Komatsu, Y., Kotani, S. and Matsumura, A.
Osaka J. Math.
34 (1997), 605–627

A PERIOD-DOUBLING BIFURCATION FOR THE DUFFING EQUATION

YUKIE KOMATSU, SHINICHI KOTANI and AKITAKA MATSUMURA

(Received September 19, 1996)

1. Introduction

We consider the periodic solutions of the Duffing equation which describes the nonlinear forced oscillation:

$$u''(t)+\mu u'(t)+\kappa u(t)+\alpha u^3(t)=f_\lambda(t),\quad t\in R \tag{1.1}$$

where μ,α are positive constants and κ is a nonnegative constant, and $f_\lambda(t)$ is a given family of T-periodic external forces parameterized by λ (>0) which somehow represents the magnitude of f_λ. It is well-known that for any λ there exists at least one T-periodic solution of (1.1), and furthermore if the magnitude λ is suitably small, then this periodic solution is unique and asymptotically stable. As λ increases, we can observe by numerical computations that the solution loses its stability and various bifurcation phenomena take place. In particular, the period-doubling bifurcations are observed as very important phenomena along the route toward a so called "Chaos". However, it is surprising that there have been no rigorous proofs of existence of these bifurcation phenomena. Recently, Komatsu-Kano-Matsumura [4] tried to detect a bifurcation phenomenon around a "linear probe" $\{(\lambda,u_\lambda)\}_{\lambda>0}$ inserted into the product space (λ,u), which is defined by

$$\begin{cases} u_\lambda(t) := \lambda U(t), & U(t) : \text{given } T\text{-periodic smooth function} \\ f_\lambda(t) := u_\lambda''(t)+\mu u_\lambda'(t)+\kappa u_\lambda(t)+\alpha u_\lambda^3(t). \end{cases} \tag{1.2}$$

Here we should note that $u=u_\lambda$ is a trivial solution of (1.1) corresponding to f_λ for any λ. Then, in the particular case $U(t)=\sin(2\pi t)$ ($T=1$), studying the linearized equation of (1.1) at $u=u_\lambda$

$$v''(t)+\mu v'(t)+\kappa v(t)+3\alpha\lambda^2U^2(t)v(t)=0 \tag{1.3}$$

by the arguments of continued fractions, they showed that T-periodic solution bifurcates from at least three points of the probe $\{u_\lambda\}_{\lambda>0}$ under some condition on μ. They also made a conjecture by numerical computations that there are countably many bifurcation points of T-periodic solution. However, they could not obtain any results on period-doubling bifurcations. On the other hand, numerical computations in the case $U(t) = \sin(2\pi t)+0.5$, indicate that there might be countably many bifurcation points of both T-periodic and $2T$-periodic solutions. In this paper, we shall try to explain these phenomena rigorously.

In fact, we show that for more general T-periodic functions $U(t)$, only T-periodic and $2T$-periodic solutions can bifurcate from $\{u_\lambda\}_{\lambda>0}$, and under some condition on μ (see (2.2) below) there exist countably many bifurcation points of T-periodic solution, and also do exist countably many bifurcation points of $2T$-periodic solution (period-doubling bifurcations) except some particular cases. We should emphasize that the condition (2.2) on μ depends only on the number of zero points of U over one period and their order, and that more zero points U has, less restrictive is the condition on μ, this somehow means, easier bifurcation phenomena take place. We also show the asymptotic stability and instability of the trivial solution $u_\lambda(t)$ alternates at each these bifurcation points. We further remark that the case $U(t) = \sin(2\pi t)$ is really a particular one where only T-periodic solutions bifurcate from $\{u_\lambda\}_{\lambda>0}$. The precise conditions and main Theorem are stated in Section 2. In Section 3, we reformulate the problem in order to apply Crandall-Rabinowitz's Theorem [2] on bifurcation theory. In this process, eigenvalue problem of (1.3) plays an essential role. In Section 4, we relate it to the Lyapunov exponent through the Floquet Theory and show the properties of the Lyapunov exponent in making use of the expansion theory by generalized eigen-functions established by Titschmarsh-Kodaira. Finally from these properties and asymptotic analysis with respect to λ, which details are stated in Section 6, we prove main Theorem in Section 5.

2. Main Theorem

To state the main Theorem precisely, we assume that

(2.1) $\quad U^2(t)$ has $N+1$ zero points $\{t_i\}_{i=0}^N$ of n-th order on $[t_0, t_0+T]$,

where $t_0 < t_1 < \cdots < t_N = t_0 + T$. We define $\nu = 1/(n+2)$ and also define $S_i = \int_{t_{i-1}}^{t_i} |U(s)|ds$. Then we have the following main theorem for the bifurcation problem of periodic solution of (1.1) with (1.2).

Theorem 2.1. *Suppose* (2.1) *and*

$$\frac{\mu}{2} < \frac{N}{T} \log\left(\cot\frac{\nu\pi}{2}\right). \tag{2.2}$$

Then it holds the followings for the bifurcation solutions from the probe $\{u_\lambda\}_{\lambda>0}$.

(1) *There exist countably many bifurcation points, whose period is* T *or* $2T$. *On the other hand,* mT*-periodic* $(m \geq 3)$ *solution does not bifurcate.*

(2) *The case* $N = 1$:

There exist λ^* *and* $\{\lambda_i\}_{i=0}^{\infty}$ $(\lambda^* < \lambda_0 < \lambda_1 \cdots \to \infty)$ *such that the sequence of bifurcation points for* $\lambda > \lambda^*$ *is coincident with* $\{\lambda_i\}_{i=0}^{\infty}$, *where* $\{\lambda_{4m}\}$, $\{\lambda_{4m+1}\}$ *are T-periodic bifurcation points and* $\{\lambda_{4m+2}\}$, $\{\lambda_{4m+3}\}$ *are period-doubling bifurcation points. Moreover, it holds that if* $\lambda \in (\lambda_{2m+1}, \lambda_{2m})$, *then* u_λ *is asymptotically stable, if* $\lambda \in (\lambda_{2m}, \lambda_{2m+1})$, *then* u_λ *is unstable.*

(3) *The case* $N = 2$:
There exist countably many T-periodic bifurcation points, and also exist countably many 2T-periodic bifurcation points except for the following two cases.
(i) *When* $S_1 = S_2$, *the set of period-doubling bifurcation points is finite.*
(ii) *When* $S_1/S_2 = (2p+1)/(2q+1)$ $(p, q \in \mathbb{N},\ S_1 \neq S_2)$, *we assume instead of* (2.2),

$$\frac{\mu}{2} < \frac{1}{T} \log \left(\frac{|\tilde{\Delta}| + \sqrt{|\tilde{\Delta}|^2 - 4}}{2} \right), \tag{2.3}$$

where

$$\tilde{\Delta} = \inf_{\lambda} \frac{2\{\cos(S_1 + S_2)\lambda + \cos(S_1 - S_2)\lambda \cos^2 \nu\pi\}}{\sin^2 \nu\pi}.$$

Then there also exist countably many period-doubling bifurcation points. The stability of u_λ *changes at any above bifurcation points.*

(3) *The case* $N \geq 3$:
There exist countably many T-periodic bifurcation points. Furthermore, if $\{S_i\}_{i=1}^{N}$ *are rationally independent, there also exist countably many period-doubling bifurcation points. The stability of* u_λ *changes at any these bifurcation points.*

REMARK 1. Throughout this paper, we use the notation "mT-periodic solution" $(m \in N)$ when the periodic solution has a period mT, but not any of iT $(1 \leq i \leq m-1)$.

REMARK 2. If $S_1/S_2 \neq (2p+1)/(2q+1)$ $(p, q \in N)$, it holds that $\tilde{\Delta} = -2(1 + \cos^2 \nu\pi)/\sin^2 \nu\pi$. Then we have

$$\log \left(\frac{|\tilde{\Delta}| + \sqrt{|\tilde{\Delta}|^2 - 4}}{2} \right) = 2 \log \left(\cot \frac{\nu\pi}{2} \right),$$

which is consistent with the condition (2.2) .

EXAMPLE 1. In the case $U(t) = \sin 2\pi t \pm 1$, $U^2(t)$ has two zero points of forth order ($N = 1$, $\nu = 1/6$). Applying Theorem, if $\mu/2 < \log(2+\sqrt{3})$, there exist countably many both 1-periodic and period-doubling bifurcation points.

EXAMPLE 2. In the case $U(t) = \sin 2\pi t + 0.5$, $U^2(t)$ has three zero points of second order, and S_1/S_2 is not rational ($N = 2$, $\nu = 1/4$). Applying Theorem, if $\mu/2 < 2\log(1+\sqrt{2})$, there exist countably many both 1-periodic and period-doubling bifurcation points.

