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SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS RELATED TO SECOND ORDER LINEAR EQUATIONS

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0. Introduction

We will construct new nonlinear dynamical systems from linear differential equations of second order. This method is first applied by Jacobi, in 1848 ([9]), in the case of the equation

$$x(1-x)\frac{d^2y}{dx^2}+(1-2x)\frac{dy}{dx}-\frac{1}{4}y=0. \tag{0.1}$$

A solution of (0.1) is given by the complete elliptic integral

$$K=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}=\frac{\pi}{2}\theta_3^2$$

and Jacobi found a nonlinear equation:

$$(y^2y'''-15yy'y''+30y'^3)^2+32(yy''-3y'^2)^3=-\pi^2y^{10}(yy''-3y'^2)^2, \tag{0.2}$$

which is satisfied by Jacobi's elliptic theta functions θ_2, θ_3 and θ_4.

Later, Halphen rewrote (0.2) as a nonlinear dynamical system ([7]):

$$\begin{cases} X'+Y'=2XY, \\ Y'+Z'=2YZ, \\ Z'+X'=2ZX. \end{cases} \tag{0.3}$$

Halphen showed that logarithmic derivatives of theta null values satisfy (0.3). The structure of (0.3) is studied in [4] and [11]. Halphen also studied nonlinear systems deduced from generic hypergeometric equations ([8]).

In this paper, we will construct nonlinear equations from general second-order linear equations following Jacobi's idea. They are equivalent to the well-known equation written by the Schwarzian derivative. One of the aim of this paper is

to rewrite the equation expressed in terms of the Schwarzian derivative into a dynamical system. By this transformation, we can study the structure of the equation more conveniently. For example, we will study initial value problems and solution spaces for these systems. In section 1, we will only treat the Fuchsian case. In section 2, we will study the case where the linear equation may have irredular singular points.

We also calculate several examples related to some special functions in section 3. In the case of Jacobi and Halphen, solutions of nonlinear equations are given by modular forms. If we take a Picard-Fuchs equation of a family of elliptic curves as the starting linear equation, solutions of our nonlinear equations are given by modular forms. Jacobi's equation (0.2) is satisfied by the logarithmic derivatives of modular forms with level two. J. Chazy studied a differential equation satisfied by the logarithmic derivatives of modular forms with level one ([3]). The author constructed a nonlinear equation which is satisfied by the logarithmic derivatives of modular forms with level three ([12]). It is an interesting problem to study nonlinear equations which are satisfied by general modular forms. The second aim of this paper is to propose a program for finding nonlinear holonomic systems for elliptic modular forms.

The equations above appear in many fields in mathematics. For example, Ehrenpreis rediscovered (0.2) in his study of scattering theory ([5]). Halphen's equation (0.3) is a specialization of self-dual Einstein equations ([6]) and is used in the study of the geometry of monopoles ([1]). We hope that our new equations may have some applications to other fields of mathematics.

1. Fuchsian equations and nonlinear equations

In this section we will generalize Jacobi's method and show how to construct a dynamical system from a Fuchsian equation of second order.

We first take a Fuchsian equation:

$$\frac{d^2y}{dz^2}+Q(z)y=0. \tag{1.1}$$

If (1.1) has regular singular points at $x=a_1,a_2,\cdots,a_m,\infty$, we have

$$Q(z)=\sum_{j=1}^{m}\frac{\alpha_j}{(z-\alpha_j)^2}+\frac{q(z)}{\Pi_{j=1}^{m}(z-a_j)},$$

where $q(z)$ is a polynomial with at most $(m-2)$-th degree. We can take $(m-1)$ constants $\beta_1,\cdots,\beta_{m-1}$ such that

$$\frac{q(z)}{\Pi_{j=1}^{m}(z-a_j)}=\sum_{j=1}^{m-1}\frac{\beta_j}{(z-\alpha_j)(z-\alpha_{j+1})}.$$

