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ON *QF*-RINGS WITH CYCLIC NAKAYAMA PERMUTATIONS

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0. Introduction

Let R be a basic Quasi-Frobenius ring (in brief, QF-ring) and $E=\{e_1,e_2,\cdots,e_n\}$ be a complete set of orthogonal primitive idempotents of R. For any e in E, there exists a unique f in E such that the top of fR is isomorphic to the bottom of eR and the top of Re is isomorphic to the bottom of Rf. Then the permutation $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ f_1 & f_2 & \cdots & f_n \end{pmatrix}$ is said to be a Nakayama permutation of R.

If R is a QF-ring, then R contains a basic QF-subring R^0 such that R is Morita equivalent to R^0. So Nakayama permutations of R^0 are considered and we call these Nakayama permutations of R.

It is well-known that Nakayama permutations of a group algebra of a finite group over a field are identity. This paper is concerned with QF-rings with cyclic Nakayama permutations. Our main result is the following:

Theorem. *If R is a basic QF-ring such that for any idempotent e in R, eRe is a QF-ring with a cyclic Nakayama permutation, then there exist a local QF-ring Q, an element c in the Jacobson radical of Q and a ring automorphism σ of Q for which R is represented as a skew-matrix ring*:

$$R \simeq \begin{pmatrix} Q \cdots Q \\ \cdots \\ Q \cdots Q \end{pmatrix}_{\sigma,c,n}.$$

Throughout this paper R will always denote an associative ring with identity and all R-modules are unitary. The notation M_R (resp. $_RM$) is used to denote that M is a right (resp. left) R-module. For a given R-module M, $J(M)$ and $S(M)$ denote its Jacobson radical and socle, respectively. For R-modules M and N, $M \underset{\sim}{\subseteq} N$ means that M is isomorphic to a submodule of N. And, for R-modules M and N, we put $(M,N)=\mathrm{Hom}_R(M,N)$ and in parrticular, we put $(e,f)=(eR,fR)$ $=\mathrm{Hom}_R(eR,fR)$ for idempotents e,f in R.

Let R be a ring which is represented as a matrix form:

$$R = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ & \cdots & \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} .$$

Then we use $\langle a \rangle_{ij}$ to denote the matrix of R whose (i,j)-position is a but other positions are zero. Consider another ring which is also represented as a matrix form:

$$T = \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ & \cdots & \\ B_{n1} & \cdots & B_{nn} \end{pmatrix} .$$

When we say $\tau = \{\tau_{ij}\}$ is a map from R to T, this word means that τ_{ij} is a map from A_{ij} to B_{ij} and $\tau(\langle a \rangle_{ij}) = \langle \tau_{ij}(a) \rangle_{ij}$. In the above ring R, we put $Q_i = A_{ii}$ for $i = 1, \cdots, n$. Consider a ring U which is isomorphic to Q_k; $\xi : U \simeq Q_k$. Then we can exchange Q_k by U and make a new ring $R(Q_k, U, \xi)$ which is canonically isomorphic to R. We often identify R with $R(Q_k, U, \xi)$.

Let R be an artinian ring. The following result due to Fuller ([2]) is useful: Let f be in E. ${}_RRf$ is injective if and only if there exists e in E such that (eR, Rf) is an i-pair, that is, ${}_RRe / J({}_RRe) \simeq {}_RS({}_RRf)$ and $fR_R / J(fR_R)_R \simeq S(eR_R)_R$. In this case, eR_R is also injective. We note that if R is a basic artinian ring and (eR, Rf) is an i-pair, then $S({}_{eRe}eRf) = S(eRf_{fRf})$ and

$$S(eR_R) = \begin{pmatrix} & 0 & \\ 0 & S(eRf) & 0 \\ & 0 & \end{pmatrix} = S({}_RRf).$$

Let R be a basic QF-ring and $E = \{e_1, e_2, \cdots, e_n\}$ be a complete set of orthogonal primitive idempotents. For each $e_i \in E$, there exists a unique $f_i \in E$ such that (e_iR, Rf_i) is an i-pair. Then $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ f_1 & f_2 & \cdots & f_n \end{pmatrix}$ is a permutation of $\{e_1, e_2, \cdots, e_n\}$. This permutation is called a Nakayama permutation of R. If there exists a ring automorphism ϕ of R satisfying $\phi(e_i) = f_i$, $i = 1, \cdots, n$, then ϕ is called a Nakayama automorphism of R.

For a ring R, End(R) and Aut(R) stand for the set of all ring endomorphisms of R and that of all ring automorphisms of R, respectively.

1. Skew matrix ring

In this section we consider some structure theorem on a skew matrix ring. After the first author published the paper [4] in which these rings are introduced, Kupish pointed out that he already introduced these rings in [3]. We note that most of the results in this section were reported in [4].

Let Q be a ring and let $c \in Q$ and $\sigma \in \mathrm{End}(Q)$ such that

$$\sigma(c)=c, \quad \sigma(q)c=cq \quad \text{for all} \quad q \in Q.$$

By R we denote the set of all $n \times n$ matrices over Q;

$$R = \begin{pmatrix} Q \cdots Q \\ \cdots \\ Q \cdots Q \end{pmatrix}.$$

We define a multiplication in R which depends on (σ,c,n) as follows: For (x_{ik}), (y_{ik}) in R,

$$(z_{ik})=(x_{ik})(y_{ik})$$

where z_{ik} is defined as follows:

(1) If $i \leq k$, $z_{ik}=\sum_{j<i} x_{ij}\sigma(y_{jk})c+\sum_{i\leq j\leq k} x_{ij}y_{jk}+\sum_{k<j} x_{ij}y_{jk}c$

(2) If $k<i$, $z_{ik}=\sum_{j\leq k} x_{ij}\sigma(y_{jk})+\sum_{k<j<i} x_{ij}\sigma(y_{jk})c+\sum_{i\leq j} x_{ij}y_{jk}$