EXAMPLE 3. In the case $U(t) = \sin 2\pi t$, $U^2(t)$ has three zero points, but $S_1 = S_2$ ($N = 2$, $\nu = 1/4$). So, Theorem implies the set of period-doubling bifurcation points is finite. However, we can show a stronger result that the period-doubling bifurcation point can not exist at all. In fact, since the period of $U^2(t)$ is 1/2 in this case, the argument in the proof of (0) implies the period of any bifurcation points can not be but 1/2 or 1. This explains why we could not detect any results on period-doubling bifurcations in [4].

3. Reformulation of the problem

We first note that any periodic solution of (1.1) should have the period $\widetilde{T} = mT$ for an $m \in \mathbb{N}$. Hence, for any fixed $m \in \mathbb{N}$, we look for the periodic solution of (1.1) in the form :

$$u(t) = u_\lambda(t) + \lambda v(t), \tag{3.1}$$

where $v(t)$ is a $\widetilde{T}$-periodic function. Then $v(t)$ must satisfy the periodic problem

$$\begin{cases} v''(t) + \mu v'(t) + \kappa v(t) + \Lambda(U^2(t)v(t) + U(t)v^2(t) + \frac{1}{3}v^3(t)) = 0 \\ v(t+\widetilde{T}) = v(t), \quad t \in R, \end{cases} \tag{3.2}$$

where we set $\Lambda = 3\alpha\lambda^2$. To study the bifurcation problem to (3.2) around the trivial solution $v = 0$, we make use of a following bifurcation theorem in Crandall-Rabinowitz [2].

Theorem 3.1 (Crandall and Rabinowitz). *Let X, Y be Banach spaces, V a neighborhood of 0 in X and*

$$F : (0,\infty) \times V \to Y$$

have the properties for a $\Lambda_0 > 0$

(a) $F(\Lambda, 0) = 0$ *for* $\Lambda \in (0,\infty)$,

(b) *The partial derivatives* F_Λ, F_x *and* $F_{\Lambda x}$ *exist and are continuous,*

(c) *$N(F_x(\Lambda_0, 0))$ and $Y/R(F_x(\Lambda_0, 0))$ are one dimensional.*
(d) *$F_{\Lambda x}(\Lambda_0, 0)x_0 \notin R(F_x(\Lambda_0, 0))$, for a nontrivial $x_0 \in N(F_x(\Lambda_0, 0))$.*

Let Z be any complement of $N(F_x(\Lambda_0, 0))$ in X. Then there is a neighborhood U of $(\Lambda_0, 0)$ in $R \times X$, an interval $(-\delta, \delta)$, and continuous functions $\varphi : (-\delta, \delta) \to R$, $\psi : (-\delta, \delta) \to Z$ such that $\varphi(0) = \Lambda_0$, $\psi(0) = 0$ and

$$F^{-1}(0) \cap U = \{\varphi(\epsilon),\ \epsilon x_0 + \epsilon\psi(\epsilon) : |\epsilon| < \delta\} \cup \{(\lambda, 0) : (\lambda, 0) \in U\}. \tag{3.3}$$

In order to apply Theorem 3.1, we define Banach spaces X and Y by

$$X = \{u \in C^2(R); u(t) = u(t + \widetilde{T}), t \in R\},$$
$$Y = \{u \in C(R); u(t) = u(t + \widetilde{T}), t \in R\},$$

with the norms

$$||u||_X = \max_{0 \le t \le \widetilde{T}} |u''(t)| + \max_{0 \le t \le \widetilde{T}} |u'(t)| + \max_{0 \le t \le \widetilde{T}} |u(t)|,$$
$$||u||_Y = \max_{0 \le t \le \widetilde{T}} |u(t)|.$$

Also define $F : (0, \infty) \times X \to Y$ by

$$F(\Lambda, v) = v'' + \mu v' + \kappa v + \Lambda \left(U^2 v + U v^2 + \frac{1}{3} v^3 \right). \tag{3.4}$$

Then we have

Lemma 3.2. *The hypotheses* (a)–(d) *of Theorem* (3.1) *are reduced to the following three conditions in the present problem* (3.2).

(i) *$\Lambda = \Lambda_0$ is a positive eigenvalue of the following linearized eigenvalue problem of* (3.2) *at $v = 0$:*

$$\begin{cases} v''(t) + \mu v'(t) + \kappa v(t) + \Lambda U^2(t) v(t) = 0 \\ v(t + \widetilde{T}) = v(t), \quad t \in R \end{cases} \tag{3.5}$$

(ii) *The solution space of* (3.5) *is one dimensional.*
(iii)

$$\int_0^{\widetilde{T}} v_0(t) v_0^*(t) U^2(t) dt \ne 0, \tag{3.6}$$

where $v_0(t)$ is an eigenfunction of (3.5) *with $\Lambda = \Lambda_0$ and $v_0^*(t)$ is a nontrivial solution of the adjoint problem to* (3.5) *with $\Lambda = \Lambda_0$:*

$$(3.7)\qquad \begin{cases} v''(t) - \mu v'(t) + \kappa v(t) + \Lambda_0 U^2(t)v(t) = 0, \\ v(t+\widetilde{T}) = v(t), \quad t \in R \end{cases}$$

REMARK 3. The condition (iii) means that the eigenvalue $\Lambda = \Lambda_0$ is simple. Since our problem here is not self-adjoint, its condition is not trivial at all. We shall give a nice criterion for it in the next section.

Proof of Lemma 3.2. It is clear that $F(\Lambda, 0) = 0$ for $\Lambda \in (0, \infty)$. Moreover, F_Λ, F_v and $F_{\Lambda v}$ are easily proved to exist and be continuous. Especially, it holds that

$$(3.8)\qquad F_v(\Lambda, 0)v = v'' + \mu v' + \kappa v + \Lambda U^2 v,$$

$$(3.9)\qquad F_{\Lambda v}(\Lambda, 0)v = U^2 v.$$

Therefore, we have $N(F_v(\Lambda_0, 0))$ coincides with the eigen space of (3.5) $\Lambda = \Lambda_0$ and the condition $F_{\Lambda v}(\Lambda_0, 0)v_0 \notin R(F_v(\Lambda_0, 0))$ is equivalent to that the equation

$$(3.10)\qquad y''(t) + \mu y'(t) + \kappa y(t) + \Lambda_0 U^2(t)y(t) = U^2(t)v_0(t).$$

has no solution. Now, let's define $F_v^*(\Lambda_0, 0) : X \to Y$ by

$$(3.11)\qquad F_v^*(\Lambda_0, 0)v = v'' - \mu v' + \kappa v + \Lambda_0 U^2 v.$$

Then, the standard argument of the ordinary differential equaitons says that

$$(3.12)\qquad \dim N(F_v(\Lambda_0, 0)) = \dim N(F_v^*(\Lambda_0, 0)) = \dim(Y/R(F_v(\Lambda_0, 0)),$$

and a necessary and sufficient condition that the equation (3.10) has no solution is that the right hand side of (3.10) is not orthogonal to $N(F_v^*(\Lambda_0, 0))$. Therefore, we can easily see the condition (c) is reduced to (ii), and the condition (d) is reduced to (iii). Thus the proof is completed. □

4. Eigenvalue problem of the linearized equation

In this section, we investigate the eigenvalue problem (3.5) in details. To do that, we generally study the linearized equation

$$(4.1)\qquad v''(t) + \mu v'(t) + \kappa v(t) + \Lambda U^2(t)v(t) = 0.$$

We set $v(t) = e^{-\mu t/2}w(t)$, then (4.1) becomes

$$(4.2)\qquad w''(t) + \left(-\frac{\mu^2}{4} + \kappa + \Lambda U^2(t)\right) w(t) = 0$$

which is a type of so called Hill's equation. The equation (4.2) also has the matrix form

$$\begin{pmatrix} w \\ w' \end{pmatrix}' = \begin{pmatrix} 0, & 1 \\ \frac{\mu^2}{4} - \kappa - \Lambda U^2(t), & 0 \end{pmatrix} \begin{pmatrix} w \\ w' \end{pmatrix} \tag{4.3}$$