Let u and v be two independent solutions of (1.1). We will take

$$\tau=\frac{v}{u}$$

as a new independent variable. Since

$$v_{zz}u-u_{zz}v=0,$$

the Wronsikian v_zu-u_zv is independent of z. If we set $v_zu-u_zv=c$,

$$\frac{d\tau}{dz}=\frac{c}{u^2}.$$

If we change the variable z into τ in (1.1), we get

$$\frac{c}{u^2}\frac{d}{d\tau}\left\{\frac{c}{u^2}\frac{du}{d\tau}\right\}+Q(x)u=0.$$

Therefore

$$\frac{u_{\tau\tau}}{u}-2\left(\frac{u_\tau}{u}\right)^2+Q(z)\frac{u^4}{c^2}=0. \tag{1.2}$$

We set

$$X_0=\frac{u_\tau}{u},\qquad X_j=\frac{u_\tau}{u}-\frac{1}{z-a_j}\frac{u^2}{c},$$

for $j=1,2,\cdots,m$. By (1.2) we get

$$\frac{dX_0}{d\tau}-X_0^2+\sum_{j=1}^{m}\alpha_j(X_j-X_0)^2+\sum_{j=1}^{m-1}\beta_j(X_j-X_0)(X_{j+1}-X_0)=0, \tag{1.3a}$$

and

$$\begin{aligned}\frac{d}{d\tau}(X_j-X_0)&=\frac{1}{(z-a_j)^2}\frac{u^4}{c^2}-\frac{1}{z-a_j}\frac{2uu_\tau}{c}\\&=(X_j-X_0)^2+2(X_j-X_0)X_0=X_j^2-X_0^2\end{aligned} \tag{1.3b}$$

for $j=1,2,\cdots,m$.

Thus we obtain nonlinear equations for $X_0,X_1,\cdots,X_m$. But (1.3) is not independent since there are algebraic relations between unknown functions. Since

$$X_j-X_0=-\frac{1}{z-a_j}\frac{u^2}{c},\qquad X_j-X_k=-\frac{a_j-a_k}{(z-a_j)(z-a_k)}\frac{u^2}{c},$$

we have

$$\frac{X_j-X_k}{a_j-a_k}\frac{X_l-X_n}{a_l-a_n}=\frac{X_j-X_n}{a_j-a_n}\frac{X_l-X_k}{a_l-a_k}, \tag{1.4a}$$

if non-zero integers j, k, l and n are distinct each other. And if non-zero integers j, k and l are distinct each other, we have

$$(X_j-X_0)\frac{X_l-X_k}{a_l-a_k}=\frac{X_j-X_k}{a_j-a_k}(X_l-X_0). \tag{1.4b}$$

Since any four of unknown functions have a quadratic relations, the dynamical system (1.3) is at most third order.

For any four complex numbers z_0, z_1, z_2 and z_3, the anharmonic ratio is defined by

$$(z_0,z_1,z_2,z_3):=\frac{(z_2-z_0)}{(z_2-z_1)}\frac{(z_3-z_1)}{(z_3-z_0)}.$$

If we set $a_0=\infty$, (1.4) means that

$$(X_j,X_k,X_l,X_n)=(a_j,a_k,a_l,a_n),$$

for any j, k, l and m.

We will define the action of the group $SL(2,\boldsymbol{C})$ on the equation (1.3–4). For any $A=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\in SL(2,\boldsymbol{C})$ and any function $f(\tau)$, we set the action ρ by

$$(\rho(A)\cdot f)(\tau)=\frac{1}{(r\tau+s)^2}f\left(\frac{p\tau+q}{r\tau+s}\right)-\frac{r}{r\tau+s}.$$

Then ρ is a right action on the solution sapce of (1.3–4).

