooth compactification. Suppose $\bar{X}$ is non-hyperelliptic of genus > 1 and the number of points in $\bar{X} - X$ at least 2. Then X is determined by the graded polarized MHS on $H^1(X,\mathbb{Z})$.*

Proof. Let $\bar{X} - X = \{p_1, \dots, p_n\}$. Consider the 'Thom-Gysin' sequence :

$$0 \longrightarrow H^1(\bar{X}, \mathbb{Z}) \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow \bigoplus_{i=1}^{n} \mathbb{Z}(-1) \longrightarrow H^2(\bar{X}, \mathbb{Z}) = \mathbb{Z}(-1)$$

where each point p_j contributes to the j-th component vector $\{e_j\}$ of $\bigoplus_{i=1}^n \mathbb{Z}(-1)$. Note that $K = \ker(\bigoplus_{i=1}^n \mathbb{Z}(-1) \longrightarrow H^2(\bar{X}, \mathbb{Z}) = \mathbb{Z}(-1))$ is just $Gr_2^{W_\bullet} H^1(X, \mathbb{Z})$ and $H^1(\bar{X}, \mathbb{Z}) = Gr_1^{W_\bullet} H^1(X, \mathbb{Z})$. Now we provide a polarization on $H^1(X, \mathbb{Z})$.

First, by the classical Torelli theorem, the polarization on $Gr_1^{W_\bullet} H^1(X, \mathbb{Z})$ determines $\bar{X}$. Second, define a map $\phi_{ij} : \mathbb{Z}(-1) \to \bigoplus_{i=1}^n \mathbb{Z}(-1)$ sending $1/2\pi\sqrt{-1}$ to $e_i - e_j$. Then these maps are all possible maps from $\mathbb{Z}(-1)$ to $\bigoplus_{i=1}^n \mathbb{Z}(-1)$ which factors through K and minimizes the length of the image of the generator $1/2\pi\sqrt{-1}$. By pulling back along ϕ_{ij}, we get an element in $\mathrm{Ext}^1(\mathbb{Z}(-1), H^1(\bar{X}, \mathbb{Z})) \cong J(\bar{X})$, which depends only on the polarized Hodge structure on $Gr_2^{W_\bullet} H^1(X, \mathbb{Z})$. This corresponds to $\alpha(p_i - p_j) \in J(\bar{X})$ under the Abel-Jacobi map α [1, Theorem 6.2]. As $\bar{X}$ is not hyperelliptic, $\alpha(p_i - p_j)$ uniquely determines p_i and p_j. Otherwise, there exists a meromorphic function f on $\bar{X}$ such that $(f) = p_i + p - p_j - q$ by the classical Abel's theorem. Therefore the graded polarized MHS on $H^1(X, \mathbb{Z})$ determines X. □

References

[1] D. Arapura and K. Oh: *Abel's theorem for twisted Jacobians*, Trans. of AMS. **342** (1994), 421–433.

[2] J. Carlson: *Extensions of mixed Hodge structures*, Journées de géométrie algébrique d'Angers 1979, Sijthoff & Noordhoff International Publishers B.V., (1980), 107–127.

[3] P. Deligne: *Théorie de Hodge II, III*, Publ. de l'Inst. des Hautes Études Scientifiques, **40** (1972), 5–57; **44** (1974), 5–77.

[4] A. Fujiki: *Duality of mixed Hodge structures of algebraic varieties*, Publ. RIMS, Kyoto Univ. **16** (1980), 635–667.

[5] W. Fulton: *Intersection theory*, Springer-Verlag, New York, (1984).

[6] P.A. Griffiths and J. Harris: Principles of algebraic geometry, John Wiley and Sons, 1978.

[7] R. Hartshorne: Algebraic geometry, Springer-Verlag, New York, 1977.

[8] M. Rosenlicht: *Generalized Jacobian varieties*, Ann. of Math. **59** (1954), 505–530.

[9] E.H. Spanier: *Algebraic topology*, McGraw-Hill, New York, (1966).

[10] J.H.M. Steenbrink: *Mixed Hodge structures associated with isolated singularities*, Singularities (Proc. Symp. Pure Math. **40**), Amer. Math. Soc. Providence, **2** (1983), 513–536.

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