To consider the original problem (3.5), we may seek the solution of (4.2) of the form $e^{\mu t/2}\tilde{w}(t)$, where $\tilde{w}$ is periodic of period $\widetilde{T} = mT$. Let $\Phi_\Lambda(t)$ be a fundamental matrix for (4.3),

$$\Phi_\Lambda(t) = \begin{pmatrix} \phi_1(t,\Lambda) & \phi_2(t,\Lambda) \\ \phi_1'(t,\Lambda) & \phi_2'(t,\Lambda) \end{pmatrix} \tag{4.4}$$

where $\{\phi_i(t,\Lambda)\}_{i=1}^2$ are given by the solutions of initial value problem to (4.2) with initial data $\Phi_\Lambda(0) = E$. By the Floquet's Theory and the fact $\Phi_\Lambda(mT) = (\Phi_\Lambda(T))^m$, we can see that the equation (4.1) has an mT-periodic solution if and only if $\Phi_\Lambda(T)$ has a characteristic root $e^{\mu T/2}\omega_m$, where ω_m is a primitive m-th root of 1, but not any i-th root for $1 \le i \le m-1$. Note that $det\Phi_\Lambda(t) = 1$ for $t \ge 0$, because the trace of the coefficient matrix of (4.3) is zero. Then, the characteristic roots of $\Phi_\Lambda(T)$ are given by the roots of characteristic equation

$$\sigma^2 - \Delta(\Lambda)\sigma + 1 = 0, \tag{4.5}$$

where $\Delta(\Lambda)$ is a trace of $\Phi_\Lambda(T)$, that is, $\Delta(\Lambda) = \phi_1(T,\Lambda) + \phi_2'(T,\Lambda)$. If $|\Delta(\Lambda)| \le 2$, then the roots of (4.5) are complex conjugates of magnitude 1. Therefore, there does not exist the root of the form $e^{\mu T/2}\omega_m$. If $\Delta(\Lambda) > 2$, then the roots of (4.5) are real and given by $e^{z(\Lambda)T}$ and $e^{-z(\Lambda)T}$ for some $z(\Lambda) > 0$. Therefore, in order for one of the roots to have the form $e^{\mu T/2}\omega_m$, $m = 1$ $(\omega_1 = 1)$, $z(\Lambda) = \mu/2$ and $\Delta(\Lambda) = e^{\mu T/2} + e^{-\mu T/2}$. Then only T-periodic solution of (4.1) exists. If $\Delta(\Lambda) < -2$, then the roots of (4.5) are real and given by $-e^{z(\Lambda)T}$ and $-e^{-z(\Lambda)T}$ for some $z(\Lambda) > 0$. In the same way as above, $m = 2$ $(\omega_2 = -1)$, $z(\Lambda) = \mu/2$ and $\Delta(\Lambda) = -(e^{\mu T/2} + e^{-\mu T/2})$ is only the case the problem (4.1) has $2T$-periodic solution, but not other periodic solutions. $z(\Lambda)$ is explicitly given by the formula

$$z(\Lambda) = \frac{1}{T}\cosh^{-1}\frac{\Delta(\Lambda)}{2} = \frac{1}{T}\log\frac{|\Delta| + \sqrt{|\Delta|^2 - 4}}{2}. \tag{4.6}$$

We also define $z(\Lambda) = 0$ for $|\Delta(\Lambda)| \le 2$. Then, $z(\Lambda)$ coincides with so called "Lyapunov exponent" of the solution of (4.1). By these consideration above, we have next lemma.

Lemma 4.1. *For the linearized equation* (4.1), *it holds the followings.*

(i) $mT(m \ge 3)$*-periodic solution does not exist.*

(ii) *T-periodic solution exists at $\Lambda = \Lambda_0$ if and only if $\Delta(\Lambda_0) = e^{\mu T/2} + e^{-\mu T/2}$.*

(iii) *$2T$-periodic solution exists at $\Lambda = \Lambda_0$ if and only if $\Delta(\Lambda_0) = -(e^{\mu T/2}+e^{-\mu T/2})$. Then, this $2T$-periodic solution is T-anti-periodic solution, i.e. $u(t) = -u(t+T)$ for $t \in R$.*

(iv) *The set of such Λ_0 as in* (ii) *and* (iii) *is discrete and countable at most.*

(v) *The solution space corresponding to* (ii) *and* (iii) *is one dimensional.*

(vi) *If $z(\Lambda) > \mu/2$* resp. $(z(\Lambda) < \frac{\mu}{2})$, *the solution of* (4.1) *grows* resp. (*decays*) *exponentially.*

REMARK 4. It is well known that if $\Lambda = \Lambda_0$ is a bifurcation point, Λ_0 must be a eigenvalue of linearized problem. Hence, (i) implies that $mT(m \geq 3)$-periodic solution does not bifurcate from $\{u_\lambda\}_{\lambda>0}$.

Proof. From the previous arguments, (i), (ii) and (iii) are clear. Since $\Delta(\Lambda)$ is a holomorphic function of λ and the characteristic roots of $\Phi_\Lambda(T)$ are distinct for $|\Delta(\Lambda)| > 2$, we can show (iv) and (v). Finally, (iv) follows from the fact that the Lyapunov exponent of the equation (4.1) is equal to $z(\Lambda) - \mu/2$. □

In the rest of this section, we further investigate the properties of $\Delta(\Lambda)$ and $z(\Lambda)$. For the Hill's equation (4.2), although the weight function U^2 is not uniformly positive, usual classical arguments such as oscillatory property of $\Delta(\Lambda)$ and expansion theory by generalized eigenfunctions for singular boundary value problem hold with proper modification (cf. Coddington-Levinson [1], Yosida [7]). For oscillatory property of $\Delta(\Lambda)$, it holds

Proposition 4.2. *There exist $\{\lambda_i\}_{i=0}^{\infty}$ and $\{\mu_i\}_{i=1}^{\infty}$ satisfying $-\infty < \lambda_0 < \mu_1 \leq \mu_2 < \lambda_1 \leq \lambda_2 < \cdots < \mu_n \leq \mu_{n+1} < \lambda_n \leq \lambda_{n+1} <\to \infty$ and the following properties*

$$
(4.7) \qquad \begin{cases} \Delta(\Lambda) < -2 & \text{for } \Lambda \in \bigcup_{i=1}^{\infty}(\mu_i, \mu_{i+1}), \\ \Delta(\Lambda) > 2 & \text{for } \Lambda \in (-\infty, \lambda_0) \cup \bigcup_{i=1}^{\infty}(\lambda_i, \lambda_{i+1}), \\ |\Delta(\Lambda)| \leq 2 & \text{for other cases} \end{cases}
$$

Now, define

$$\Sigma = \{\Lambda \in R; |\Delta(\Lambda)| \leq 2\},$$

and let L be a operator in $L^2_{U^2}(R)$ defined by

$$L = \frac{1}{U^2}\left(-\frac{d^2}{dt^2} + \left(\frac{\mu^2}{4} - \kappa\right)\right),$$

where $L^2_{U^2}$ denotes the weighted L^2-space defined by

$$L^2_{U^2}(R) = \left\{h(t); \int_R |h(s)|^2 U^2(s) ds < \infty\right\}.$$

Then, we can see that L is a self-adjoint operator in $L^2_{U^2}$, the spectrum of L coincides with Σ, and the resolvent set coincides with $R \setminus \Sigma$. In particular, if $\Lambda \notin \Sigma$, by the above argument on $\Phi_\Lambda(T)$ and $\Delta(\Lambda)$, there are two independent solution of (4.2) $w^{\pm}_\Lambda(t)$ $(t \in R)$ such that $w^{+}_\Lambda(t)$ (resp. $w^{-}_\Lambda(t)$) decays at the rate $e^{-z(\Lambda)t}$ (resp. $e^{z(\Lambda)t}$) as $t \to +\infty$ (resp. $t \to -\infty$), and $({}^t w^{\pm}_\Lambda(0),\ w^{\pm'}_\Lambda(0))$ is an eigenvector of $\Phi_\Lambda(T)$. Then, the solution of

$$(L - \Lambda I)g = f \quad in \quad L^2_{U^2} \tag{4.8}$$

which is equivalent to

$$-\frac{d^2 g}{dt^2} + \left(\frac{\mu^2}{4} - \kappa - \Lambda U^2\right) g = U^2 f \tag{4.9}$$

is concretely constructed by the Green function in the form

$$g(t) = \int_R G_\Lambda(t,s) U^2(s) f(s) ds, \tag{4.10}$$

where

$$G_\Lambda(t,s) = G_\Lambda(s,t) = \frac{w^{+}_\Lambda(t) w^{-}_\Lambda(s)}{[w^{+}_\Lambda, w^{-}_\Lambda]} \quad ; \quad t \geq s,$$

and $[w^{+}_\Lambda, w^{-}_\Lambda]$ is the Wronskian.

Now, we are ready to state the key lemma in this paper.

Lemma 4.3. *For $\Lambda \notin \Sigma$, $dz/d\Lambda$ can be represented in the form*

$$\frac{dz}{d\Lambda} = -\frac{1}{T}\int_0^T G_\Lambda(\tau,\tau) U^2(\tau) d\tau. \tag{4.11}$$

REMARK 5. This formula was first given by Johnson-Moser [3]. They analyzed the corresponding formula in the case of the Schrödinger operator $L = -d^2/dt^2 + q(t)$ for almost periodic $q(t)$.