We may understand this operation as follows:

$$\langle a\rangle_{ij}\langle b\rangle_{jk}=\begin{cases} \langle a\sigma(b)\rangle_{ik} & (j\leq k<i) \\ \langle a\sigma(b)c\rangle_{ik} & (k<j<i \quad \text{or} \quad j<i\leq k) \\ \langle ab\rangle_{ik} & (i=j) \\ \langle abc\rangle_{ik} & (i\leq k<j) \\ \langle ab\rangle_{ik} & (k<i<j \quad \text{or} \quad i<j\leq k). \end{cases}$$

Note that this operation satisfies associative law, i.e.,

$$(\langle x\rangle_{ij}\langle y\rangle_{jk})\langle z\rangle_{kl}=\langle x\rangle_{ij}(\langle y\rangle_{jk}\langle z\rangle_{kl}).$$

Therefore R becomes a ring by this multiplication together with the usual sum of matrices. We call R the skew matrix ring over Q with respect to (σ,c,n) and denote it by

$$R = \begin{pmatrix} Q \cdots Q \\ \cdots \\ Q \cdots Q \end{pmatrix}_{\sigma,c,n}$$

or

$$R = \begin{pmatrix} Q \cdots Q \\ \cdots \\ Q \cdots Q \end{pmatrix}_{\sigma,c}$$

if there are no confusions.

When $n=2$, the multiplication is:

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}=\begin{pmatrix} x_1y_1+x_2y_3c & x_1y_2+x_2y_4 \\ x_3\sigma(y_1)+x_4y_3 & x_3\sigma(y_2)c+x_4y_4 \end{pmatrix}.$$

Now, in the skew-matrix ring R above, we put $e_i=\langle 1\rangle_{ii}$, $i=1,\cdots,n$. Then $\{e_1,\cdots,e_n\}$ is a set of orthogonal idempotents with $1=e_1+\cdots+e_n$, and

$$e_iR = \begin{pmatrix} 0 \\ Q \cdots Q \\ 0 \end{pmatrix} < i$$

$$Re_j = \begin{pmatrix} & Q & \\ 0 & \vdots & 0 \\ & Q & \end{pmatrix}, \quad \text{(column } j\text{)}.$$

If Q is a local ring, then each e_i is a primitive idempotent.

Proposition 1. *The mapping $\tau: R \to R$ given by*

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & \cdots & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \to \begin{pmatrix} x_{nn} & x_{n1} & \cdots & x_{n,n-1} \\ \sigma(x_{1n}) & \sigma(x_{11}) & \cdots & \sigma(x_{1,n-1}) \\ & \cdots & & \\ \sigma(x_{n-1,n}) & \sigma(x_{n-1,1}) & \cdots & \sigma(x_{n-1,n-1}) \end{pmatrix}$$

is a ring homomorphism; in particular if $\sigma \in \mathrm{Aut}(Q)$, then $\tau \in \mathrm{Aut}(R)$.

Proof. Straightforward.

We put

$$W_i = \begin{pmatrix} & & & \overset{i}{\vee} & & & \\ & & & 0 & & & \\ Q & \cdots & Q & Qc & Q & \cdots & Q \\ & & & 0 & & & \end{pmatrix} < i.$$

Then W_i is a submodule of e_iR_R. For $i=2,\cdots,n$, let $\phi_i: e_iR \to W_{i-1}$ be a map given by

$$\begin{pmatrix} & 0 & \\ x_1 \cdots x_{i-1} & x_i & \cdots x_n \\ & 0 & \end{pmatrix} < i \to \begin{pmatrix} & 0 & \\ x_1 \cdots x_{i-1}c & x_i & \cdots x_n \\ & 0 & \end{pmatrix} < i-1$$

and let $\phi_1: e_1R \to W_n$ be a map given by

$$\begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \cdots & \cdot & 0 \\ & \cdots & \cdot & \\ 0 & \cdots & \cdot & 0 \\ \sigma(x_1) & \cdots & \sigma(x_{n-1}) & \sigma(x_n)c \end{pmatrix}.$$

Then it is easy to check the following

Proposition 2. *Each ϕ_i is a homomorphism. In particular, if $\sigma \in \mathrm{Aut}(Q)$, then each ϕ_i is an onto homomorphism and*

$$\mathrm{Ker}\,\phi_1 = \begin{pmatrix} 0 & \cdots & 0 & (0:c) \\ 0 & \cdots & 0 & 0 \\ & \cdots & & \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$\mathrm{Ker}\,\phi_i = \begin{pmatrix} & \overset{i-1}{\vee} & \\ & 0 & \\ 0 & (0:c) & 0 \\ & 0 & \end{pmatrix} < i \quad \textit{for} \quad i=2,\cdots,n.$$

where $(0:c)$ is a right (or left) annihilator ideal of c.

Theorem 1. *If Q is a local QF-ring, $\sigma \in \mathrm{Aut}(Q)$ and $c \in J(Q)$, then the skew matrix ring R over Q with respect to (σ,c,n) is a basic indecomposable QF-ring* *and* $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ e_n & e_1 & \cdots & e_{n-1} \end{pmatrix}$ *is a Nakayama permutation where* $e_i = <1>_{ii}$, $i=1,2,\cdots,n$; *whence R has a Nakayama automorphism. Furthermore, for any idempotent e in R, eRe is represented as a skew-matrix ring over Q with respect to* $(\sigma, c, (k \leq n))$; *so eRe is a QF-ring with a cyclic Nakayama permutation.*

Proof. Put $X=S(Q_Q)\,(=S(_QQ))$. Noting $cX=Xc=0$, we can easily see that

$$S(e_1R)=\begin{pmatrix} 0 & \cdots & 0 & X \\ 0 & \cdots & 0 & 0 \\ & \cdots & & \\ 0 & \cdots & 0 & 0 \end{pmatrix}=S(Re_n)$$

$$S(e_iR)=\begin{pmatrix} 0 & & & \overset{\overset{i-1}{\vee}}{} & & & \\ 0 & \cdot & & & & \mathbf{0} & \\ & \cdot & \cdot & & & & \\ & & 0 & 0 & & & \\ & & & X & 0 & & \\ & \mathbf{0} & & & 0 & \cdot & \\ & & & & & \cdot & \cdot \\ & & & & & 0 & 0 \end{pmatrix}=S(Re_{i-1})$$

for $i=2,\cdots,n$. Hence it follows that $(e_1R, Re_n), (e_2R, Re_1), \cdots, (e_nR, Re_{n-1})$ are i-pairs. Therefore R is a QF-ring with a Nakayama automorphism (cf. Proposition 1). For any subset $\{f_1,\cdots,f_k\} \subseteq E$, clealy, fRf is represented as a skew matrix ring over Q with respect to (σ,c,k) where $f=f_1+\cdots+f_k$; whence so is represented eRe for any idempotent e in R.

