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DEHN SURGERY ON A KNOT WITH THREE BRIDGES CANNOT YIELD P^3

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Introduction

We work in the P.L. category, with compact manifolds. Two knots k and k' are equivalen if and only if there exists a homeomorphism $h: S^3 \to S^3$ such that $h(k)=k'$. They are equivalent if and only if their complements S^3-k and S^3-k' are homeomorphic. This is a direct consequence of Theorem 2 (p.371) of [9]: "*nontrivial Dehn surgery on a nontrivial knot never yield* S^3".

Similarly, can the complement of a knot k in S^3 and that of a knot in P^3 be homeomorphic? This question is equivalent to: can a Dehn surgery on a knot k in S^3 yield P^3? C. McA. Gordon has conjectured that this is not possible if k is not trivial (Conjecture 5.6 of [7], p.12).

Trivial surgery always yields S^3 and non-trivial surgeries on a trivial knot yield lens spaces. The $2/1$-surgery along the trivial knot give P^3. From now on, here all knots are nontrivial. In this paper, we shall prove that P^3 cannot be obtained by Dehn surgery on a class of knots.

Theorem 1. *P^3 can not be obtained by Dehn surgery on a non-trivial knot with at most* 3 *bridges.*

For many kinds of knots the same result is known:

(i) torus knots by L. Moser ([12] 1971);
(ii) satellite knots by C. McA. Gordon ([8] 1990) and the "Cyclic Surgery Theorem" ([2] 1987);
(iii) symmetric knots by S. Bleiler with R. Litherland ([1] 1989) and by S. Wang with Q. Zhou ([17] 1992),
(iv) knots with genus 1 by M. Domergue ([3] 1991),
(v) knots with 2 bridges; this is a consequence of the note by N. Sayari, G. Hoquenghem and myself (in [11] 1995, this is in the thesis of N. Sayari) see also [1] and [16].

But for a knot with 3 bridges the answer remained unknown.

Let k be a knot in S^3, $N(k)$ a regular neighbourhood of k in S^3, and

$X = S^3 - \operatorname{Int} N(k)$ the *exterior* of k in S^3. The unoriented isotopy class of a nontrivial simple closed curve on $\partial X = T$ will be called its *slope*. They are parametrised by $\boldsymbol{Q} \cup \{\pm\infty\}$ (see [14], we note that ∞ is the meridian slope of $N(k)$). Let r be a slope; $S^3(k,r)$ (also noted $k(r)$) is the 3-manifold obtained by attaching a solid torus $V_0(r)$ to X so that r bounds a meridian disk in $V_0(r)$.

This paper is devoted to the proof of Theorem 1. We follow the path used by C. McA. Gordon and J. Luecke to prove Theorem 2 of [9]. The *bridge number* of a knot, introduced by H. Schubert [15], is a well-known knot invariant. It can be viewed as the minimal number of maxima of the knot, after being put in vertical position in S^3. The concept of *thin presentation* developed by D. Gabai [5], has proved to be very useful, playing a key-role in D. Gabai's proof of property $\mathscr{R}$ as well as the solution of the complement problem of C. McA. Gordon and J. Luecke.

Section I is devoted to the proof of the following proposition, (see also [11]).

Proposition 1. *If there exists a slope r on T such that $S^3(k,r)$ is homeomorphic to P^3 then, there exist two surfaces P and Q properly embedded in X satisfying the three following conditions.*

(i) *the components of ∂P (resp. ∂Q) are parallel copies of r (resp. ∞);*
(ii) *P and Q intersect transversely, and each component of ∂P intersects each component of ∂Q in exactly one point;*
(iii) *no arc of $P \cap Q$ is boundary-parallel in either P or Q.*

The construction of P is based on a projective plane pierced a minimal number of times by the core of surgery, and that of Q on Lemma 4.4 of D. Gabai ([5], using the thin presentation of a knot).