Proof. We first consider the left hand side of (4.11). Because

$$\frac{dz}{d\Lambda} = \frac{1}{T(e^{Tz(\Lambda)} - e^{-Tz(\Lambda)})} \frac{d\Delta}{d\Lambda}, \tag{4.12}$$

we may consider $d\Delta/d\Lambda (= \partial\phi_1/\partial\Lambda + \partial\phi_2'/\partial\Lambda)$. Substituting ϕ_1 to (4.2) and differentiating with respect to Λ, we obtain

$$\begin{cases} -\dfrac{\partial\phi_1''}{\partial\Lambda} + \left(\kappa - \dfrac{\mu^2}{4} - \Lambda U^2\right) \dfrac{\partial\phi_1}{\partial\Lambda} = U^2\phi_1, \\ \dfrac{\partial\phi_1}{\partial\Lambda}(0) = 0, \ \dfrac{\partial\phi_1'}{\partial\Lambda}(0) = 0. \end{cases} \tag{4.13}$$

Hence, from the variation-of-constants formula, $\partial\phi_1/\partial\Lambda(T)$ is given by

$$\frac{\partial\phi_1}{\partial\Lambda}(T) = \int_0^T \{\phi_1(T)\phi_2(s) - \phi_2(T)\phi_1(s)\} U^2(s)\phi_1(s)ds, \tag{4.14}$$

and in the same way, it holds

$$\frac{\partial\phi_2'}{\partial\Lambda}(T) = \int_0^T \{\phi_1'(T)\phi_2(s) - \phi_2'(T)\phi_1(s)\} U^2(s)\phi_2(s)ds. \tag{4.15}$$

Thus we have

$$\begin{aligned} \frac{dz}{d\Lambda} = \frac{1}{T(e^{Tz(\Lambda)} - e^{-Tz(\Lambda)})} \times \int_0^T \{&-\phi_2(T)\phi_1^2(s) \\ &+ (\phi_1(T) - \phi_2'(s))\phi_1(s)\phi_2(s) + \phi_1'(T)\phi_2^2(s)\} U^2(s)ds. \end{aligned} \tag{4.16}$$

Next we consider the right hand side of (4.11). If $w_\Lambda^\pm(0) \neq 0$, we can normalize $w_\Lambda^\pm$ so that $w_\Lambda^\pm(0) = 1$, and we can represent $w_\Lambda^\pm$ in terms of $\{\phi_i\}_{i=1}^2$ as

$$w_\Lambda^\pm(t) = \phi_1(t, \Lambda) + c^\pm(\Lambda)\phi_2(t, \Lambda) \tag{4.17}$$

for some constants $c^\pm(\Lambda)$. Recalling the fact that ${}^t(w_\Lambda^\pm(0), w_\Lambda^{\pm'}(0))$ is the eigen vector of $\Phi_\Lambda(T)$, we have $w_\Lambda^\pm(T) = e^{\mp Tz(\Lambda)}$ and the coefficient $c^\pm(\Lambda)$ is given by

$$c^\pm(\Lambda) = \frac{e^{\mp Tz(\Lambda)} - \phi_1(T, \Lambda)}{\phi_2(T, \Lambda)}. \tag{4.18}$$

Substituting the relations (4.17) and (4.18) into (4.16), we can show the right hand side of (4.11) and (4.16) coincides each other. In the case $w_\Lambda^+(0) = 0$, we normalize $w_\Lambda^\pm$ so that $w_\Lambda^{+'}(0) = 1$, $w_\Lambda^-(0) = 1$. Then we have $w_\Lambda^+(t) = \phi_2(t, \Lambda)$ and $w_\Lambda^-(t) = \phi_1(t, \Lambda)$. And it holds that $\phi_1(T, \Lambda) = e^{Tz}$, $\phi_1'(T, \Lambda) = 0$, $\phi_2(T, \Lambda) = 0$ and

$\phi_2'(T, \Lambda) = e^{-Tz}$. Therefore we also have the equality (4.11). The case $w_\Lambda^-(0) = 0$ can be similarly treated. Thus the proof is completed. □

According to the expansion theory by generalized eigen-functions established by Weyl, Stone, Titschmarsh and Kodaira, $G_\Lambda(s,t)$ has the following representation;

(4.19) $$G_\Lambda(s,t) = \int_\Sigma \frac{\Sigma_{1\le i,j\le 2}\phi_i(s,\xi)\phi_j(t,\xi)\sigma_{ij}(d\xi)}{\xi - \Lambda},$$

where $\{\sigma_{ij}\}$ is a matrix valued Stieltjes measure which is nonnegative definite. Substituting this to (4.11), we have

Lemma 4.4. *For any $\Lambda \notin \Sigma$, $dz/d\Lambda$ also has a representation*

(4.20) $$\frac{dz}{d\Lambda} = -\int_\Sigma \frac{\sigma(d\xi)}{\xi - \Lambda},$$

where $\sigma(d\xi)$ is a nonnegative Stieltjes measure satisfying $\int_\Sigma 1/(1+|\xi|)\sigma(d\xi) < \infty$.

By this lemma, we have

(4.21) $$\frac{d^2z}{d\Lambda^2} = -\int_\Sigma \frac{\sigma(d\xi)}{(\xi - \Lambda)^2} < 0$$

for $\Lambda \notin \Sigma$, that is, $z(\Lambda)$ is a convex function on $R \setminus \Sigma$. Finally, we give a nice criterion for the condition (3.6).

Lemma 4.5. *For any eigenvalues $\Lambda = \Lambda_0$ of (4.1), it holds that*

(4.22) $$\frac{dz}{d\Lambda}(\Lambda_0) \neq 0 \Longleftrightarrow \int_0^T v_0(t)v_0^*(t)U^2(t)dt \neq 0,$$

where v_0 and v_0^* are as in Lemma 3.2.

Proof. Put $v_0(t) = e^{-\mu t/2}w_0(t)$, then $w_0(t)$ satisfies (4.2). So, $v_0(t)$ is equal to $e^{-\mu t/2}w_{\Lambda_0}^-(t)$ except for constant factor. In the same way, $v_0^*(t)$ is equal to $e^{\mu t/2}w_{\Lambda_0}^+(t)$ up to constant factor. Therefore, we have

(4.23) $$\int_0^{\widetilde{T}} v_0(t)v_0^*(t)U^2(t)dt \neq 0 \Longleftrightarrow \int_0^{\widetilde{T}} w_{\Lambda_0}^+(t)w_{\Lambda_0}^-(t)U^2(t)dt \neq 0.$$

Hence, noting v_0 and v_0^* are T-anti-periodic function for $m = 2$, Lemma 4.3 implies Proposition 4.5. □

5. Nonlinear Problem and Proof of Theorem

We turn to the nonlinear equation (3.2). By the last two lemmas and Theorem 3.1, we obtain the following basic properties of the bifurcation points of (3.2).

Theorem 5.1. *On each interval I of $R \setminus \Sigma$, say $I = (\lambda_{i_0}, \lambda_{i_0+1})$* (resp. $I = (\mu_{i_0}, \mu_{i_0+1})$), *if $\mu/2 < \max_{\Lambda \in I} z(\Lambda)$, there exist exactly two bifurcation points of nonlinear problem* (3.2) *with $m = 1$* (resp. $m = 2$). *The bifurcating solution of* (3.2) *is T-periodic* (resp. *$2T$-periodic*). *Furthermore, at each of two eigenvalues does alternate the asymptotic stability of the trivial solution $v = 0$.*

Proof. If $\mu/2 < \max_{\Lambda \in I} z(\Lambda)$, the convexity of $z(\Lambda)$ implies that the graph of $z(\Lambda)$ transversally intersects the line $z = \mu/2$ at exactly two points on I. The all hypotheses of Lemma 3.2 holds for $m = 1$ (resp. $m = 2$). Hence, these two points are bifurcation points of the solution with period T (resp. $2T$). For $m = 2$, note that the bifurcating solution is really $2T$-periodic, but not T-periodic. In fact, since the eigenfunction v_0 of (4.1) with $m = 2$ and $\Delta(\Lambda_0) < -2$, is T-anti-periodic, so the bifurcating solution of the form $\epsilon v_0 + \epsilon\psi(\epsilon)$ is $2T$-periodic when δ is small enough. □

In order to prove the main Theorem, Theorem 5.1 suggests that all we need is to study the asymptotic properties of $\Delta(\Lambda)$ as $\Lambda \to \infty$. In fact, if we can prove

$$\limsup_{\Lambda \to \infty} \Delta(\Lambda) > 2 \tag{5.1}$$

$$\left(\text{resp. } \liminf_{\Lambda \to \infty} \Delta(\Lambda) < -2\right) \tag{5.2}$$

then the Theorem implies there exist countably many bifurcation points of T-periodic solution (resp. $2T$-periodic solution) of (3.2) provided

$$\frac{\mu}{2} < \frac{1}{T} \log\left(\frac{|\tilde{\Delta}| + \sqrt{|\tilde{\Delta}|^2 - 4}}{2}\right) \tag{5.3}$$

where $\tilde{\Delta}$ is the left hand side of (5.1) (resp. (5.2)). For the asymptotic properties of $\Delta(\Lambda)$, we admit the following Proposition for the moment. The proof will be given in the next section.