By Theorem 1 and Propositon 2, we obtain

Corollary 1 (cf. [3]). *If Q is a local Nakayama ring (artinian serial ring),* $\sigma \in \mathrm{Aut}(Q)$ *and* $cQ=J(Q)$, *then the skew matrix ring R over Q with respect to* (σ,c,n) *is a basic indecomposable QF-serial ring such that* $\{e_nR, e_{n-1}R, \cdots, e_1R\}$ *is a Kupisch series and* $\begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ e_n & e_1 & \cdots & e_{n-1} \end{pmatrix}$ *is a Nakayama permutation. Furthermore, R has a Nakayama automorphism.*

2. Main Theorem

In this section we prove the following main theorem which is the converse of Theorem 1 above.

Theorem 2. *If R is a basic QF-ring such that for any idempotent e in R, eRe is a QF-ring with a cyclic Nakayama permutation, then there exist a local QF-ring Q, an element c in the Jacobson radical of Q and a ring automorphism σ of Q for which R is represented as a skew-matrix ring:*

$$R \simeq \begin{pmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{pmatrix}_{\sigma,c,n}.$$

Proof. Let E be a complete set of orthogonal primitive idempotents of R with $1=\Sigma\{e \mid e\in E\}$. First we consider the case that the cardinal $|E|$ of E is 2; let $E=\{e, f\}$. We represent R as

$$R=\begin{pmatrix} Q & A \\ B & T \end{pmatrix}$$

where $Q=(e,e)$, $A=(f, e)$, $B=(e,f)$, $T=(f,f)$. Since e is a primitive idempotent, $eRe=Q$ is a local ring and by the assumption, Q is a QF-ring. Since $\begin{pmatrix} e & f \\ f & e \end{pmatrix}$ is a Nakayama permutation, we see that

$$S(eR)=S(Rf)=\begin{pmatrix} 0 & S(A) \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S(fR)=S(Re)=\begin{pmatrix} 0 & 0 \\ S(B) & 0 \end{pmatrix}.$$

Noting these facts, we can easily prove the following:

Lemma 1. (1) $\{a\in A \mid aB=0\}=\{a\in A \mid Ba=0\}$.
(2) $\{b\in B \mid bA=0\}=\{b\in B \mid Ab=0\}$.

We denote the sets in 1) and 2) by A^* and B^*, respectively. Note that A^* and B^* are submodules of ${}_QA_T$, and ${}_TB_Q$, respectively, and

$$\begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ B^* & 0 \end{pmatrix}$$

are ideals of R.

Now, we denote the factor ring $\bar{R}=\begin{pmatrix} Q & A \\ B & T \end{pmatrix}/\begin{pmatrix} 0 & 0 \\ B^* & 0 \end{pmatrix}$ by $\begin{pmatrix} Q & A \\ \bar{B} & T \end{pmatrix}$, and

$r+\begin{pmatrix} 0 & 0 \\ B^* & 0 \end{pmatrix}$ by $\bar{r}$ for each $r\in R$. Then $\{\bar{e},\bar{f}\}$ is a complete set of orthogonal primitive idempotens of $\bar{R}$ and

$$S(\bar{f}\bar{R})=\begin{pmatrix} 0 & 0 \\ 0 & S(T) \end{pmatrix}.$$

Since eR_R is injective and $S(\bar{f}\bar{R})_R$ is simple, we see $\begin{pmatrix} Q & A \\ 0 & 0 \end{pmatrix} \supseteq \begin{pmatrix} 0 & 0 \\ \bar{B} & T \end{pmatrix}$ as R (and as $\bar{R}$)-module. Since $S(A_T)_T$ is simple, it follows

$$A_T \simeq T_T.$$

Hence $\alpha T=A$ for some $\alpha\in A$. If $Q\alpha \subsetneqq {}_Q Q$, then $S(Q)\alpha=S(Q)Q\alpha=0$; whence $S(Q)A=0$, which is a contradiction. Hence

$$Q\alpha=\alpha T=A.$$

If $q\in Q$, then there exists $t\in T$ such that $q\alpha=\alpha t$. Then the mapping $\psi: Q\to T$ given by $\psi(q)=t$ is a ring isomorphism. We exchange T by Q with respect to the isomorphism ψ;

$$R=\begin{pmatrix} Q & A \\ B & Q \end{pmatrix}.$$

Then

$$q\alpha=\alpha q \quad \text{for all} \quad q\in Q.$$

Next, considering the factor ring $\begin{pmatrix} Q & A \\ B & Q \end{pmatrix} / \begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix}$, we can obtain $\beta\in B$, $\sigma\in \mathrm{Aut}(Q)$ such that $B=Q\beta=\beta Q$ and

$$\beta q=\sigma(q)\beta \quad \text{for all} \quad q\in Q.$$

We put $c=\alpha\beta$. Noting $\langle\beta\rangle_{21}(\langle\alpha\rangle_{12}\langle\beta\rangle_{21})=(\langle\beta\rangle_{21}\langle\alpha\rangle_{12})\langle\beta\rangle_{21}$, we see that

