Teragaito, M.
Osaka J. Math.
34 (1997), 145–150

CYCLIC SURGERY ON GENUS ONE KNOTS

MASAKAZU TERAGAITO

(Received May 2, 1996)

0. Introduction

The real projective 3-space, denoted by RP^3, is identified with the lens space of type (2,1). Then one can ask: when can RP^3 be obtained by Dehn surgery on a knot in the 3-sphere S^3? Clearly RP^3 is obtained by Dehn surgery on a trivial knot. However, it is conjectured that no Dehn surgery on a nontrivial knot K in S^3 yields RP^3 (cf. [1,4]). It is known to be true if K is a composite knot [3], a torus knot [9], an alternating knot [10], a satellite knot [1,12,13], or a symmetric knot [1].

In this paper we prove the conjecture for genus one knots.

Theorem 0.1. *Real projective 3-space RP^3 cannot be obtained by Dehn surgery on a genus one knot in S^3.*

This will be proved by applying the combinatorial techniques developed in [2,5,6,8].

1. Preliminaries

Let K be a genus one knot which is neither a torus knot nor a satellite knot. Let $N(K)$ be a tubular neighborhood of K and let $E(K)=S^3-\operatorname{int}N(K)$. Suppose that some surgery on K yields RP^3, that is, $E(K)\cup J=RP^3$ where J is a solid torus. By [2], the surgery coefficient is ± 2.

Let $P^2\subset RP^3$ be a projective plane which intersects J in a disjoint union of meridian disks of J. We assume that $|P^2\cap J|$ is minimal among all projective planes in RP^3 that intersect J in a family of meridian disks of J. Let $p=|P^2\cap J|$ and $P=P^2\cap E(K)$. Then P is incompressible in $E(K)$ by the minimality of p. If p is even, then $E(K)$ would contain a closed non-orientable surface by attaching tubes to ∂P. Hence p is odd. Furthermore, if $p=1$ then K is either a torus knot or a $(2,\pm 1)$-cable knot. Thus $p\neq 1$.

Let Q be a genus one Seifert surface for K. We may assume that P and Q intersect transversely, and ∂Q intersects each component of ∂P exactly twice. By the incompressibility of P and Q, we can assume that no circle component of

$P \cap Q$ bounds a disk in P or Q.

Let $\hat{P}$, $\hat{Q}$ be the closed surfaces obtained by capping off the components of ∂P and ∂Q with disks. We can identify $\hat{P}$ with P^2. We obtain a graph G_P in $\hat{P}$ by taking the disks $\mathrm{cl}(\hat{P}-P)$ as the (fat) vertices of G_P, and the arc components of $P \cap Q$ in P as the edges of G_P. Similarly, we obtain the graph G_Q in $\hat{Q}$.

Number the components of ∂P, $\{1,2,\cdots,p\}$, in the order in which they appear on $\partial E(K)$. The endpoints of edges of G_Q are labelled by the numbers of the corresponding components of ∂P. Thus around the only vertex v of G_Q, we will consecutively meet the labels $1,2,\cdots,p$, $1,2,\cdots,p$ (repeated twice). Since each vertex of G_P has valency two, G_P consists of disjoint cycles.

2. Proof of Theorem 0.1

A *trivial loop* is a length one cycle which bounds a disk face of the graph.

Lemma 2.1. *Neither G_P nor G_Q contains trivial loops.*

Proof. Let e be a trivial loop in G_P, and let D be a regular neighborhood of e in Q. Given the orientation of ∂Q induced by some orientation of D, the points of intersection of ∂Q with the component of ∂P meeting e have opposite signs, a contradiction. If G_Q contains a trivial loop, P would be compressible in $E(K)$, a contradiction.

An edge of G_Q is said to be *level* if its endpoints have the same label.

Lemma 2.2. *G_Q cannot contain two level edges on distinct labels.*

Proof. Let e be a level edge in G_Q with label i. Then e is a loop in G_P based at the vertex V_i corresponding to the component of ∂P with label i. We see that a regular neighborhood of $e \cup V_i$ in $\hat{P}$ is homeomorphic to a Möbius band. Since a projective plane cannot contain two disjoint Möbius bands, we have the conclusion.