Theorem 1.1. *Let m is an integer such that $m\geq 2$. We take nonlinear equations on $X_0,X_1,\cdots,X_m$*

$$\begin{cases} \dfrac{dX_k}{d\tau}=X_k^2-\displaystyle\sum_{j=1}^{m}\alpha_j(X_j-X_0)^2-\sum_{j=1}^{m-1}\beta_j(X_j-X_0)(X_{j+1}-X_0), \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for any } k=0,1,\cdots,m, \\ (X_j,X_k,X_l,X_n)=(a_j,a_k,a_l,a_n), \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for any } j,\ k,\ l,\ n. \end{cases} \tag{1.5}$$

1) *The equation* (1.5) *can be solved in the following way. Let u and v be two solutions of*

$$\frac{d^2y}{dz^2}+\left\{\sum_{j=1}^{m}\frac{\alpha_j}{(z-a_j)^2}+\sum_{j=1}^{m-1}\frac{\beta_j}{(z-a_j)(z-a_{j+1})}\right\}y=0. \tag{1.6}$$

Since v_zu-vu_z *is independent of* z, *we set* $v_zu-vu_z=c$. *If we set* $\tau=\dfrac{v}{u}$,

$$X_0=\frac{u_\tau}{u},\qquad X_j=\frac{u_\tau}{u}-\frac{1}{z-a_j}\frac{u^2}{c},\qquad (j=1,2,\cdots,m)$$

satisify (1.5), *where we set* $a_0=\infty$.

2) (*SL*(2, **C**)-*symmetry*) *If* $X_j(\tau)$ $(j=0,1,\cdots,m)$ *satisify* (1.5), $(\rho(A)\cdot X_j)(\tau)$ *also satisfy* (1.5) *for any* $A\in SL(2,\mathbf{C})$.

3) (*initial value problem*) *We will solve* (1.5) *with the initial condition* $X_j(t_0)=x_j$ $(j=0,1,\cdots,m)$. *Since the second equation of* (1.5) *is an algebraic eqation,* x_j *should satisfy the second equation of* (1.5).

Let x_j $(j=0,1,\cdots,m)$ *be distinct complex numbers. Let u and v be independent solutions of* (1.6). *Then there exists a matrix* $A=\begin{pmatrix}p & q\\ r & s\end{pmatrix}\in SL(2,\mathbf{C})$ *such that*

$$Y_0(t)=\frac{y_t}{y},\qquad Y_j(t)=\frac{y_t}{y}-\frac{1}{z-a_j}\frac{y^2}{c},\qquad (j=1,2,\cdots,m)$$

satisfy (1.5) *with the initial condition* $Y_j(t_0)=x_j$, *where* $t=\dfrac{w}{y}$ *and*

$$w(z)=sv(z)-qu(z),$$
$$y(z)=-rv(z)+pu(z).$$

If $x_k=x_l$ *for some* k *and* l $(k\neq l)$, *then* $(m-1)$ *functions of* X_k*'s coincide. Therefore the nonlinear equation is reduced to*

$$\begin{cases}\dfrac{dX_j}{d\tau}=X_j^2-a(X_k-X_j)^2,\\[2ex] \dfrac{dX_k}{d\tau}=X_k^2-a(X_k-X_j)^2.\end{cases} \tag{1.7}$$

In this case, all of solutions are rational functions.

Proof. The assertion (1) is already proved and (2) can be shown by direct calculations. We will only show (3).

We will first define a complex number z_0 by

$$(x_1-x_0)(a_1-z_0)=(x_2-x_0)(a_2-z_0).$$

By the second equation of (1.5), we have

$$(x_j-x_0)(a_j-z_0)=(x_1-x_0)(a_1-z_0),$$

for any $j=1,2,\cdots,m$.

We will take constants p, q, r and s so that

$$-rv(z_0)+pu(z_0)=\sqrt{c(x_1-x_0)(a_1-z_0)}, \tag{1.8}$$

$$-r\frac{dv}{dz}(z_0)+p\frac{du}{dz}(z_0)=\frac{cx_0}{\sqrt{c(x_1-x_0)(a_1-z_0)}}, \tag{1.9}$$

$$sv(z_0)-qu(z_0)=\frac{t_0}{\sqrt{c(x_1-x_0)(a_1-z_0)}}, \tag{1.10}$$

$$ps-qr=1. \tag{1.11}$$

Since u and v are linearly independent, r and p are uniquely determined by (1.8) and (1.9). Since the right hand side of (1.8) is not zero, $-rv(z_0)+pu(z_0)\neq 0$. Hence s and q are uniquely determined by (1.10) and (1.11).