In the section 2, we prove the following proposition:

Proposition 2. *Suppose that X contains properly embedded surfaces P and Q satisfying the three conditions of Proposition* 1, *where Q and P are the intersections with X of respectively, a level* 2*-sphere in a thin presentation of k and a minimal projective plane in P^3; then k is a knot with at least* 4 *bridges.*

The proof of Proposition 2 is based on a combinatorial analysis of the intersection of the surfaces P and Q. Capping off the boundary components of P and Q with disks, we regard these disks as forming the "fat" vertices of graphs H and G in P^2 and S^2 respectively; the edges of G (resp. H) correspond to the arcs of $P \cap Q$ in Q (resp. P). The (disk) faces of H correspond to subdisks of P, which we may regard as lying in $S^3(k,\infty)$ with their boundaries contained in $Q \cup \partial N(k)$. Similarly, the faces of G may be regarded as lying in $S^3(k,r)$. This allows us to infer topological properties of $S^3(k,\infty)$ (resp. $S^3(k,r)$) from graph-theoretic properties of H (resp. G).

With Theorem 2 in [11] we know there exists a kind of edge in H that allows us to obtain a representation of H in a disk. Using Proposition 2.0.1 in [9] and Theorem I.1 in [4], we give combinatorial results on G and H; in particular that neither G or H contains a kind of cycles (great cycles). Moreover, with Theorem of W. Parry [13], H cannot contain a "special" set of faces (representating all types). An immediate consequence of the fact that the knot has at most 3 bridges is that the number of boundary-components of Q is at most 6 (this follows from the definitions of the thin presentation of a knot and of its bridge number). In this case, a more precise analysis of these graphs yields a contradiction to one of the two previous claims.

We conclude this introduction with the proof of Theorem 1.

Proof of Theorem 1. If there exists a slope r such that $S^3(k,r)=P^3$ then, we can choose the two surfaces P and Q from Proposition 1, the intersections with X of respectively a level 2-sphere in a thin presentation of k and a minimal projective plane (see section I); Then we conclude with Proposition 2. $\square$

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1. Preliminaries

This section is devoted to the proof of Proposition 1 stated in the Introduction. All submanifolds are properly embedded and in general position unless otherwise specified. By [12] we may suppose that k is not a torus knot. We choose $\hat{P}$ a projective plane in $S^3(k,r)=P^3$ which intersects the core of surgery in a minimal number of points p. In this case, we say that $\hat{P}$ is *minimal*. We remark that p is nonzero (because there is no projective plane in S^3), odd (because there is no non-orientable closed surface neither); and $r=2/1$, by the "Cyclic Surgery Theorem" in [2]. But in this paper we only need that r is an integer (see Corollary 1 of [2] p.238 or Theorem 1 of [10] p.97). We note $P=\hat{P}\cap X=\hat{P}-\operatorname{Int}N(k)$. Since p is minimized P is incompressible in X. And since P is a 1-sided surface such that ∂P lies in the torus component $T=\partial X$, then it is not hard to see that P is also ∂-incompressible in X.

For the convenience of the reader we recall the definition of a thin presentation, introduced by D. Gabai [5]. Let $\pm\infty$ be the north and south poles of S^3. Then $S^3-\{\pm\infty\}$ is naturally homeomorphic to $S^2\times \boldsymbol{R}^1$, and we have an associated height function $h: S^3-\{\pm\infty\}\to \boldsymbol{R}^1$. The *level 2-spheres* are the spheres $S^2\times\{t\}$, $t\in \boldsymbol{R}^1$.

Let k be a nontrivial knot in S^3. By an isotopy of k we may assume that $k\subset S^3-\{\pm\infty\}$ and that $h_{|k}$ is a Morse function (that is, $h_{|k}$ has only finitely many critical points, all nondegenerate, and with all critical values distinct)

Given such a Morse presentation of k, let $S_1, \cdots, S_m$ be level 2-spheres, one between each consecutive pair of critical levels of $h_{|k}$. One then calls the number $\Sigma_{i=1}^{m} |S_i \cap k|$ the *complexity* of the Morse presentation. A *thin presentation* of k is a Morse presentation of minimal complexity.