A pair of edges $\{e_1,e_2\}$ in G_Q is called an *S-cycle* if it is a Scharlemann cycle of length two. That is, e_1 and e_2 are adjacent parallel edges, and have the same two labels at their endpoints. Note that in this case the two labels are successive (see Figure 1).

Lemma 2.3. *G_Q cannot contain an S-cycle.*

Proof. Let $\{e_1,e_2\}$ be an S-cycle in G_Q with labels $\{i,i+1\}$. Let D be the disk face between e_1 and e_2. Let H be the annulus in $\partial E(K)$ cobounded by the components of ∂P with labels i and $i+1$, whose interior is disjoint from P. Set $P' = (\hat{P} - V_i \cup V_{i+1}) \cup H$, where V_i and V_{i+1} are the vertices corresponding to the components of ∂P with labels i and $i+1$, respectively. Then $\operatorname{int} D \cap P' = \emptyset$ and $\partial D \subset P'$ is non-separating in P'. Compressing P' along D gives a new projective plane in RP^3 which intersects J in $p-2$ meridian disks of J. This contradicts the minimality of p.

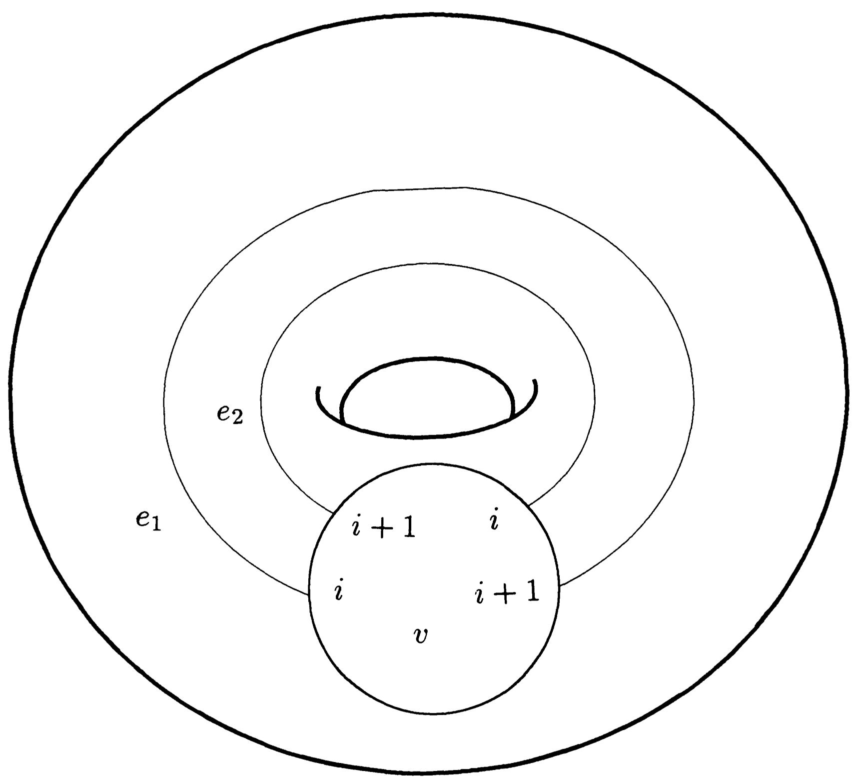

Figure 1

The reduced graph $\bar{G}_Q$ of G_Q is defined to be the graph obtained from G_Q by amalgamating each set of mutually parallel edges of G_Q to a single edge. By Lemma 2.1, $\bar{G}_Q$ consists of essential loops in $\hat{Q}$. Thus $\bar{G}_Q$ is a subgraph of the graph illustrated in Figure 2 (after a homeomorphism of $\hat{Q}$).