We set

$$w(z)=sv(z)-qu(z),$$
$$y(z)=-rv(z)+pu(z).$$

If we will take $t=\dfrac{w}{y}$, we have

$$\tau=\frac{v}{u}=\frac{pt+q}{rt+s}.$$

Moreover if we take

$$Y_0=\frac{y_t}{y}, \qquad Y_j=\frac{y_t}{y}-\frac{1}{z-a_j}\frac{y^2}{c},$$

$Y_0(t), Y_1(t), \cdots, Y_m(t)$ satisfy (1.5) as functions of t and $Y_j(t_0)=x_j$ for $j=0,\cdots,m$.

Let $X_j(\tau)$ $(j=0,1,\cdots,m)$ be functions defined in (1). We will show $Y_j(t)$ are written by $X_j(\tau)$.

$$Y_0(t)=\frac{d}{dt}\log(-rv+pu)$$

$$=\frac{1}{(rt+s)^2}\frac{d}{d\tau}\log u(-r\tau+p)$$

$$=\frac{1}{(rt+s)^2}X_0(\tau)-\frac{1}{(rt+s)^2}\frac{r}{-r\tau+p}$$

$$=\frac{1}{(rt+s)^2}X_0\left(\frac{pt+q}{rt+s}\right)-\frac{r}{rt+s}.$$

Since

$$Y_j(t)-Y_0(t)=-\frac{1}{z-a_j}\frac{(-rv+pu)^2}{c}$$

$$=-\frac{1}{z-a_j}\frac{u^2}{c}(r\tau+p)^2$$

$$=-\frac{1}{(rt+s)^2}\frac{1}{z-a_j}\frac{u^2}{c},$$

we obtain

$$Y_j(t)=\frac{1}{(rt+s)^2}X_0\left(\frac{pt+q}{rt+s}\right)-\frac{r}{rt+s}-\frac{1}{(rt+s)^2}\frac{1}{z-a_j}\frac{u^2}{c}$$

$$=\frac{1}{(rt+s)^2}X_j\left(\frac{pt+q}{rt+s}\right)-\frac{r}{rt+s}.$$

In the case where $x_k=x_l$ for some k and l $(k\neq l)$, $(m-1)$ of X_k's coincide by the second equation of (1.5). We will study solutions of (1.7) in the EXAMPLE 1 in the section 3. □

REMARK. By the $SL(2,C)$-symmetry, we get a three-parameter family of solutions of (1.5). This orbit gives generic solutions. □

It is well known that the quotient of any independent solutions of (1.1) satisfies the equation

$$\{\tau,z\}=2Q(z). \tag{1.12}$$

Here $\{\tau,z\}$ is the Schwarzian derivative

$$\{\tau,z\}=\frac{\tau'''}{\tau'}-\frac{3}{2}\left(\frac{\tau''}{\tau'}\right)^2,$$

where $'$ means a differentiation by z. The equation (1.12) is a complicated equation of third order. We will show that our new dynamical system (1.5) is equivalent to (1.12).

Proposition 1.2. *We can transform* (1.5) *to* (1.12).

Proof. We set $z_j=X_j-X_0$ for $j=1,2,\cdots,m$. By (1.3a) we get

$$\frac{dX_0}{d\tau}-X_0^2=-\sum_{j=1}^{m}\alpha_j z_j^2-\sum_{j=1}^{m-1}\beta_j z_j z_{j+1}. \tag{1.13}$$

By (1.3b) we have

$$\frac{d}{d\tau}\log z_j=X_j+X_0. \tag{1.14}$$

By the algebraic relations (1.4b) we have

$$z_j\frac{z_l-z_k}{a_l-a_k}=z_l\frac{z_j-z_k}{a_j-a_k}.$$

And by (1.4a) we have

$$(z_j,z_k,z_l,z_n)=(a_j,a_k,a_l,a_n), \tag{1.15}$$

which is true for any $j=0,1,2,\cdots,m$ if we set $z_0=0$ and $a_0=\infty$.