Since P is incompressible, by D. Gabai's lemma (see the Lemma 4.4 of [5] p.491) putting k in a thin presentation, there exists a level 2-sphere $\hat{Q}$ in $S^3 = S^3(k, \infty)$ such that each arc of $P \cap Q$ is essential in Q and in P, where $Q = \hat{Q} \cap X = \hat{Q} - \mathrm{Int}\, N(k)$. By an isotopy on Q we may assume that $P \cap Q$ has the minimal number of components.

In conclusion, one has found two surfaces P and Q satisfying the three conditions of Proposition 1, as is required in Proposition 2.

We remark that with a standard innermost disk argument, no circular component of $P \cap Q$ bounds a disk on Q because P is incompressible, and X is irreducible.

2. Combinatorial analysis

This section is devoted to the proof of Proposition 2 stated in the Introduction (independently of section I). We note by X the exterior of the knot k in S^3, and we assume that X contains two surfaces P and Q satisfying the three claims of Proposition 1; where the surfaces P and Q are the intersections with X of respectively, a level 2-sphere in a thin presentation of k and a minimal projective plane in $S^3(k,r) = P^3$.

We divided this section into three sub-sections. In the first, we recall the basic definitions for a pair of graphs coming from the intersection $P \cap Q$. In the second, we give some general combinatorial results on these graphs. Finally in the third, we show that the combinatorial properties imposed by the hypothesis that the knot has at most 3 bridges, contradict the results of the second sub-section.

2.1. Basic definitions

The torus $T = \partial X$ contains the slopes r and ∞ (in the previous section we have seen that r is an integer) with intersection number 1. The surfaces P and Q are compact connected and properly embedded in X with ∂P, $\partial Q \subset T$. Furthermore, each component of ∂P (resp. ∂Q) represents r (resp. ∞); P and Q intersect transversely, and each component of ∂P intersects each component of ∂Q exactly once. Finally, no arc of $P \cap Q$ is boundary-parallel in either P or Q. We follow the definitions and constructions of [9] in the beginning of Chapter 2 (p.385 –394).

Number the components of ∂P:

$$X_{-(p-1)/2},\cdots,X_{-1},X_0,X_1,\cdots,X_{(p-1)/2}$$

and the components of ∂Q:

$$V_1,V_2,\cdots,V_q$$

in the order in which they appear on T.

This allows us to label the endpoints of arcs of $P\cap Q$ in P (resp. Q) with the number of the corresponding boundary of Q (resp. P). Thus around each component of ∂P (resp. ∂Q) we see the labels $\{1,2,\cdots,q\}$ (resp. $\{-(p-1)/2,\cdots,-1,0,1,\cdots,(p-1)/2\}$) appearing in this order (either clockwise or anti-clockwise).

Capping off the boundary components of Q (resp. P) with meridian disks of $N(k)$ (resp. $V_0(r)$), we regard these as forming the *"fat" vertices* of the graphs G (resp. H) in S^2 (resp. P^2). The *edges* of G (resp. H) are the arcs components of $P\cap Q$ in Q (resp. P). Each endpoint of each edge is labeled with the label of the corresponding arc of $P\cap Q$. The *faces* of G (resp. H) are the components of $P-Q$ (resp. $Q-P$).

Let Λ be such a graph and let F be a face of Λ. Each component of ∂F gives rise to an alternating sequence of edges and *corners*; these are the arcs of ∂F projected on the adjacent vertices to F.