$$\beta(\alpha\beta)=(\beta\alpha)\beta.$$

Further $\alpha\beta=\beta\alpha$. For, if $\alpha\beta-\beta\alpha\neq 0$, then $(\alpha\beta-\beta\alpha)A\neq 0$; so $0\neq(\alpha\beta-\beta\alpha)\alpha=\alpha\beta\alpha-\beta\alpha\alpha$ $=\alpha\beta\alpha-\alpha\beta\alpha$, contradiction. Thus $\alpha\beta=\beta\alpha$ and hence

$$\sigma(c)=c.$$

And we can see easily that $c\in J(Q)$ and $\sigma(q)c=cq$ for any $q\in Q$. Now, for

$$X=\begin{pmatrix} x_1 & x_2\alpha \\ x_3\beta & x_4 \end{pmatrix},\quad Y=\begin{pmatrix} y_1 & y_2\alpha \\ y_3\beta & y_4 \end{pmatrix}\in R=\begin{pmatrix} Q & Q\alpha \\ Q\beta & Q \end{pmatrix},$$

we calculate XY and see

$$XY=\begin{pmatrix} x_1y_1+x_2y_3c & (x_1y_2+x_2y_4)\alpha \\ (x_3\sigma(y_1)+x_4y_3)\beta & x_3\sigma(y_2)c+x_4y_4 \end{pmatrix}.$$

Thus we see that R is isomorphic to the skew matrix ring $\begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}_{\sigma,c}$ by the mapping

$$\begin{pmatrix} x_1 & x_2\alpha \\ x_3\beta & x_4 \end{pmatrix}\rightarrow\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

We note that in the above the mappings $\begin{pmatrix} 0 & 0 \\ x\beta & y \end{pmatrix}\rightarrow\begin{pmatrix} xc & y\alpha \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} x & y\alpha \\ 0 & 0 \end{pmatrix}$ $\rightarrow\begin{pmatrix} 0 & 0 \\ x\beta & y \end{pmatrix}$ are onto right R-homomorphisms from $\begin{pmatrix} 0 & 0 \\ B & Q \end{pmatrix}$ to $\begin{pmatrix} Qc & A \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} Q & A \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & 0 \\ B & Qc \end{pmatrix}$ with kernels $\begin{pmatrix} 0 & 0 \\ B^* & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix}$ respectively, so $Qc_Q\simeq\bar{A}_Q\simeq\bar{B}_Q$.

Next, consider the case of $|E|=3$; put $E=\{e_1,e_2,e_3\}$. We may assume that $\begin{pmatrix} e_1 & e_2 & e_3 \\ e_3 & e_1 & e_2 \end{pmatrix}$ is a Nakayama permutation. We represent R as

$$R=\begin{pmatrix} (e_1,e_1) & (e_2,e_1) & (e_3,e_1) \\ (e_1,e_2) & (e_2,e_2) & (e_3,e_2) \\ (e_1,e_3) & (e_2,e_3) & (e_3,e_3) \end{pmatrix}=\begin{pmatrix} Q_1 & A_{12} & A_{13} \\ A_{21} & Q_2 & A_{23} \\ A_{31} & A_{32} & Q_3 \end{pmatrix}.$$

We put $Q=Q_1$. Considering $\begin{pmatrix} Q_1 & A_{12} \\ A_{21} & Q_2 \end{pmatrix}$, $\begin{pmatrix} Q_1 & A_{13} \\ A_{31} & Q_3 \end{pmatrix}$ and $\begin{pmatrix} Q_2 & A_{23} \\ A_{32} & Q_3 \end{pmatrix}$, we can assume that $Q=Q_2=Q_3$ by the argument above;

$$R=\begin{pmatrix} Q & A_{12} & A_{13} \\ A_{21} & Q & A_{23} \\ A_{31} & A_{32} & Q \end{pmatrix},$$

and then note that $(A_{ij})_Q\simeq Q_Q$ for each ij.

Noting that

$$S(e_1R)=S(Re_3)=\begin{pmatrix} 0 & 0 & S(A_{13}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S(e_2R)=S(Re_1)=\begin{pmatrix} 0 & 0 & 0 \\ S(A_{21}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S(e_3R)=S(Re_2)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S(A_{32}) & 0 \end{pmatrix},$$

we prove the following

Lemma 2.

(1)
$$\begin{aligned}\{x\in A_{32}\mid xA_{23}=0\}&=\{x\in A_{32}\mid A_{23}x=0\}\\ &=\{x\in A_{32}\mid xA_{21}=0\}\\ &=\{x\in A_{32}\mid A_{13}x=0\}.\end{aligned}$$

(2)
$$\begin{aligned}\{x\in A_{21}\mid xA_{12}=0\}&=\{x\in A_{21}\mid xA_{13}=0\}\\ &=\{x\in A_{21}\mid A_{12}x=0\}\\ &=\{x\in A_{21}\mid A_{32}x=0\}.\end{aligned}$$

(3)
$$\begin{aligned}\{x\in A_{13}\mid xA_{31}=0\}&=\{x\in A_{13}\mid A_{31}x=0\}\\ &=\{x\in A_{13}\mid xA_{32}=0\}\\ &=\{x\in A_{13}\mid A_{21}x=0\}.\end{aligned}$$

Proof. 1) By Lemma 1, $\{x\in A_{32}\mid xA_{23}=0\}=\{x\in A_{32}\mid A_{23}x=0\}$. Let $x\in A_{32}$ such that $xA_{23}=0$. If $xA_{21}\neq 0$, then $A_{23}xA_{21}\neq 0$; whence $A_{23}x\neq 0$, a contradiction. If $A_{13}x\neq 0$, then $A_{13}xA_{23}\neq 0$; whence $xA_{23}\neq 0$, a contradiction.Thus $\{x\in A_{32}\mid xA_{23}=0\}\subseteq\{x\in A_{32}\mid xA_{21}=0\}$ and $\{x\in A_{32}\mid xA_{23}=0\}\subseteq\{x\in A_{32}\mid A_{13}x=0\}$.