We define a new variable ξ by

$$z_1=-\frac{d}{d\tau}\log(\xi-b_1).$$

Here b_1 is any complex number. By (1.15)

$$\frac{d}{d\tau}\log\left(\frac{z_j}{z_1}\right)=X_j-X_1=z_j-z_1.$$

If we set $y_j=\dfrac{z_j}{z_1}$ for $j=2,3,\cdots,m$, we have

$$\frac{1}{y_j}\frac{dy_j}{d\tau}=z_1(y_j-1).$$

Therefore

$$\frac{d}{d\tau}\log\left(\frac{y_j-1}{y_j}\right)=\left(\frac{1}{y_j-1}-\frac{1}{y_j}\right)\frac{dy_j}{d\tau}=-\frac{d}{d\tau}\log(\xi-b_1). \tag{1.16}$$

Hence we can represent y_j by ξ. Integrating (1.16), we can take a constant b_j such that

$$\frac{y_j-1}{y_j}=\frac{b_j-b_1}{\xi-b_1}.$$

Therefore

$$y_j=\frac{\xi-b_1}{\xi-b_j},$$

which means $z_j=-\dfrac{\xi_\tau}{\xi-b_j}$. Since $z_0=0$, we should take $b_0=\infty$.

Since

$$z_j-z_k=-\xi_\tau\frac{b_j-b_k}{(\xi-b_j)(\xi-b_k)},$$

we obtain

$$(b_j,b_k,b_l,b_n)=(a_j,a_k,a_l,a_n),$$

by (1.15). Therefore there exists a linear fractional transformation T such that

$$b_j=Ta_j$$

for any $j=0,1,\cdots,m$. Since $a_0=b_0=\infty$, T is a linear transformation. Hence there are constants p and q such that $b_j=pa_j+q$ for any $j=1,2,\cdots,m$.

Since

$$\frac{dX_1}{d\tau}-\frac{dX_2}{d\tau}=X_1^2-X_2^2,$$

we have

$$\frac{d}{d\tau}\log(z_1-z_2)=X_1+X_2.$$

Hence

$$\begin{aligned}2X_0&=(X_0+X_1)+(X_0+X_2)-(X_1+X_2)\\&=\frac{d}{d\tau}\log z_1+\frac{d}{d\tau}\log z_2-\frac{d}{d\tau}\log(z_1-z_2)\end{aligned}$$

$$=\frac{d}{d\tau}\log\left(\frac{1}{z_2}-\frac{1}{z_1}\right)$$

$$=-\frac{d}{d\tau}\log\frac{b_2-b_1}{\xi_\tau}$$

$$=\frac{\xi_{\tau\tau}}{\xi_\tau}.$$

Therefore

$$\frac{dX_0}{d\tau}-X_0^2=\frac{1}{2}\{\xi,\tau\}.$$

By (1.13) we obtain

$$\{\xi,\tau\}=-2\left\{\sum_{j=1}^{m}\alpha_j z_j^2+\sum_{j=1}^{m-1}\beta_j z_j z_{j+1}\right\}$$

$$=-2(\xi_\tau)^2\left\{\sum_{j=1}^{m}\frac{\alpha_j}{(\xi-b_j)^2}+\sum_{j=1}^{m-1}\frac{\beta_j}{(\xi-b_j)(\xi-b_{j+1})}\right\}$$

$$=-2(\xi_\tau)^2\left\{\sum_{j=1}^{m}\frac{\alpha_j}{(\xi-pa_j-q)^2}+\sum_{j=1}^{m-1}\frac{\beta_j}{(\xi-pa_j-q)(\xi-pa_{j+1}-q)}\right\}$$