Assigning (arbitrary) orientations to Q and k allows us to refer $+$ or $-$ boundary components of Q, according to the direction of the induced orientation of a boundary component as lying on T. We denote by $\hat{Q}$ (resp. $\hat{P}$) the closed compact connected surface obtained by capping off the boundary components of Q (resp. of P) by disks. Thus, we can refer to $+$ or $-$ vertices of G, according to the sign of the boundary of the corresponding disk in $\hat{Q}$. By convention, if around a vertex we see the labels appearing in order anti-clockwise (resp. clockwise) it is a *positive vertex* (resp. *negative vertex*). If two vertices have the same sign we say they are *parallel*, otherwise *antiparallel*. If an edge connects two parallel vertices (or a loop) we say it is a *positive edge*

Let e be a positive edge (of G) with the same label i at both ends. The circuit $X_i\cup e$ of $\hat{P}$ has to reverse the local orientation of $\hat{P}$. Such an edge is called a *double edge*. Since $\hat{P}\backslash(X_i\cup e)$ is an open disk, any other such edge must have the same label i. This particular label i will be noted 0 (zero). In [11] Theorem 2 states that G contains a double edge. Thus, the vertex X_0 of H is this one where are attached all the double edges. We note by $\hat{D}$ the closure of this open disk. Then, we have a representation of H on the disk $\hat{D}$ (see [4] p.9). The vertex X_0 is split into two "half-vertices" X_0^+ and X_0^- oriented $+$ and $-$ (resp.) on $\hat{D}$. We regard these two half-vertices as vertices of the "new" graph H in $\hat{D}$.

We attribute signs to the vertices of H in $\hat{D}$ as we did for the vertices of G. The labels 0 at the endpoints of the edges of G are split into two kinds: the labels 0^+ and the labels 0^- corresponding to the vertices X_0^+ and X_0^- (resp.) of H in $\hat{D}$.

A graph is *taut* if it does not contain a *trivial loop* (a loop that bounds a disk-face of the graph). Thus, we obtain two oriented labeled and taut graphs (no arc boundary-parallel in either P or Q) in a 2-sphere $\hat{Q}$ and a disk $\hat{D}$. From now on, we will note by G the graph obtained as described above on the 2-sphere $\hat{Q}$ (the "G" to recall this is a 2-sphere coming from the thin presentation of k by D. Gabai); and by H the graph on $\hat{D}$. Thus, the two graphs satisfy the *parity rule*: an edge is a positive edge in one graph if and only if it is not one in the other graph.

We define the *representation of types* (see [9] p.387) for the graph H as follows.

A q-type is an ordered q-tuple $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_q)\in\{+,-\}^q$.

Let V be a vertex of H. The edges that are attached at V divide ∂V into q corners. Thus, each corner c at each vertex V of H is naturally labelled by the labels of the adjacent edges to c. We label c by $(i,i+1)$ where i and $i+1$ are the labels of the adjacent edges to c ($i\in\{1,\cdots,q\}$ and $q+1\equiv 1$).

Moreover, let $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_q)$ be a q-type, we attribute a sign $\varepsilon(c)=\pm$ to c corresponding to the sign of V and the type ε as follows:

$$\varepsilon(c)=\operatorname{sign}(V)\times\varepsilon_i,$$

where c is labelled $(i,i+1)$. We define $\varepsilon_{|c}=\varepsilon_i$.

Let F be a disk face of H and τ be a q-type. Let L_F be the set of corners c of F. We say that F represents $\tau=(\varepsilon_1,\cdots,\varepsilon_q)$ if:

i) For each corner c of F, the vertices of H that have this corner, all have the same sign, say $\varepsilon(c)$.

ii) $\exists\delta\in\{\pm\}$ such that $\forall c\in L_F\quad \varepsilon(c)=\delta\times\tau_{|c}$.

We remark that $\operatorname{sign}(V)=\varepsilon(c)$, so ii) means

$$\exists\delta\in\{\pm\}\ \forall c\in L_F\quad \tau(c)=\delta\times\tau_{|c}\times\tau_{|c}=\delta.$$

So we can give another formulation:

F represents τ if the corners of F have all the same sign.

We say *H represents a type* if it contains a disk-face that represents this type. We call the *trivial type* the type for which $\varepsilon_1=\cdots=\varepsilon_q$.