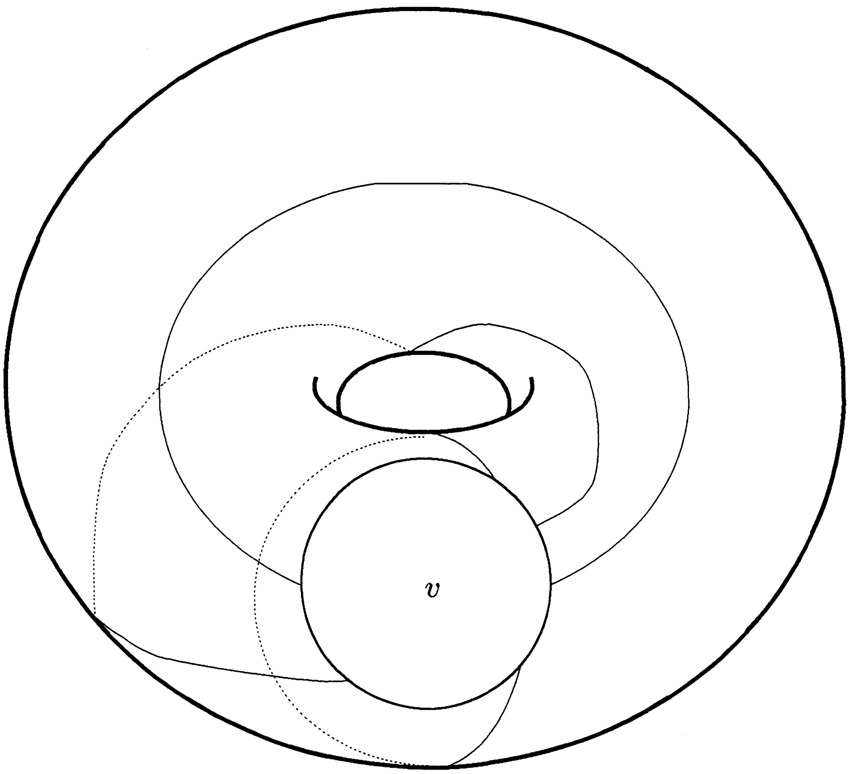

Figure 2

Therefore, the edges in G_Q are partitioned into at most three parallel families of edges. Let U, V, W be the parallel families of edges. We denote by $|U|$ the number of edges in U, etc. Then $|U|+|V|+|W|=p$.

Suppose that $|U|\neq 0$ and $|U|$ is even. Let $e_1, e_2, \cdots, e_{2t}$ be the edges of U, numbered consecutively, where $|U|=2t$. Then e_1 and e_{2t} have the same two labels at their endpoints. Therefore, e_t and e_{t+1} form an S-cycle. But this contradicts Lemma 2.3. Thus $|U|$ is odd, unless $U=\emptyset$. Similarly for V and W.

We now distinguish two cases.

Case 1. *G_Q consists of at least two parallel families of edges.*

We may assume that U and V are non-empty. Then U and V each contain a level edge, since $|U|$ and $|V|$ are odd. But these two level edges have distinct labels, which contradicts Lemma 2.2.

Case 2. *G_Q consists of one parallel family of edges.*

Let $e_1, e_2, \cdots, e_p$ be the edges in G_Q, numbered consecutively. We can assume that their endpoints are labelled as shown in Figure 3. Then e_i and e_{p+1-i} have the same two labels at their endpoints, for $1 \leq i \leq (p-1)/2$, and $e_{(p+1)/2}$ is level.