Proposition 5.2. *Suppose $U(t)$ satisfies the hypotheses of Theorem* 2.1. *Then it holds the followings.*

(1) *The case $N = 1$:*

$$\Delta(\Lambda) = \frac{2\cos(S_1\sqrt{\Lambda})}{\sin \nu\pi}(1 + o(1)) \quad as \quad \Lambda \to \infty \tag{5.4}$$

(2) *The case* $N = 2$:

$$\Delta(\Lambda) = \frac{2\{\cos((S_1 + S_2)\sqrt{\Lambda}) + \cos((S_1 - S_2)\sqrt{\Lambda})\cos^2 \nu\pi\}}{\sin^2 \nu\pi} \times (1 + o(1)) \quad as \quad \Lambda \to \infty \tag{5.5}$$

(3) *The case* $N \geq 3$:

$$\limsup_{\Lambda\to\infty} \Delta(\Lambda) \geq \frac{1}{\sin^N \nu\pi}\{(1 + \cos\nu\pi)^N + (1 - \cos\nu\pi)^N\} \tag{5.6}$$

and if $\{S_i\}_{i=1}^N$ *are rationally independent, then*

$$\liminf_{\Lambda\to\infty} \Delta(\Lambda) \leq \frac{-1}{\sin^N \nu\pi}\{(1 + \cos\nu\pi)^N + (1 - \cos\nu\pi)^N\} \tag{5.7}$$

In the case $N = 1$, (5.4) implies

$$\begin{cases} \limsup_{\Lambda\to\infty} \Delta(\Lambda) = \dfrac{2}{\sin\nu\pi} \\ \liminf_{\Lambda\to\infty} \Delta(\Lambda) = -\dfrac{2}{\sin\nu\pi} \end{cases} \tag{5.8}$$

Since $|\widetilde{\Delta}|$ equals to $2/\sin\nu\pi$ for both cases, if $\mu/2 < 1/T\log(\cot(\nu\pi/2))$, there exsit countably many bifurcating points of not only T-periodic solution, but also $2T$-periodic solution. Furthermore, the sequence of bifurcating points $\{\lambda_i\}_{i=0}^\infty$ have the property for $\lambda > \exists\lambda^*$,

$$\lambda^* < \overbrace{\lambda_i < \lambda_{i+1}}^{T-periodic} < \underbrace{\lambda_{i+2} < \lambda_{i+3}}_{2T-periodic} < \overbrace{\lambda_{i+4} < \lambda_{i+5}}^{T-periodic} < \underbrace{\lambda_{i+6} < \lambda_{i+7}}_{2T-periodic} \cdots \to \infty$$

In the case $N = 2$, (5.5) implies

$$\limsup_{\Lambda\to\infty} \Delta(\Lambda) = \frac{2(1 + \cos^2 \nu\pi)}{\sin^2 \nu\pi} \tag{5.9}$$

$$\liminf_{\Lambda\to\infty} \Delta(\Lambda) = \begin{cases} -\dfrac{2(1 + \cos^2 \nu\pi)}{\sin^2 \nu\pi}, & \dfrac{S_1}{S_2} \neq \dfrac{2p+1}{2q+1}, \ p, q \in N, \\ -2, & S_1 = S_2 \\ \inf_\lambda \dfrac{2(\cos(S_1 + S_2)\lambda + \cos(S_1 - S_2)\lambda\cos^2 \nu\pi)}{\sin^2 \nu\pi} & \text{other cases} \end{cases}$$

Therefore, if $\mu/2 < 2/T \log(\cot(\nu\pi/2))$, there exists countably many bifurcating points of T-periodic solution. And if $S_1/S_2 \neq (2p+1)/(2q+1)$ $(p, q \in N)$, there also exists countably many period-doubling bifurcation points. But, if $S_1 = S_2$, then the set of period-doubling bifurcation points is finite. In other cases, under the weak condition (2.3), we can show that there exists countably many period-doubling bifurcation points. In the case $N \geq 3$, (5.6) and (5.7) imply that

$$|\widetilde{\Delta}| \geq \frac{1}{\sin^N \nu\pi}\{(1+\cos\nu\pi)^N + (1-\cos\nu\pi)^N\}. \tag{5.10}$$

Hence, if $\mu/2 < N/T \log(\cot(\nu\pi/2))$, there exists countably many bifurcating points of T-periodic solution. And if $\{S_i\}_{i=1}^N$ are rationally independent, then there also exists countably many period-doubling bifurcation points. Thus, main Theorem can be proved.

6. Proof of Proposition

Before the proof of the Proposition, we introduce some notations. Let $R_\Lambda[t,s]$ be a 2-by-2 matrix defined by $\Phi_\Lambda(t)\Phi_\Lambda^{-1}(s)$. And let's denote $U^2(t)$ by $\rho(t)$. Then we may assume that zero points of $\rho(t)$ are $0 = t_0 < t_1 < \cdots < t_N = T$, without loss of generality. We would like to investigate the asymptotic behavior of $\Delta(\Lambda)$, making use of the order at zero points of $\rho(t)$. In the case $N = 1$, $\rho(t)$ has two zero points on $[0,T]$. From (2.1), there exist $\beta, \widehat{\beta} \geq 1$ such that

$$\begin{aligned} &\rho(t) = C_1 t^n(1 + C_2 t^\beta + O(t^{2\beta})) \quad as \quad t \to 0, \\ &\rho(t) = \widehat{C_1}(T-t)^n(1 + \widehat{C_2}(T-t)^{\widehat{\beta}} + O((T-t)^{2\widehat{\beta}})) \quad as \quad t \to T \end{aligned} \tag{6.1}$$

In order to decrease zero points of $\rho(t)$ on [0,T], we separate the interval $[0,T]$ by $T/2$. We define $\widehat{\rho}(t) = \rho(T-t)$, and the fundamental matrix for

$$w''(t) - \frac{\mu^2}{4}w(t) + \kappa w(t) + \Lambda\widehat{\rho}(t)w(t) = 0 \tag{6.2}$$

by $\widehat{\Phi}_\Lambda(t) = \begin{pmatrix} \widehat{\phi}_1(t,\Lambda) & \widehat{\phi}_2(t,\Lambda) \\ \widehat{\phi}_1'(t,\Lambda) & \widehat{\phi}_2'(t,\Lambda) \end{pmatrix}$, with initial data $\widehat{\Phi}_\Lambda(0) = E$.

Then, making use of $\widehat{\Phi}_\Lambda(t)$, we have

$$\begin{aligned} \Phi_\Lambda(T) &= R_\Lambda\left[\frac{T}{2}, T\right]^{-1} R_\Lambda\left[\frac{T}{2}, 0\right]. \\ &= \begin{pmatrix} \widehat{\phi}_1(T/2,\Lambda) & -\widehat{\phi}_2(T/2,\Lambda) \\ -\widehat{\phi}_1'(T/2,\Lambda) & \widehat{\phi}_2'(T/2,\Lambda) \end{pmatrix}^{-1} \begin{pmatrix} \phi_1(T/2,\Lambda) & \phi_2(T/2,\Lambda) \\ \phi_1'(T/2,\Lambda) & \phi_2'(T/2,\Lambda) \end{pmatrix} \\ &= \begin{pmatrix} \widehat{\phi}_2'(T/2,\Lambda) & \widehat{\phi}_2(T/2,\Lambda) \\ \widehat{\phi}_1'(T/2,\Lambda) & \widehat{\phi}_1(T/2,\Lambda) \end{pmatrix} \begin{pmatrix} \phi_1(T/2,\Lambda) & \phi_2(T/2,\Lambda) \\ \phi_1'(T/2,\Lambda) & \phi_2'(T/2,\Lambda) \end{pmatrix} \end{aligned} \tag{6.3}$$

which implies

$$(6.4)\qquad \Delta(\Lambda) = \phi_1\left(\frac{T}{2}\right)\widehat{\phi_2}'\left(\frac{T}{2}\right) + \phi_1'\left(\frac{T}{2}\right)\widehat{\phi_2}\left(\frac{T}{2}\right) + \phi_2\left(\frac{T}{2}\right)\widehat{\phi_1}'\left(\frac{T}{2}\right) + \phi_2'\left(\frac{T}{2}\right)\widehat{\phi_1}\left(\frac{T}{2}\right).$$