Let $x\in A_{32}$ such that $xA_{21}=0$. If $xA_{23}\neq 0$, then we see from ${}_QQ\simeq{}_QA_{31}$ that $xA_{23}A_{31}\neq 0$; so $xA_{21}\neq 0$, a contradiction. Hence $\{x\in A_{32}\mid xA_{23}=0\}=\{x\in A_{32}\mid xA_{21}=0\}$. Let $x\in A_{32}$ such that $A_{13}x=0$. If $xA_{23}\neq 0$, then $A_{13}xA_{23}\neq 0$; so $A_{13}x\neq 0$, a contradiction. Hence $\{x\in A_{32}\mid xA_{23}=0\}=\{x\in A_{32}\mid A_{13}x=0\}$. Similarly we can prove 2) and 3).

We put the sets in 1), 2) and 3) above by A^*_{32}, A^*_{21}, A^*_{13}, respectively. We see ${}_Q(A_{32}^*)_Q$, ${}_Q(A_{21}^*)_Q$, ${}_Q(A_{13}^*)_Q$ are submodules of ${}_Q(A_{32})_Q$, ${}_Q(A_{21})_Q$, ${}_Q(A_{13})_Q$, respectively. Further we put

$$X_{13}=\begin{pmatrix} 0 & 0 & A_{13}^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{21}=\begin{pmatrix} 0 & 0 & 0 \\ A_{21}^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{32}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_{32}^* & 0 \end{pmatrix}.$$

These are ideals of R. Consider the factor rings $\bar{R}(1)=R/X_{13}$, $\bar{R}(2)=R/X_{21}$ and $\bar{R}(3)=R/X_{32}$ and put $\bar{R}=\bar{R}(i)$ if no confusion occurs and put $\bar{r}=r+X_{ij}$ for each $r\in R$. We can easily see that

$$S(\bar{e}_1\bar{R})_{\bar{R}}=S(\bar{e}_1\bar{R})_R=\begin{pmatrix} 0 & S(A_{12}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S(\bar{e}_2\bar{R})_{\bar{R}}=S(\bar{e}_2\bar{R})_R=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & S(A_{23}) \\ 0 & 0 & 0 \end{pmatrix},$$

$$S(\bar{e}_3\bar{R})_{\bar{R}}=S(\bar{e}_3\bar{R})_R=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ S(A_{31}) & 0 & 0 \end{pmatrix}.$$

Therefore there are left multiplications $\langle\theta_{23}\rangle_{23}:\bar{e}_3\bar{R}_R\rightarrow e_2R_R$, $\langle\theta_{12}\rangle_{12}:\bar{e}_2\bar{R}_R\rightarrow e_1R_R$ and $\langle\theta_{31}\rangle_{31}:\bar{e}_1\bar{R}_R\rightarrow e_3R_R$, which are monomorphisms. We put $\gamma_1=\langle\theta_{31}\rangle_{31}\eta_1$, $\gamma_2=\langle\theta_{12}\rangle_{12}\eta_2$ and $\gamma_3=\langle\theta_{23}\rangle_{23}\eta_3$, where η_i is a canonical homomorphism: $e_iR_R\rightarrow\bar{e}_i\bar{R}_R$.

Noting

$$\gamma_1(\begin{pmatrix} 0 & A_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_{32} & 0 \end{pmatrix},$$

$$\gamma_2(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{pmatrix})=\begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & 0 & 0 \end{pmatrix})=\begin{pmatrix} 0 & 0 & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and using Lemma 1, we can prove the following

Lemma 3.

(1)
$$\{x\in A_{31}\mid xA_{12}=0\}=\{x\in A_{31}\mid xA_{13}=0\}$$
$$=\{x\in A_{31}\mid A_{13}x=0\}$$
$$=\{x\in A_{31}\mid A_{23}x=0\}.$$

(2)
$$\{x\in A_{23}\mid xA_{31}=0\}=\{x\in A_{23}\mid xA_{32}=0\}$$
$$=\{x\in A_{23}\mid A_{32}x=0\}$$
$$=\{x\in A_{23}\mid A_{12}x=0\}.$$

(3)
$$\{x\in A_{12}\mid xA_{21}=0\}=\{x\in A_{12}\mid xA_{23}=0\}$$
$$=\{x\in A_{12}\mid A_{31}x=0\}$$
$$=\{x\in A_{12}\mid A_{21}x=0\}.$$

Proof. (1) We put $K_1=\{x\in A_{31}\mid xA_{12}=0\}$, $K_2=\{x\in A_{31}\mid xA_{13}=0\}$, $K_3=\{x\in A_{31}\mid A_{13}x=0\}$ and $K_4=\{x\in A_{31}\mid A_{23}x=0\}$. By Lemma 1, we see $K_2=K_3$, and using γ_2, we see $K_3=K_4$. To show $K_1=K_2$, let $x_{31}\in K_1$. If $x_{31}A_{13}\neq 0$, then $x_{31}A_{13}A_{32}\neq 0$, since $S(e_3R)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S(A_{32}) & 0 \end{pmatrix}$. But $x_{31}A_{13}A_{32}\subseteq x_{31}A_{12}=0$, a contradiction. So, $x_{31}A_{13}=0$ and $x_{31}\in K_2$. Conversely, let $x_{31}\in K_2$; $x_{31}A_{13}=0$. If $0\neq x_{31}A_{12}$ $(\subseteq A_{32})$, then $\langle\theta_{31}\rangle_{31}^{-1}(x_{31}A_{12})\subseteq A_{12}$. So, $0\neq\langle\theta_{31}\rangle_{31}^{-1}(x_{31}A_{12})A_{23}$. But $\langle\theta_{31}\rangle_{31}^{-1}(x_{31}A_{12})A_{23}=\langle\theta_{31}\rangle_{31}^{-1}(x_{31})A_{12}A_{23}\subseteq\langle\theta_{31}\rangle_{31}^{-1}(x_{31})A_{13}=0$, contradiction. So, $x_{31}A_{12}=0$ and hence $x_{31}\in K_1$ as desired. (2) and (3) can be proved by the same arguments.