$$=-2\left(\frac{\xi_\tau}{p}\right)^2 Q\left(\frac{\xi-q}{p}\right).$$

Therefore if we set $\eta=\dfrac{\xi-q}{p}$, we obtain

$$\{\eta,\tau\}=-2(\eta_\tau)^2 Q(\eta),$$

which is equivalent to

$$\{\tau,\eta\}=2Q(\eta).$$

Thus we obtain (1.12). □

REMARK. The idea of the proof of Proposition 1.2 is due to Brioschi ([2]). Brioschi showed that Halphen's equation (0.3) is equivalent to the equation

$$\{x,\tau\}=-\frac{1}{2}\frac{1-x+x^2}{x^2(1-x)^2}\left(\frac{dx}{d\tau}\right)^2. \tag{1.17}$$

It is known that ratio τ of any two particular solutions of (0.1) satisfy (1.17). □

By Proposition 1.2, we can reconstruct a linear equations of the type (1.1) from our dynamical system (1.5), up to a linear transformation.

2. Case of irregular singularities

In the section 1, we study the Fuchsian case. But even if the starting linear equations may have irregular singularities, we can construct nonlinear equations. We will study general case in this section.

We take an equation:

$$\frac{d^2y}{dz^2}+Q(z)y=0, \tag{2.1}$$

whose singular points are $x=a_1,a_2,\cdots,a_m,\infty$. We can write

$$Q(z)=\sum_{j=1}^{m}\sum_{k=1}^{v_j}\frac{\alpha_{j,k}}{(z-a_j)^k}+\sum_{k=1}^{v_0}k\alpha_{0,k}z^{k-1},$$

where v_j is a natural number and $q(z)$ is a polynomial. Then we obtain nonlinear equations similar to (1.5).

Theorem 2.1. 1) *Let u and v be two solutions of*

$$\frac{d^2y}{dz^2}+\left\{\sum_{j=1}^{m}\sum_{k=1}^{v_j}\frac{\alpha_{j,k}}{(z-a_j)^k}+\sum_{k=1}^{v_0}k\alpha_{0,k}z^{k-1}\right\}y=0,$$

where $m\geq 2$. We assume $v_zu-vu_z=c$, where c is a non-zero constant. We set $\tau=\dfrac{v}{u}$, and

$$X_0^{(n)}=\frac{u_\tau}{u}+nz^{n-1}\frac{u^2}{c}\qquad(n=0,1,2,\cdots,v_0),$$

$$X_j^{(n)}=\frac{u_\tau}{u}-\frac{1}{(z-a_j)^{n+1}}\frac{u^2}{c}\qquad(1\leq j\leq m,\ n=0,1,\cdots,v_j-2).$$

Then $X_j^{(n)}$'s satisfy the following equations:

$$\frac{d}{d\tau}X_0^{(0)}=(X_0^{(0)})^2-\sum_{j=1}^{m}\sum_{k=2}^{v_j}\alpha_{j,k}(X_j^{(k-2)}-X_0^{(0)})(X_j^{(0)}-X_0^{(0)})$$

$$-\sum_{j=1}^{m}\alpha_{j,1}(X_j^{(0)}-X_0^{(0)})(X_0^{(1)}-X_0^{(0)})-\sum_{k=1}^{\nu_0}\alpha_{0,k}(X_0^{(k)}-X_0^{(0)})(X_0^{(1)}-X_0^{(0)}),$$

$$\frac{d}{d\tau}X_0^{(n)}=\frac{d}{d\tau}X_0^{(0)}+n(X_0^{(n-1)}-X_0^{(0)})(X_0^{(1)}-X_0^{(0)})+2(X_0^{(n)}-X_0^{(0)})X_0^{(0)},$$

for $n>0$,

$$\frac{d}{d\tau}X_j^{(n)}=\frac{d}{d\tau}X_0^{(0)}+(X_j^{(n)}-X_0^{(0)})((n+1)X_j^{(0)}-(n-1)X_0^{(0)}),$$

for $j>0$.