We may consider $\{\phi_i(T/2,\Lambda)\}_{i=1,2}$, since similar arguments hold for $\{\widehat{\phi}_i(T/2,\Lambda)\}_{i=1,2}$. On the interval $[0,T/2]$, we introduce the following change of variable and function, so called Liouville transformation:

$$(6.5)\qquad \textit{variable}: \quad x = \int_0^t \sqrt{\rho(s)}ds,$$

$$(6.6)\qquad \textit{function}: \quad g(x) = \rho(t)^{1/4}w(t).$$

By this transformation, (4.2) is reduced to

$$(6.7)\qquad g''(x) + (\Lambda - Q(x))g(x) = 0,$$

where $Q(x) = (\mu^2/4 - \kappa)\rho^{-1}(t) - \rho^{-3/4}(t)(\rho^{-1/4}(t))''$. From (6.1), it holds that

$$(6.8)\qquad Q(x) = -\frac{n(n+4)}{16}C_1^{-1}t^{-(n+2)} \times\left\{1 - \frac{(n-\beta)^2 + 3\beta^2 + 4(n-\beta)}{n(n+4)}C_2t^{\beta} + O(t^{2\beta})\right\}$$

as $t \to 0$. According to (6.5), we have the relation t and x

$$(6.9)\qquad t = C_1^{-\frac{1}{n+2}}\left(\frac{2}{n+2}\right)^{-\frac{2}{n+2}}x^{\frac{2}{n+2}} \times\left\{1 - \frac{C_2}{n+2\beta+2}C_1^{-\frac{\beta}{n+2}}\left(\frac{2}{n+2}\right)^{-\frac{2\beta}{n+2}}x^{\frac{2\beta}{n+2}} + O(x^{\frac{4\beta}{n+2}})\right\}$$

as $x \to 0$. Combining (6.8) with (6.9), we have the behavior of $Q(x)$ near $x = 0$,

$$(6.10)\qquad Q(x) = Q_0(x) \times\left\{1 - \frac{8\beta(\beta^2-1)C_2}{n(n+4)(n+2\beta+2)}C_1^{-\nu\beta}(2\nu)^{-2\nu\beta}x^{2\nu\beta} + O(x^{4\nu\beta})\right\},$$

as $x \to 0$, where $Q_0(x) = -n(n+4)\nu^2/(4x^2)$ and $\nu = 1/(n+2)$.

Let's set $\Phi_i(x) = \rho^{1/4}(t)\phi_i(t)$ $(i = 1,2)$, then $\{\Phi_i(x)\}_{i=1,2}$ satisfy (6.7). Especially, if $Q(x) = Q_0(x)$, the solutions of (6.7) are explicitly given by $A_n\sqrt{x}J_{-\nu}(\sqrt{\Lambda}x)$

and $B_n\sqrt{x}J_\nu(\sqrt{\Lambda}x)$, where J_ν is a ν-th Bessel function and A_n, B_n are determined, so that $\{\phi_i(t)\}_{i=1,2}$ satisfy the initial condition $\Phi_\Lambda(0)=E$ by the form,

$$A_n = \frac{1}{\sqrt{2}}\Gamma(1-\nu)(n+2)^{n\nu/2}C_1^{\nu/2},$$
$$B_n = \frac{1}{\sqrt{2}}\Gamma(1+\nu)(n+2)^{-n\nu/2}C_1^{-\nu/2}. \tag{6.11}$$

Making use of these solutions, we should note that $\{\Phi_i(x)\}_{i=1,2}$ also satisfy the following integral equation:

$$\begin{cases} \Phi_1(x) = \Lambda^{-\frac{1-2\nu}{4}}A(\sqrt{\Lambda}x) \\ \qquad + \dfrac{1}{\sqrt{\Lambda}}\displaystyle\int_0^x (A(\sqrt{\Lambda}s)B(\sqrt{\Lambda}x) - (A(\sqrt{\Lambda}x)B(\sqrt{\Lambda}s))\widetilde{Q}(s)\Phi_1(s)ds, \\ \Phi_2(x) = \Lambda^{-\frac{1+2\nu}{4}}B(\sqrt{\Lambda}x) \\ \qquad + \dfrac{1}{\sqrt{\Lambda}}\displaystyle\int_0^x (A(\sqrt{\Lambda}s)B(\sqrt{\Lambda}x) - (A(\sqrt{\Lambda}x)B(\sqrt{\Lambda}s))\widetilde{Q}(s)\Phi_2(s)ds. \end{cases} \tag{6.12}$$

Here $\widetilde{Q}(x)=Q(x)-Q_0(x)$, $A(y)=A_n\sqrt{y}J_{-\nu}(y)$ and $B(y)=B_n\sqrt{y}J_\nu(y)$. Taking notice that

$$A_nB_n\frac{2}{\pi} = \frac{1}{\sin\nu\pi} \tag{6.13}$$

and $A(y)$, $B(y)$ have the asymptotic properties

$$A(y) = A_n\sqrt{\frac{2}{\pi}}\cos\left(y-\frac{1-2\nu}{4}\pi\right)(1+o(1)), \quad y\to\infty,$$
$$B(y) = B_n\sqrt{\frac{2}{\pi}}\cos\left(y-\frac{1+2\nu}{4}\pi\right)(1+o(1)), \quad y\to\infty, \tag{6.14}$$

the following lemma holds.

Lemma 6.1. *$\Phi_1(x)$ satisfies that*

$$|\Phi_1(x) - \Lambda^{-\frac{1-2\nu}{4}}A(\sqrt{\Lambda}x)| = o(\Lambda^{-\frac{1-2\nu}{4}}) \quad \Lambda\to\infty, \tag{6.15}$$

for any fixed x. $\Phi_2(x)$ satisfies that

$$|\Phi_2(x) - \Lambda^{-\frac{1+2\nu}{4}}B(\sqrt{\Lambda}x)| = o(\Lambda^{-\frac{1+2\nu}{4}}) \quad \Lambda\to\infty, \tag{6.16}$$

for any fixed x.

Proof. Consider the function $\Phi_1(x)$, since similar arguments hold for $\Phi_2(x)$. Let's set $y = \sqrt{\Lambda}x$ and define the successive approximations $\{\Phi_1^{(m)}(y)\}_{m=0}^{\infty}$ by

$$(6.17)\quad \begin{cases} \Phi_1^{(0)}(y) = \Lambda^{-\frac{1-2\nu}{4}} A(y), \\ \Phi_1^{(m)}(y) = \dfrac{1}{\Lambda}\displaystyle\int_0^y (A(z)B(y) \\ \qquad -A(y)B(z))\, \widetilde{Q}\left(\dfrac{z}{\sqrt{\Lambda}}\right) \Phi_1^{(m-1)}(z)dz, \quad \text{for } m \geq 1. \end{cases}$$

First, we deal with the case $\beta > 1$. From (6.17) with $m = 1$, it follows that

$$(6.18)\quad \begin{aligned} |\Phi_1^{(1)}(y)| &= \frac{1}{\Lambda}\left|\int_0^y (A(z)B(y) - A(y)B(z))\widetilde{Q}\left(\frac{z}{\sqrt{\Lambda}}\right) \Lambda^{-\frac{1-2\nu}{4}} A(z)dz\right| \\ &\leq \Lambda^{-\frac{1-2\nu}{4}}\Lambda^{-\nu\beta} \\ &\quad \times \left|\int_0^y (A(z)B(y) + A(y)B(z))\widetilde{C_Q} z^{-2+2\nu\beta} A(z)dz\right|, \end{aligned}$$

where $\widetilde{C_Q}$ is some positive constant. From the definition of $A(z)$ and $B(z)$, it holds for any $0 < y \leq 1$ that

$$(6.19)\quad \begin{aligned} |\Phi_1^{(1)}(y)| &\leq \Lambda^{-\frac{1-2\nu}{4}}\Lambda^{-\nu\beta}\widetilde{C_Q} C_J \frac{1}{2\nu} \\ &\quad \times \left|\int_0^y \left(z^{\frac{1}{2}-\nu}y^{\frac{1}{2}+\nu} + y^{\frac{1}{2}-\nu}z^{\frac{1}{2}+\nu}\right) z^{-2+2\nu\beta} z^{\frac{1}{2}-\nu} dz\right| \\ &\leq \Lambda^{-\frac{1-2\nu}{4}}\Lambda^{-\nu\beta}\widetilde{C_Q} C_J \frac{1}{2\nu}\frac{2\beta - 1}{2\nu\beta(\beta-1)} y^{\frac{1}{2}-\nu+2\nu\beta} \\ &< \Lambda^{-\frac{1-2\nu}{4}}\Lambda^{-\nu\beta} C_J \frac{\widetilde{C_Q}}{2\nu^2(\beta-1)} y^{\frac{1}{2}-\nu+2\nu\beta}, \end{aligned}$$

where C_J is a positive constant. In the same way, an induction implies that

$$(6.20)\quad |\Phi_1^{(m)}(y)| \leq \Lambda^{-\frac{1-2\nu}{4}}\Lambda^{-m\nu\beta} C_J \left(\frac{\widetilde{C_Q}}{2\nu^2(\beta-1)}\right)^m \frac{1}{m!} y^{\frac{1}{2}-\nu+2m\nu\beta}$$