We denote the sets in 1), 2) and 3) by A_{31}^*, A_{23}^* and A_{12}^*, respectively, and put

$$X_{31}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31}^* & 0 & 0 \end{pmatrix}, \quad X_{23}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23}^* \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_{12}=\begin{pmatrix} 0 & A_{12}^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

(4)
$$\gamma_3(X_{31})=X_{21},\ \gamma_2(X_{23})=X_{13} \quad\text{and}\quad \gamma_1(X_{12})=X_{32}.$$

Lemma 4. *There exist $\alpha_{12}\in A_{12}$, $\alpha_{21}\in A_{21}$, $c\in J(Q)$ and $\sigma\in\mathrm{Aut}(Q)$ such that*

(1)
$$c=\alpha_{12}\alpha_{21}=\alpha_{21}\alpha_{12}$$
$$\alpha_{12}q=q\alpha_{12} \quad \text{for all} \quad q\in Q$$
$$\sigma(q)\alpha_{21}=\alpha_{21}q \quad \text{for all} \quad q\in Q$$

(2)
$$\begin{pmatrix} Q & A_{12} \\ A_{21} & Q \end{pmatrix} \simeq \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}_{\sigma,c}$$

by the mapping:

$$\begin{pmatrix} q_{11} & q_{12}\alpha_{12} \\ q_{21}\alpha_{21} & q_{22} \end{pmatrix} \to \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

(3)
$$\operatorname{Im}\langle\theta_{23}\rangle_{23}=\begin{pmatrix} 0 & 0 & 0 \\ A_{21} & cQ & A_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\operatorname{Im}\langle\theta_{12}\rangle_{12}=\begin{pmatrix} cQ & A_{12} & A_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\operatorname{Im}\langle\theta_{31}\rangle_{31}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & A_{32} & cQ \end{pmatrix}.$$

(4) $\operatorname{Im}\langle\theta_{31}\rangle_{31}$, $\operatorname{Im}\langle\theta_{12}\rangle_{12}$, $\operatorname{Im}\langle\theta_{23}\rangle_{23}$, $\operatorname{Im}\eta_3$, $\operatorname{Im}\eta_2$ *and* $\operatorname{Im}\eta_1$ *are quasi-injective (or equivalently, fully invariant) submodules of* e_3R_R, e_1R_R, e_2R_R, e_1R_R, e_3R_R *and* e_2R_R, *respectively.*

Proof. Considering $\begin{pmatrix} Q & A_{12} \\ A_{21} & Q \end{pmatrix}$, we get $\alpha_{12}\in A_{12}$, $\alpha_{21}\in A_{21}$, $c\in J(Q)$ and $\sigma\in \operatorname{Aut}(Q)$ for which 1) and 2) hold. Furthermore, considering $\begin{pmatrix} Q & A_{23} \\ A_{32} & Q \end{pmatrix}$ and $\begin{pmatrix} Q & A_{13} \\ A_{31} & Q \end{pmatrix}$, we get $c_2, c_3\in J(Q)$ and $\sigma_2,\sigma_3\in \operatorname{Aut}(Q)$ for which

$$\begin{pmatrix} Q & A_{23} \\ A_{32} & Q \end{pmatrix} \simeq \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}_{\sigma_2,c_2}, \quad \begin{pmatrix} Q & A_{13} \\ A_{31} & Q \end{pmatrix} \simeq \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}_{\sigma_3,c_3}.$$

By the remark above:

$$(\bar{A}_{12})_Q \simeq cQ_Q, \quad (\bar{A}_{21})_Q \simeq cQ_Q, \quad (\bar{A}_{13})_Q \simeq c_3Q_Q,$$
$$(\bar{A}_{31})_Q \simeq c_3Q_Q, \quad (\bar{A}_{32})_Q \simeq c_2Q_Q, \quad (\bar{A}_{23})_Q \simeq c_2Q_Q,$$

where $\bar{A}_{ij}=A_{ij}/A^*_{ij}$.

Further, as

$$e_1R/X_{12}+X_{13}=\begin{pmatrix} Q & \bar{A}_{12} & \bar{A}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{A}_{31} & \bar{A}_{32} & c_3Q \end{pmatrix} \subseteq e_3R/X_{32}$$

$$e_2R/X_{21}+X_{23}=\begin{pmatrix} 0 & 0 & 0 \\ \bar{A}_{21} & Q & \bar{A}_{23} \\ 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} cQ & A_{12} & \bar{A}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subseteq e_1R/X_{13}$$

$$e_3R/X_{31}+X_{32}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{A}_{31} & \bar{A}_{32} & Q \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 \\ \bar{A}_{21} & c_2Q & A_{23} \\ 0 & 0 & 0 \end{pmatrix} \subseteq e_2R/X_{21}$$

we see that $(\bar{A}_{ij})_Q \simeq (\bar{A}_{kj})_Q$ for $i \neq k$ and $cQ_Q \simeq c_2Q_Q \simeq c_3Q_Q$. Since cQ_Q, c_2Q_Q and c_3Q_Q are fully invariant submodules of Q, it follows that $cQ=c_2Q=c_3Q$. Hence 3) is proved. 4) is clear.

Lemma 5. 1) *For any* $\psi \in (e_3,e_2)$, $\mathrm{Im}\,\psi \subseteq \mathrm{Im}\langle\theta_{23}\rangle_{23}$. *For any* $\psi \in (e_2,e_1)$, $\mathrm{Im}\,\psi \subseteq \mathrm{Im}\langle\theta_{12}\rangle_{12}$. *For any* $\psi \in (e_1,e_3)$, $\mathrm{Im}\,\psi \subseteq \mathrm{Im}\langle\theta_{31}\rangle_{31}$.

2) *For any* $\psi \in (e_3,e_1)$, $\mathrm{Im}\,\psi \subseteq \mathrm{Im}\,\eta_3$. *For any* $\psi \in (e_2,e_3)$, $\mathrm{Im}\,\psi \subseteq \mathrm{Im}\,\eta_2$. *For any* $\psi \in (e_1,e_2)$, $\mathrm{Im}\,\psi \subseteq \mathrm{Im}\,\eta_1$.