As the same manner, we can talk of the representation of types by G, considering the p-types.

Let Λ be G or H. A *cycle* of Λ is a subgraph homeomorphic to a circle. Let X be a vertex of H (resp. G). A *X-cycle* of G (resp. H) is a cycle σ such that the vertices of σ all have the same sign; and for some consistent orientation of the edges in σ, each edge has the label corresponding to the vertex X at its beginning. A *great X-cycle* of Λ is a X-cycle that bounds a disk Δ such that the vertices in the closure of Δ all have the same sign. We can talk of a great cycle without specifing the vertex X.

2.2. Representation of types by H.

In this section, we establish some general combinatorial results on G and H; in particular we show that there exists a nontrivial type which cannot be represented by H.

Lemma 2.2. (i) *H does not represent all types;*
(ii) *Neither G or H contains a great cycle;*
(iii) *H represents the trivial type.*

Proof. (i) The details of the proof is given in [11] (see Lemma III.3.) If H represents all types then (since H is taut and from the Theorem of W. Parry [13]), $H_1(S^3(k,\infty))$ contains a subgroup with torsion wich is not possible. The small technical hitch that there can exist a circular component of $P\cap Q$ that bounds a disk in P is solved by (A) of Proposition 3.2 in [9]. We remark there is no circular component of $P\cap Q$ that bounds a disk in Q (see the end of section I).

(ii) By Lemma 2.6.2 in [2], if H contains a great cycle then, it contains a *Scharlemann cycle* (great cycle that bounds a disk-face) and so represents all types which is impossible by (i). Moreover, since P comes from a minimal projective plane, G cannot contain a great cycle (by Corollary I.2 in [4]).

(iii) Let x be a vertex of $H-\{X_0^+,X_0^-\}$. If x is attached at most at one positive edge in H, then each edge in G, except at most one, that has a label corresponding to x, is a positive edge by the parity rule. Thus, they have two distinct labels, and there exists at most one vertex V in G that is not connected to a parallel vertex at the label corresponding to x. Begining with a vertex of opposite sign to V, we construct an x-cycle σ with these positive edges. In the 2-sphere $\hat{Q}$, σ bounds a disk that does not contain V (if one exists); if it is not a great x-cycle then we can construct, in the same manner another x-cycle in the interior of this disk, and by induction we obtain a great x-cycle (see Lemma 2.6.3 in [2]), contradicts (ii). So, all the vertices of $H-\{X_0^+,X_0^-\}$ meet at least 2 positive edges. Let x be a vertex of $H-\{X_0^+,X_0^-\}$, and consider Γ be the subgraph of H with all the parallel vertices of x and the positive edges incident to these ones. Only one of the two vertices X_0^+ and X_0^- is in Γ. Suppose this is X_0^+. Since all the vertices of $\Gamma-X_0^+$, are incident to two edges in Γ, then Γ contains a cycle. So we can construct cycles with only positive edges, and by induction there exists a disk-face in H bounded only by positive edges, which means that this face represents the trivial type. □

2.3. Special case: $q\leq 6$.

In the following, we suppose that k has at most three bridges and so $q\leq 6$. A label i (of an edge of H) is a *switch* of a type $\tau=(\tau_1,\cdots,\tau_q)$ if $\tau_{i-1}\neq\tau_i$, where

$i \in \boldsymbol{Z}_q$. Let c and c' be the two corners of a fat vertex of H labelling $(i-1,i)$ and $(i,i+1)$ respectively then, $\tau(c) = -\tau(c')$ (see Fig. 1).