for any $0 < y \leq 1$. On the other hand, for large y (> 1), according to the asymptotic property (6.14), we have

$$(6.21)\quad \begin{aligned} |\Phi_1^{(1)}(y)| < \Lambda^{-\frac{1-2\nu}{4}}\Lambda^{-\nu\beta} \Bigg\{ & C_J \frac{\widetilde{C_Q}}{2\nu^2(\beta-1)} \\ & + \left|\int_1^y (A(z)B(y) + A(y)B(z))\widetilde{C_Q} z^{-2+2\nu\beta} A(z)dz\right| \Bigg\} \end{aligned}$$

$$
= \begin{cases} \Lambda^{-\frac{1-2\nu}{4}} \Lambda^{-\nu\beta} \\ \quad \times \left\{ C_J \dfrac{\widetilde{C_Q}}{2\nu^2(\beta-1)} + C_A \widetilde{C_Q} \left| \dfrac{1}{-1+2\nu\beta}(y^{-1+2\nu\beta}-1) \right| \right\} & \left(\nu\beta \neq \dfrac{1}{2}\right) \\ \Lambda^{-\frac{1-2\nu}{4}} \Lambda^{-\nu\beta} \\ \quad \times \left\{ C_J \dfrac{\widetilde{C_Q}}{2\nu^2(\beta-1)} + C_A \widetilde{C_Q} \log y \right\} & \left(\nu\beta = \dfrac{1}{2}\right) \end{cases}
$$

Hence, there exist some constants C_0 and C such that the following inequality holds for any δ $(0 < \delta < 1)$.

$$
(6.22) \qquad |\Phi_1^{(1)}(y)| < \begin{cases} \Lambda^{-\frac{1-2\nu}{4}} \Lambda^{-\nu\beta} C_0 C & \left(\nu\beta < \dfrac{1}{2}\right) \\ \Lambda^{-\frac{1-2\nu}{4}} \Lambda^{-\nu\beta} C_0 C y^{\delta} & \left(\nu\beta = \dfrac{1}{2}\right) \\ \Lambda^{-\frac{1-2\nu}{4}} \Lambda^{-\nu\beta} C_0 C y^{-1+2\nu\beta} & \left(\nu\beta > \dfrac{1}{2}\right) \end{cases}
$$

In the same way, from an induction, we have

$$
(6.23) \qquad |\Phi_1^{(m)}(y)| < \begin{cases} \Lambda^{-\frac{1-2\nu}{4}} \Lambda^{-m\nu\beta} C_0 C^m & \left(\nu\beta < \dfrac{1}{2}\right) \\ \Lambda^{-\frac{1-2\nu}{4}} \Lambda^{\frac{-m}{2}} C_0 \delta \left(\dfrac{C}{\delta}\right)^m y^{\delta} & \left(\nu\beta = \dfrac{1}{2}\right) \\ \Lambda^{-\frac{1-2\nu}{4}} \Lambda^{-m\nu\beta} C_0 \dfrac{C^m}{m!} y^{m(-1+2\nu\beta)} & \left(\nu\beta > \dfrac{1}{2}\right) \end{cases}
$$

Therefore, the series $\sum_{m=0}^{\infty} \Phi_1^{(m)}(y)$ is absolutely and uniformly convergent on any compact interval in $[0, \infty)$. Hence, we can obtain the following estimate.

$$
(6.24) \qquad \begin{aligned} &|\Phi_1(x) - \Lambda^{-\frac{1-2\nu}{4}} A(\sqrt{\Lambda}x)| \\ &\leq \sum_{m=1}^{\infty} |\Phi_1^{(m)}(\sqrt{\Lambda}x)| \\ &< \begin{cases} \Lambda^{-\frac{1-2\nu}{4}} \Lambda^{-\nu\beta} 2C_0 C & \left(\nu\beta < \dfrac{1}{2}\right) \\ \Lambda^{-\frac{1-2\nu}{4}} C_0 C \Lambda^{\frac{1}{2}(\delta-1)} x & \left(\nu\beta = \dfrac{1}{2}\right) \\ \Lambda^{-\frac{1-2\nu}{4}} C_0 (\exp(C\Lambda^{-\frac{1}{2}} x^{-1+2\nu\beta}) - 1) & \left(\nu\beta > \dfrac{1}{2}\right) \end{cases} \end{aligned}
$$

Thus, we can show that

$$
(6.25) \qquad |\Phi_1(x) - \Lambda^{-\frac{1-2\nu}{4}} A(\sqrt{\Lambda}x)| = o(\Lambda^{-\frac{1-2\nu}{4}}) \quad \Lambda \to \infty,
$$

for any fixed x. Next we consider the case $\beta = 1$. From (6.10), the coefficient of $x^{-2+2\nu}$ of $\widetilde{Q}(x)$ is equal to 0. Therefore $\widetilde{Q}(x) = 0(x^{-2+4\nu})$ as $x \to 0$. It means that the proof of $\beta = 2$ is consistent with one of the case $\beta = 1$. Thus the proof is completed. □

Now, we return to the proof of Proposition 5.2. From Lemma 7.1 and (6.14), we have

$$
\begin{aligned}
\phi_1\left(\frac{T}{2}\right) &= \Lambda^{-\frac{1-2\nu}{4}} \rho\left(\frac{T}{2}\right)^{-\frac{1}{4}} A_n \sqrt{\frac{2}{\pi}} \\
&\quad \times \cos\left(\int_0^{\frac{T}{2}} \sqrt{\rho(y)}dy\sqrt{\Lambda} - \frac{1-2\nu}{4}\pi\right)(1+o(1)) \\
\phi_2\left(\frac{T}{2}\right) &= \Lambda^{-\frac{1+2\nu}{4}} \rho\left(\frac{T}{2}\right)^{-\frac{1}{4}} B_n \sqrt{\frac{2}{\pi}} \\
&\quad \times \cos\left(\int_0^{\frac{T}{2}} \sqrt{\rho(y)}dy\sqrt{\Lambda} - \frac{1+2\nu}{4}\pi\right)(1+o(1))
\end{aligned} \tag{6.26}
$$

as $\Lambda \to \infty$. In the same way,

$$
\begin{aligned}
\widehat{\phi_1}\left(\frac{T}{2}\right) &= \Lambda^{-\frac{1-2\nu}{4}} \rho\left(\frac{T}{2}\right)^{-\frac{1}{4}} A_n \sqrt{\frac{2}{\pi}} \\
&\quad \times \cos\left(\int_{\frac{T}{2}}^{T} \sqrt{\rho(y)}dy\sqrt{\Lambda} - \frac{1-2\nu}{4}\pi\right)(1+o(1)) \\
\widehat{\phi_2}\left(\frac{T}{2}\right) &= \Lambda^{-\frac{1+2\nu}{4}} \rho\left(\frac{T}{2}\right)^{-\frac{1}{4}} B_n \sqrt{\frac{2}{\pi}} \\
&\quad \times \cos\left(\int_{\frac{T}{2}}^{T} \sqrt{\rho(y)}dy\sqrt{\Lambda} - \frac{1+2\nu}{4}\pi\right)(1+o(1))
\end{aligned} \tag{6.27}
$$

as $\Lambda \to \infty$. According to (6.4), we have

$$
\begin{aligned}
\Delta(\Lambda) &= A_n B_n \frac{4}{\pi}\cos(S_1\sqrt{\Lambda})(1+o(1)) \\
&= \frac{2\cos(S_1\sqrt{\Lambda})}{\sin \nu\pi}(1+o(1))
\end{aligned} \tag{6.28}
$$

as $\Lambda \to \infty$. Thus, we can prove the case $N = 1$. Next we consider the case $N = 2$.