Proof. Let $\psi \in (e_3,e_2)$. If $x \in A^*_{32}$ and $\psi(\langle x\rangle_{32})) \neq 0$, then

$$\psi(\langle x\rangle_{32})\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{pmatrix} \neq 0, \text{ but } \langle x\rangle_{32}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{pmatrix} = 0, \text{ which is impossible.}$$

Hence $\psi(\{\langle x\rangle_{32} \mid x \in A^*_{32}\})=0$ and there exists an epimorphism from $\mathrm{Im}\langle\theta_{23}\rangle_{23}$

$$=\begin{pmatrix} 0 & 0 & 0 \\ A_{21} & cQ & A_{23} \\ 0 & 0 & 0 \end{pmatrix} \text{ to } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & A_{32} & Q \end{pmatrix} /\mathrm{Ker}\,\psi \simeq \mathrm{Im}\,\psi.$$ Since $\mathrm{Im}\langle\theta_{23}\rangle_{23}$ is a fully invariant submodule of e_2R, we see $\mathrm{Im}\,\psi \subseteq \mathrm{Im}\langle\theta_{23}\rangle_{23}$.

Similarly we can see the rest parts of 1).

Next for $\psi \in (e_3,e_1)$, we see $\psi(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A^*_{31} & A^*_{32} & 0 \end{pmatrix})=0$. Hence it follows that

$\operatorname{Im}\psi \subseteq \operatorname{Im}\langle\theta_{12}\rangle_{12}\langle\theta_{23}\rangle_{23}$. The other parts of 2) can be similarly proved.

Now consider the factor ring $\bar{R}=R/X_{32}$ and denote $r+X_{32}$ by $\bar{r}$ for $r\in R$. We represent $\bar{R}$ as

$$\begin{aligned}\bar{R}&=\bar{e}_1\bar{R}\oplus\bar{e}_2\bar{R}\oplus\bar{e}_3\bar{R}\\ &=\begin{pmatrix}(e_1,e_1) & (e_2,e_1) & (\bar{e}_3,e_1)\\ (e_1,e_2) & (e_2,e_2) & (\bar{e}_3,e_1)\\ (e_1,\bar{e}_3) & (e_2,\bar{e}_3) & (\bar{e}_3,\bar{e}_3)\end{pmatrix}\\ &=\begin{pmatrix}Q & A_{12} & A_{13}\\ A_{21} & Q & A_{23}\\ A_{31} & \bar{A}_{32} & Q\end{pmatrix}\end{aligned}$$

where $\bar{A}_{32}=A_{32}/A^*_{32}$.

Lemma 6. *The mapping*

$$\tau=\begin{pmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\ \tau_{21} & \tau_{22} & \tau_{23}\\ \tau_{31} & \tau_{32} & \tau_{33}\end{pmatrix}:\begin{pmatrix}Q & A_{12} & A_{12}\\ A_{21} & Q & Q\\ A_{21} & I & Q\end{pmatrix}\to\bar{R}=\begin{pmatrix}Q & A_{12} & A_{13}\\ A_2 & Q & A_{23}\\ A_{31} & \bar{A}_{32} & Q\end{pmatrix}$$

where $I=\theta_{23}\bar{A}_{32}$, *given by*

$$\begin{pmatrix}q_{11} & q_{12} & p_{12}\\ q_{21} & q_{22} & p_{22}\\ t_{21} & t_{22} & y_{22}\end{pmatrix}\to\begin{pmatrix}q_{11} & q_{12} & p_{12}\theta_{23}\\ q_{21} & q_{22} & p_{22}\theta_{23}\\ \theta_{23}^{-1}t_{21} & \theta_{23}^{-1}t_{22} & \theta_{23}^{-1}y_{22}\theta_{23}\end{pmatrix}$$

is a ring isomorphism.

Proof. By Lemma 5, τ is well-defined and furthermore it is a ring monomorphism. Noting e_1R_R is injective, we can see that τ_{13} is an onto mapping. And noting e_2R_R is injective, we see that τ_{23} and τ_{33} are onto mapping. It is easy to see that τ_{31} is an onto mapping. τ_{32} is a clearly onto mapping. Hence τ is a ring isomorphism.

By the lemma above, we see $(\bar{A}_{32})_Q\simeq I_Q$ and hence we see that $I=cQ$. In the isomorphism

$$\tau:\begin{pmatrix} Q & A_{12} & A_{12} \\ A_{12} & Q & Q \\ A_{21} & I & Q \end{pmatrix} \simeq \begin{pmatrix} Q & A_{12} & A_{13} \\ A_{21} & Q & A_{23} \\ A_{31} & \bar{A}_{32} & Q \end{pmatrix}$$

we put $\langle\alpha_{31}\rangle_{31}=\tau(\langle\alpha_{21}\rangle_{31})$, $\langle\alpha_{13}\rangle_{13}=\tau(\langle\alpha_{12}\rangle_{13})$, $\alpha_{32}=\alpha_{31}\alpha_{12}$ and $\langle\alpha_{23}\rangle_{23}=\tau(\langle 1\rangle_{23})$. Since A_{32}^* is a small submodule of A_{32}, we see that $\alpha_{32}Q=A_{32}$.