Lemma 2.3.1. *If $q \leq 6$ then it follows that:*
(i) *G cannot contain a loop;*
(ii) *There exists a type τ with exactly two switches which is not represented by H.*

Proof. (i) Suppose that G contains a loop. We denote by b this edge and by B the vertex of G to which it is attached. Then, $\hat{Q}-(b \cup B)$ is the union of two disks. By the Lemma 2.2 (ii), b is not a great cycle, so these two disks have in their interior at least one vertex of the opposite sign to that of B. Since $\hat{Q}$ is a level 2-sphere in S^3 (so separating), G has as many positive vertices as negative vertices, and since $q \leq 6$ one of these disks contains in its interior one vertex A with an opposite sign to that of B. Since there is no loop based at A then, A is connected only to antiparallel vertices in G, which gives rise to a great A-cycle in H (see Lemma 2.6.3 in [2] or the proof of Lemma 2.2 (iii)). So by (ii) of Lemma 2.2 there cannot exist a loop in G.

(ii) Since $\hat{Q}$ is separating, the faces of H are divided into two families: the white faces representing those lying on one side of $\hat{Q}$ and the black faces representing these ones that are in the other side. We can do the same thing for the corners, and for the types (see Fig. 1).

Thus, each type $\mu = (\mu_1, \mu_2, \cdots, \mu_q)$ can be seen as the "reunion" of a white type $\mu_W = (\mu_1, \mu_3, \cdots, \mu_{q-1})$ and a black type $\mu_B = (\mu_2, \mu_4, \cdots, \mu_q)$, we note $\mu = (\mu_W, \mu_B)$ (representing resp. the white and black corners).

If H does not represent $\tau = (\tau_W, \tau_B)$ then H does not represent the type $\tau' = (\tau_W, -\tau_B)$. Indeed if there exists a disk-face representing τ' then, it represents τ_W or τ_B (according to its colour) and so it represents τ too. Moreover, each switch of τ is not one of τ' and vice-versa. The number of switches of a type is even and $q \leq 6$. By the Lemma 2.2 (iii), neither τ or τ' is trivial, so each has at least 2 switches. In any case, there exists a type (τ or τ') with exactly 2 switches which is not represented by H. □

We now construct an *oriented dual graph* $\Gamma = (H, \omega)$ (in the same manner as it made in [9] pp.392–395), where ω is an orientation (oriented edges) on each corner of each face of H. We proceed as follows. For each disk-face F of H, choose a *dual vertex* $v \in \operatorname{Int} F$. Then, $\{$vertices of $\Gamma\} = \{$fat vertices of $H\} \cup \{$dual vertices$\}$. For each corner c of each disk-face F of H at a fat vertex V, put an edge joining V to v at this corner c; where v is the dual vertex of Γ in F.

Finally, orient these edges according to the dual orientation ω. They go from V towards v if $\tau(c)$ is positive (V and $\tau_{|c}$ have the same sign), otherwise they go from v towards V. The *edges* of Γ are these oriented edges.

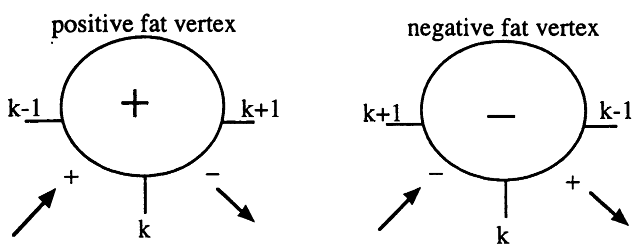
Let $S(\mu)$ be the number of switches of μ. Then $S(\mu)$ is even and we divide the set of the switches of μ into 2 kinds:

$$C(\mu)=\{s_1(\mu),s_3(\mu),s_5(\mu),\cdots\} \quad \text{et} \quad A(\mu)=\{s_2(\mu),s_4(\mu),s_6(\mu),\cdots\},$$

where: $1\leq s_1(\mu)\leq s_2(\mu)\leq s_3(\mu)\cdots\leq s_{s(\mu)}(\mu)$, is the set of the swiches of μ.

Let k be a switch of μ. The graph Λ induces a circuit on each fat vertex S say, at the label k, since the oriented edges incident to S on the right side and on the left side of k do not have the same direction. By definition the sens of the circuit at k is the same for all the fat vertices. Indeed, since the labels are numbered $\cdots,k-1,k,k+1,\cdots$ in the anticlockwise sens on the boundary of a positive vertex and in the clockwise sens on the boundary of a negative vertex, the sens of this circuit does not depend on the sign of the fat vertex (see Fig.1).