Moreover any four of $X_j^{(n)}$*'s satisfy an algebraic relation.*

2) *(*$SL(2,\boldsymbol{C})$*-symmetry) If* $X_j^{(n)}(\tau)$*'s are solutions of* (2.2), $(\rho(A)\cdot X_j^{(n)})(\tau)$ *also satisfy* (2.2) *for any* $A\in SL(2,\boldsymbol{C})$.

Proof. We can show that

$$\frac{u_{\tau\tau}}{u}-2\left(\frac{u_\tau}{u}\right)^2+Q(z)\frac{u^4}{c^2}=0, \tag{2.3}$$

in the same way as (1.2). The equation (2.3) can be written as

$$\frac{d}{d\tau}X_0^{(0)}=(X_0^{(0)})^2-\sum_{j=1}^{m}\sum_{k=2}^{\nu_j}\alpha_{j,k}(X_j^{(k-2)}-X_0^{(0)})(X_j^{(0)}-X_0^{(0)})$$

$$-\sum_{j=1}^{m}\alpha_{j,1}(X_j^{(0)}-X_0^{(0)})(X_0^{(1)}-X_0^{(0)})-\sum_{k=1}^{\nu_0}\alpha_{0,k}(X_0^{(k)}-X_0^{(0)})(X_0^{(1)}-X_0^{(0)}),$$

We have

$$\begin{aligned}\frac{d}{d\tau}(X_j^{(n)}-X_0^{(0)})&=\frac{n+1}{(z-a_j)^{n+2}}\frac{u^4}{c^2}-\frac{2}{(z-a_j)^{n+1}}\frac{u^2}{c}\frac{u_\tau}{u}\\&=(n+1)(X_j^{(n)}-X_0^{(0)})(X_j^{(0)}-X_0^{(0)})+2(X_j^{(n)}-X_0^{(0)})X_0^{(0)}\\&=(X_j^{(n)}-X_0^{(0)})((n+1)X_j^{(0)}-(n-1)X_0^{(0)}).\end{aligned}$$

And in the same way

$$\frac{d}{d\tau}(X_0^{(n)}-X_0^{(0)})=n(X_0^{(n-1)}-X_0^{(0)})(X_0^{(1)}-X_0^{(0)})+2(X_0^{(n)}-X_0^{(0)})X_0^{(0)}.$$

Let X, Y, Z and W be any for functions of $X_j^{(n)}$. Since the difference of any two functions has the form

$$f(z)\frac{u^2}{c}$$

for a rational function $f(z)$, both $A=\dfrac{X-Y}{X-Z}$ and $B=\dfrac{W-Y}{W-Z}$ are rational functions of z. Hence there is an algebraic relation between A and B:

$$g(A,B)=0. \tag{2.4}$$

By direct calculations we can show (2.2) is invariant under $SL(2,\boldsymbol{C})$-action on $X_j^{(n)}$. Since A and B are invariant under $SL(2,\boldsymbol{C})$-action in (2.4), the algebraic relation (2.4) is also invariant. □

3. Examples

We will show some examples of Jacobi's method for several Fuchsian equations.

EXAMPLE 1: the case of two singularities

The simplest case is that the singularities of the starting equation are 0 and ∞. In this case the linear equation is as follows:

$$\frac{d^2y}{dz^2}=\frac{a}{z^2}y. \tag{3.1}$$

If $a\neq -\dfrac{1}{4}$, solutions of (3.1) are

$$u=z^{\alpha},\ v=z^{\beta},$$

where α and β are distinct solutions of

$$t(t-1)=a.$$

If $a=-\dfrac{1}{4}$, solutions of (3.1) are

$$u=z^{1/2},\ v=z^{1/2}\log z.$$

In any case, we set $\tau=\dfrac{v}{u}$ and

$$X=\frac{u_\tau}{u},\quad Y=\frac{u_\tau}{u}-\frac{1}{z}\frac{dz}{d\tau}.$$

Then the deduced nonlinear equations are

$$\left\{\begin{array}{l} \dfrac{dX}{d\tau}=X^2+a(Y-X)^2, \\ \dfrac{dY}{d\tau}=Y^2+a(Y-X)^2, \end{array}\right. \tag{3.2}$$

which is a second-order equation.