For $N = 2$, it holds that

$$
(6.29)\quad \begin{aligned}\Phi_\Lambda(T) &= R_\Lambda[T, t_1]R_\Lambda[t_1, 0] \\ &= R_\Lambda\left[\frac{t_1+T}{2}, T\right]^{-1} R_\Lambda\left[\frac{t_1+T}{2}, t_1\right] R_\Lambda\left[\frac{t_1}{2}, t_1\right]^{-1} R_\Lambda\left[\frac{t_1}{2}, 0\right].\end{aligned}
$$

As in the proof of the case $N = 1$, we have

$$
\begin{aligned}(6.30)\quad & R_\Lambda[t_i, t_{i-1}] \\ &= R_\Lambda\left[\frac{t_{i-1}+t_i}{2}, t_i\right]^{-1} R_\Lambda\left[\frac{t_{i-1}+t_i}{2}, t_{i-1}\right] \\ &= \begin{pmatrix} A_nB_n\frac{2}{\pi}\cos(S_i\sqrt{\Lambda}), & \Lambda^{-\nu}B_nB_n\frac{2}{\pi}\cos(S_i\sqrt{\Lambda}-\nu\pi) \\ \Lambda^{\nu}A_nA_n\frac{2}{\pi}\cos(S_i\sqrt{\Lambda}+\nu\pi), & A_nB_n\frac{2}{\pi}\cos(S_i\sqrt{\Lambda}) \end{pmatrix}(1+o(1))\end{aligned}
$$

as $\Lambda \to \infty$, for $i = 1, 2$ $(0 = t_0 < t_1 < t_2 = T)$. Calculating from (6.29) and (6.30), we have

$$
(6.31)\quad \Delta(\Lambda) = \frac{2\{\cos((S_1+S_2)\sqrt{\Lambda}) + \cos((S_1-S_2)\sqrt{\Lambda})\cos^2\nu\pi\}}{\sin^2\nu\pi}(1+o(1)) \quad \text{as} \quad \Lambda \to \infty.
$$

Finally we consider the case $N \geq 3$. Note that for any $\{S_i\}_{i=1}^N$ there exist a sequence $\{\Lambda_j\}_{j=1}^\infty \nearrow \infty$ such that

$$
(6.32)\qquad \lim_{j\to\infty}\cos(S_i\sqrt{\Lambda_j}) = 1 \quad \text{for any } 1 \leq i \leq N.
$$

In fact, such a sequence can be taken as follows. Reordering the indicies, $\{S_{1,1}, S_{2,1}, \cdots S_{r,1}\}$ $(1 \leq r \leq N)$ denote a maximal subset of $\{S_i\}_{i=1}^N$ whose components are rationally independent. For each $1 \leq k \leq r$, $\{S_{k,1}, S_{k,2}, \cdots, S_{k,n_k}\}$ denote a subset of $\{S_i\}_{i=1}^N$ whose components are rationally dependent for $S_{k,1}$. Since $\{(e^{i2\pi S_{1,1}m}, e^{i2\pi S_{2,1}m}, \cdots, e^{i2\pi S_{r,1}m}); m \in N\}$ is dense in $\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{r}$, there exist a sequence $\{\widetilde{\Lambda}_j\}_{j=1}^\infty \nearrow \infty$ such that

$$
(6.33)\qquad \lim_{j\to\infty}\cos\left(S_i\sqrt{\widetilde{\Lambda}_j}\right) = 1 \quad \text{for any } 1 \leq i \leq r.
$$

However, since $S_{k,l}$ is rationally dependent for $S_{k,1}$, there exist $p_{k,l}$, $q_{k,l} \in N$ such that $S_{k,l}/S_{k,1} = q_{k,l}/p_{k,l}$. Therefore, we define $\{\Lambda_j\}_{j=1}^\infty \nearrow \infty$ so that $\sqrt{\Lambda_j} = \prod_{1\leq k\leq r}\prod_{2\leq l\leq n_k} p_{k,l}(\widetilde{\Lambda}_j)^{1/2}$, then $\{\Lambda_j\}_{j=1}^\infty$ satisfies the equality (6.32). Now, similarly as in (6.30), we have for such a sequence,

$$
(6.34)\quad R_{\Lambda_j}[t_i, t_{i-1}] = \begin{pmatrix} A_nB_n\frac{2}{\pi}, & \Lambda_j^{-\nu}B_nB_n\frac{2}{\pi}\cos\nu\pi \\ \Lambda_j^{\nu}A_nA_n\frac{2}{\pi}\cos\nu\pi, & A_nB_n\frac{2}{\pi} \end{pmatrix}(1+o(1))
$$

as $\Lambda \to \infty$, for any $1 \le i \le N$. If we assume that

$$(6.35)\quad R_{\Lambda_j}[t_k,0] = \frac{1}{2}\left(A_nB_n\frac{2}{\pi}\right)^{k-1} \times \begin{pmatrix} A_nB_n\frac{2}{\pi}C^1(\nu,k), & \Lambda_j^{-\nu}B_nB_n\frac{2}{\pi}C^2(\nu,k) \\ \Lambda_j^{\nu}A_nA_n\frac{2}{\pi}C^2(\nu,k), & A_nB_n\frac{2}{\pi}C^1(\nu,k) \end{pmatrix}(1+o(1))$$

as $\Lambda \to \infty$, for $1 \le k \le N-1$, where $C^1(\nu,k) = (1+\cos\nu\pi)^k + (1-\cos\nu\pi)^k$ and $C^2(\nu,k) = (1+\cos\nu\pi)^k - (1-\cos\nu\pi)^k$. Then it holds that

$$(6.36)\quad \begin{aligned} R_{\Lambda_j}[t_{k+1},0] &= R_{\Lambda_j}[t_{k+1},t_k]R_{\Lambda_j}[t_k,0], \\ &= \frac{1}{2}\left(A_nB_n\frac{2}{\pi}\right)^{k-1} \\ &\quad\times \begin{pmatrix} A_nB_n\frac{2}{\pi}, & \Lambda_j^{-\nu}B_nB_n\frac{2}{\pi}\cos\nu\pi \\ \Lambda_j^{\nu}A_nA_n\frac{2}{\pi}\cos\nu\pi, & A_nB_n\frac{2}{\pi} \end{pmatrix} \\ &\quad\times \begin{pmatrix} A_nB_n\frac{2}{\pi}C^1(\nu,k), & \Lambda_j^{-\nu}B_nB_n\frac{2}{\pi}C^2(\nu,k) \\ \Lambda_j^{\nu}A_nA_n\frac{2}{\pi}C^2(\nu,k), & A_nB_n\frac{2}{\pi}C^1(\nu,k) \end{pmatrix}(1+o(1)) \\ &= \frac{1}{2}\left(A_nB_n\frac{2}{\pi}\right)^{k} \\ &\quad \begin{pmatrix} A_nB_n\frac{2}{\pi}C^1(\nu,k+1), & \Lambda_j^{-\nu}B_nB_n\frac{2}{\pi}C^2(\nu,k+1) \\ \Lambda_j^{\nu}A_nA_n\frac{2}{\pi}C^2(\nu,k+1), & A_nB_n\frac{2}{\pi}C^1(\nu,k+1) \end{pmatrix}(1+o(1)) \end{aligned}$$

as $\Lambda \to \infty$. From an induction, we have

$$(6.37)\quad \begin{aligned} \Phi_{\Lambda_j}(T) &= R_{\Lambda_j}[t_N,0] \\ &= \frac{1}{2}\left(A_nB_n\frac{2}{\pi}\right)^{N-1} \\ &\quad\times \begin{pmatrix} A_nB_n\frac{2}{\pi}C^1(\nu,N), & \Lambda_j^{-\nu}B_nB_nC^2(\nu,N) \\ \Lambda_j^{\nu}A_nA_nC^2(\nu,N), & A_nB_n\frac{2}{\pi}C^1(\nu,N) \end{pmatrix}(1+o(1)) \end{aligned}$$

as $\Lambda \to \infty$, which implies

$$\limsup_{\Lambda\to\infty} \Delta(\Lambda) \geq \frac{1}{\sin^N \nu\pi}\{(1+\cos\nu\pi)^N + (1-\cos\nu\pi)^N\}. \tag{6.38}$$

If $\{S_i\}$ are rationally independent, by the similar arguments above, there exist a sequence $\{\Lambda_j\}_{j=1}^{\infty} \nearrow \infty$ such that

$$\lim_{j\to\infty} \cos S_i\sqrt{\Lambda_j} = \begin{cases} =1 & \text{for} \quad 1 \leq i \leq N-1, \\ =-1 & \text{for} \quad i = N \end{cases} \tag{6.39}$$

As in the previous case, we have

$$\begin{aligned} \Phi_{\Lambda_j}(T) &= \frac{-1}{2}\left(A_n B_n \frac{2}{\pi}\right)^{N-1} \\ &\times \begin{pmatrix} A_n B_n \frac{2}{\pi} C^1(\nu, N), & \Lambda_j^{-\nu} B_n B_n \frac{2}{\pi} C^2(\nu, N) \\ \Lambda_j^{\nu} A_n A_n \frac{2}{\pi} C^2(\nu, N), & A_n B_n \frac{2}{\pi} C^1(\nu, N) \end{pmatrix} (1+o(1)) \end{aligned} \tag{6.40}$$

as $\Lambda \to \infty$, which implies

$$\liminf_{\Lambda\to\infty} \Delta(\Lambda) \leq -\frac{1}{\sin^N \nu\pi}\{(1+\cos\nu\pi)^N + (1-\cos\nu\pi)^N\}. \tag{6.41}$$

Thus the proof is completed. □

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Y. Komatsu
Department of Mathematics
Graduate school of Science
University of Osaka
Toyonaka, Osaka 560, Japan

S. Kotani
Department of Mathematics
Graduate school of Science
University of Osaka
Toyonaka, Osaka 560, Japan

A. Matsumura
Department of Mathematics
Graduate school of Science
University of Osaka
Toyonaka, Osaka 560, Japan