To justify the terminology, we may assume that the sens of the circuit is the clockwise sens for a label-switch of $C(\mu)$, and the anticlockwise sens for a label-switch of $A(\mu)$ (see Fig.1).



The sens of the circuit does not depend on the sign of the fat vertex

$$\begin{array}{c} (\ 1.2,\ 2.3\ ,\ 3.4\ ,\ 4.5\ ,\ 5.6\ ,\ \dots\ ,\ q.1\ ) \\ \mu = (\ +\ ,\ +\ ,\ -\ ,\ +\ ,\ -\ ,\ \cdots\ ,\ -\ ) \end{array}$$

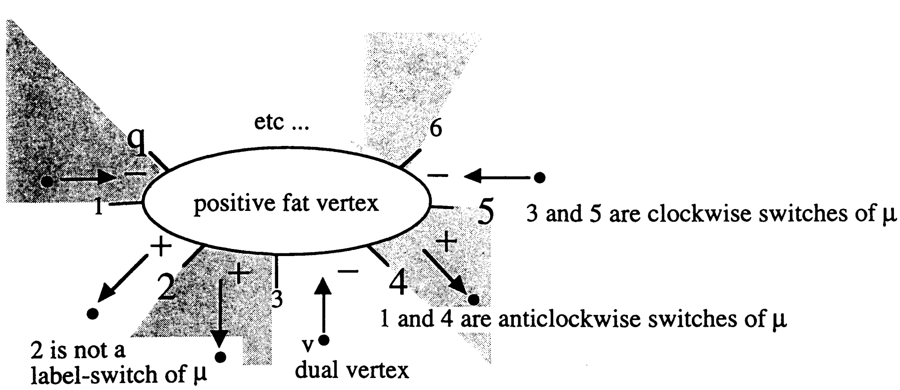


Fig. 1.

A *switch* around a face F of Γ is a pair of adjacent edges of ∂F incident to a vertex v, say, whose orientations agree at v.

A *switch* at a vertex v of Γ is a pair of adjacent edges incident to v, whose orientations are opposite at v.

The *index of a face* F and *of a vertex* v of Γ are respectively

$$I(F)=\chi(F)-s(F)/2 \quad and \quad I(v)=1-s(v)/2$$

where $s(F)$ (resp. $s(v)$) is the number of switches around F (resp. at v). The following lemma and its proof are taken from [6] or from the Lemma 2.3.1 in [9].

Lemma 2.3.2. $\Sigma_{vertices}I(v)+\Sigma_{faces}I(F)=1$.

Proof. This is the same as the proof of Lemma 2.3.1 in [9]. Here we have $\Sigma_{vertices}I(v)+\Sigma_{faces}I(F)=\chi(\hat{D})=1$. □

PROOF OF THE PROPOSITION 2. We suppose $q\leq 6$. By Lemma 2.3.1 above, there exists a type τ with exactly two switches which is not represented by H; if v is a fat vertex then $s(v)=2$ and $I(v)=0$. Moreover $C(\tau)$ and $A(\tau)$ have only one element, this means $C(\tau)=\{x\}$ and $A(\tau)=\{y\}$. For each dual vertex v, we have $I(v)\leq 0$, otherwise v has no switch and the disk-face corresponding to v represents τ (see Lemma 2.2.1 in [9]). Thus Lemma 2.3.2 gives $1\leq\Sigma_{faces}I(F)$. This means that there exists a disk-face of Γ coming from an edge of H whose labels are all in $C(\tau)$ or all in $A(\tau)$. This edge is called a *switch-edge* in [9] (for the details see p.394). So e is a loop in G attached at the vertex corresponding to the label x (or y), which contradicts (i) of the Lemma 2.3.1. □

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