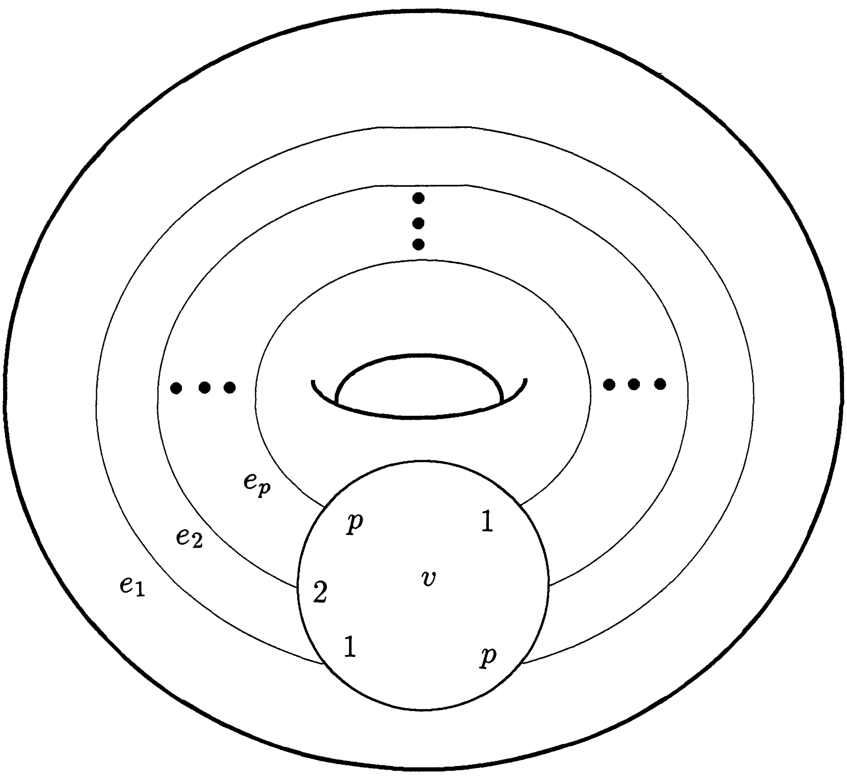

Figure 3

On the other hand, in G_P, e_i and e_{p+1-i} form a length two cycle, if $i \neq (p+1)/2$. Note that these cycles bound disks in $\hat{P}$, since G_P has a nontrivial loop $e_{(p-1)/2}$. Hence we can choose an innermost one among the cycles $\{e_i, e_{p+1-i}\}$, $i \neq (p+1)/2$. Let $\{e_s, e_{p+1-s}\}$ be an innermost cycle in G_P. Then e_s and e_{p+1-s} are parallel in G_P. Let $D_1 \subset P$ be the disk between e_s and e_{p+1-s}. Let $D_2 \subset Q$ be the disk between e_s and e_{p+1-s}, containing $e_{(p+1)/2}$. Then $D_1 \cap D_2 = e_s \cup e_{p+1-s}$. Let $A = D_1 \cup D_2$. Then A is a Möbius band in $E(K)$. By moving ∂A slightly into general position with respect to ∂Q, we see that ∂A has algebraic (and geometric) intersection number two with ∂Q. Hence ∂A has slope $2/n$ on $\partial E(K)$ for some n (cf. [11]). Then the resulting manifold M obtained by $(2/n)$-surgery on K contains a projective plane, and hence M is either reducible or RP^3. In any case, $|n|=1$ by [2,7]. But this implies that K is either a torus knot or a $(2, \pm 1)$-cable knot.

This completes the proof of Theorem 0.1.

References

[1] S. Bleiler and R. Litherland: *Lens spaces and Dehn surgery*, Proc. Amer. Math. Soc. **107** (1989), 1127–1131.

[2] M. Culler, C. McA. Gordon, J. Luecke and P.B. Shalen: *Dehn surgery on knots*, Ann. of Math. **125** (1987), 237–300.

[3] C. McA. Gordon: *Dehn surgery and satellite knots*, Tras. Amer. Math. Soc. **275** (1983) 687–708.

[4] C. McA. Gordon: *Dehn surgery on knots*, Proc. of the International Congress of Mathematicians, The Mathematical Society of Japan, Tokyo, 1990, 631–642.

[5] C. McA. Gordon: *Boundary slopes of punctured tori in 3-manifolds*, preprint.

[6] C. McA. Gordon and R. Litherland: *Incompressible planar surfaces in 3-manifolds*, Topology and its Appl. **18** (1984), 121-144.

[7] C. McA. Gordon and J. Luecke: *Only integral Dehn surgeries can yield reducible manifolds*, Math. Proc. Camb. Phil. Soc. **102** (1987), 97–101.

[8] C. McA. Gordon and J. Luecke: *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), 371–415.

[9] L. Moser. *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 734–745.

[10] J.P. Neuzil: *Elementary surgery manifolds and the elementary ideals*, Proc. Amer. Math. Soc. **68** (1978), 225–228.

[11] D. Rolfsen: Knots and Links, Mathematical Lecture Series **7**, Publish or Perish, Berkeley, 1976.

[12] S. Wang: *Cyclic surgery on knots*, Proc. Amer. Math. Soc. **107** (1989), 1091–1094.

[13] Y.Q. Wu: *Cyclic surgery and satellite knots*, Topology and its Appl. **36**. (1990), 205–208.

Department of Mathematics
Faculty of Science
Hiroshima University
Kagamiyama 1–3–1
Higashi-Hiroshima 739
Japan
teragai@top2.math.sci.hiroshima-u.ac.jp