Hence R is represented as $R\simeq \begin{pmatrix} Q & \alpha_{12}Q & \alpha_{13}Q \\ \alpha_{21}Q & Q & \alpha_{23}Q \\ \alpha_{31}Q & \alpha_{32}Q & Q \end{pmatrix}$ with relations:

$$c=\alpha_{21}\alpha_{12}=\alpha_{12}\alpha_{21}$$

$$\sigma(c)=c$$

$$\alpha_{12}q=q\alpha_{12} \quad \text{for all} \quad q\in Q$$

$$\sigma(q)\alpha_{21}=\alpha_{21}q \quad \text{for all} \quad q\in Q.$$

Putting $\alpha_{ii}=1$ for $i=1,2,3$, we further see that the following relations $(*)$ hold for $1\leq i,j\leq 3$:

$$(*)\quad \begin{cases} \text{If } i>j,\ \sigma(q)\alpha_{ij}=\alpha_{ij}q \text{ for all } q\in Q \\ \qquad \alpha_{ij}\alpha_{jk}=\begin{cases}\alpha_{ik} & (i>k\geq j) \\ \alpha_{ik}c & (k\geq i \text{ or } j>k)\end{cases} \\ \text{If } i=j,\ q\alpha_{ij}=\alpha_{ij}q \text{ for all } q\in Q \\ \qquad \alpha_{ij}\alpha_{jk}=\alpha_{ik} \\ \text{If } i<j,\ q\alpha_{ij}=\alpha_{ij}q \text{ for all } q\in Q \\ \qquad \alpha_{ij}\alpha_{jk}=\begin{cases}\alpha_{ik}c & (i\leq k<j) \\ \alpha_{ik} & (k<i \text{ or } j\leq k).\end{cases} \end{cases}$$

By these relations, we see that R is isomorphic to the skew-matrix ring

$\begin{pmatrix} Q & Q & Q \\ Q & Q & Q \\ Q & Q & Q \end{pmatrix}_{\sigma,c}$ by the mapping

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \to \begin{pmatrix} q_{11}\alpha_{11} & q_{12}\alpha_{12} & q_{13}\alpha_{13} \\ q_{21}\alpha_{21} & q_{22}\alpha_{22} & q_{23}\alpha_{23} \\ q_{31}\alpha_{31} & q_{32}\alpha_{32} & q_{33}\alpha_{33} \end{pmatrix}.$$

For induction on $|E|$, we assume that our statement is true for $n-1=|E|$ and consider the case $n=|E|$, let $E=\{e_1,e_2,\cdots,e_n\}$.

We may assume that $\begin{pmatrix} e_1, & e_2 & \cdots & e_n \\ e_n & e_1 & \cdots & e_{n-1} \end{pmatrix}$ is a Nakayama permutation.

We represent R as

$$R=\begin{bmatrix} Q_1 & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & Q_2 & A_{23} & \cdots & A_{2n} \\ & \cdots & & \cdots & \\ A_{n1} & A_{n2} & A_{n3} & \cdots & Q_n \end{bmatrix}$$

where $Q_i=(e_i,e_i)$ and $A_{ij}=(e_j,e_i)$. By the argument above, we may assume that $Q_1=Q_2=\cdots=Q_n$; put $Q=Q_i$. And we see that $(A_{ij})_Q\simeq Q_Q$ for each ij.

Now, specially we look at the first minor matrix

$$R_0=\begin{bmatrix} Q & A_{12} & \cdots & A_{1,n-1} \\ A_{21} & Q & \cdots & A_{2,n-1} \\ & \cdots & & \cdots \\ A_{n-1,1} & A_{n-1,2} & \cdots & Q \end{bmatrix} .$$

By induction hypothesis, R_0 is isomorphic to a skew matrix ring over a local ring Q with respect to a certain $(\sigma,c,n-1)$ where $\sigma\in\mathrm{Aut}(Q)$ and $c\in J(Q)$. So there exist $\alpha_{ij}\in A_{ij}$ and $\alpha_{ii}\in Q$ for $1\leq i, j\leq n-1$ for which the relations (*) hold.

Now we consider an extension ring R_1 of R_0,

$$R_1=\left(\begin{array}{ccc|c} & & & A_{1,n-1} \\ & R_0 & & \vdots \\ & & & A_{n-2,n-1} \\ & & & Q \\ \hline A_{n-1,1} & \cdots & A_{n-1,n-2}\ cQ & Q \end{array}\right) .$$

By the similar argument which is used for the case $n=3$, we see that there is a ring isomorphism $\tau=(\tau_{ij})$ from R_1 to

$$R_2=\left(\begin{array}{ccc|c} & & & A_{1n} \\ & R_0 & & \vdots \\ & & & A_{n-2,n} \\ & & & A_{n-1,n} \\ \hline A_{n1} & \cdots & A_{n,n-2}\ \bar{A}_{n,n-1} & Q \end{array}\right)$$

where $\bar{A}_{n,n-1}=A_{n,n-1}/A^*_{n,n-1}$ and

$$A^*_{n,n-1}=\{x\in A_{n,n-1}\mid xA_{n-1,j}=0,\ j=1,2,\cdots,n-2,n\}$$
$$=\{x\in A_{n,n-1}\mid A_{in}x=0,\ i=1,2,\cdots,n-1\}$$

We put $\langle\alpha_{in}\rangle_{in}=\tau(\langle\alpha_{i,n-1}\ \rangle_{in})$ and $\langle\alpha_{nj}\rangle_{nj}=\tau(\langle\alpha_{n-1,j}\rangle_{nj})$ for $i=1,2,\cdots,n-2$ and $j=1,2,\cdots,n-2$, put $\alpha_{n,n-1}=\alpha_{n,n-2}\alpha_{n-2,n-1}\in A_{n,n-1}$ and $\langle\alpha_{n-1,n}\rangle_{n-1,n}=\tau(\langle 1\rangle_{n-1,n})$. Since $A^*_{n,n-1}$ is a small submodule of $A_{n,n-1}$, we see that $\alpha_{n,n-1}Q=A_{n,n-1}$.

As the relations (*) hold for $\{\alpha_{ij}\mid 1\leq i,\ j\leq n-1\}$ with respect to σ, c, we can also see that the relations (*) hold for $\{\alpha_{ij}\mid 1\leq i,\ j\leq n\}$ with respect to σ, c. Accordingly R is isomorphic to the skew matrix ring

$$\begin{pmatrix} Q & \cdots & Q \\ & \cdots & \\ Q & \cdots & Q \end{pmatrix}_{\sigma,c,n}$$

by the mapping

$$(q_{ij})\leftrightarrow(q_{ij}\alpha_{ij}).$$

Corollary 2. *If R is a basic QF-ring such that for any idempotent e in R, eRe is a QF-ring with a cyclic Nakayama permutation, then R has a Nakayama automorphism.*

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