Proposition 3.1. *If* $a\neq -\dfrac{1}{4}$, *the solution space of* (3.2) *is* $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$. *If* $a=-\dfrac{1}{4}$, *the solution space of* (3.2) *is the line bundle of degree two on* $\boldsymbol{P}^1(\boldsymbol{C})$. *Therefore the solution space is compact in the former case, and is non-compact in the latter case.*

Proof. We set

$$P=\alpha X+\beta Y,$$
$$Q=\beta X+\alpha Y.$$

Then we have

$$\frac{d}{d\tau}P=P^2, \quad \frac{d}{d\tau}Q=Q^2. \tag{3.3}$$

Hence if $a\neq -\dfrac{1}{4}$, (3.2) is equivalent to (3.3). If $a=-\dfrac{1}{4}$, $P=Q$ and (3.2) is equivalent to

$$\frac{dP}{d\tau}=P^2, \quad \frac{dX}{d\tau}=2PX-P^2. \tag{3.4}$$

The general solutions of (3.3) are given by

$$P=-\frac{c}{c\tau+d}, \qquad Q=-\frac{p}{p\tau+q}.$$

Hence the solution space is $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$.

The general solutions of (3.4) are given by

$$P=-\frac{c}{c\tau+d}, \qquad X=\frac{p}{(c\tau+d)^2}-\frac{c}{c\tau+d}.$$

Since p can be considered as a section of the line bundle of degree two on $\boldsymbol{P}^1(\boldsymbol{C})$, the solution space is the total space of the line bundle. □

EXAMPLE 2: hypergeometric case ([8])

We take the hypergeometric equation as the starting equation:

$$z(1-z)\frac{d^2u}{dz^2}+(\gamma-(\alpha+\beta+1)z)\frac{du}{dz}-\alpha\beta u=0.$$

If we set

$$y=z^{\gamma/2}(1-z)^{(\alpha+\beta-\gamma+1)/2}u,$$

y satisfies the following equation:

$$\frac{d^2y}{dz^2}=\left\{\frac{a}{z^2}+\frac{b}{(z-1)^2}+\frac{c}{z(z-1)}\right\}y, \tag{3.5}$$

where

$$a=\frac{\gamma(\gamma-2)}{4},\quad b=\frac{(\alpha+\beta-\gamma)^2-1}{4},\quad c=\frac{\alpha\gamma+\beta\gamma-2\alpha\beta-\gamma^2+\gamma}{2}.$$

Let τ be the ratio of two solutions of (3.5). We set

$$X=\frac{d}{d\tau}\log y,\quad Y=\frac{d}{d\tau}\log\frac{y}{z},\quad Z=\frac{d}{d\tau}\log\frac{y}{z-1}.$$

Then X, Y and Z satisfy the following equations:

$$\begin{cases}\dfrac{dX}{d\tau}=X^2+a(X-Y)^2+b(X-Z)^2+c(X-Y)(X-Z),\\[2ex] \dfrac{dY}{d\tau}=Y^2+a(X-Y)^2+b(X-Z)^2+c(X-Y)(X-Z),\\[2ex] \dfrac{dZ}{d\tau}=Z^2+a(X-Y)^2+b(X-Z)^2+c(X-Y)(X-Z).\end{cases}$$

Since

$$(X-Y)(X-Z)=\frac{1}{2}(X-Y)^2+\frac{1}{2}(X-Z)^2-\frac{1}{2}(Y-Z)^2,$$

we have

$$\text{(3.6)}\qquad \begin{cases} \dfrac{dX}{d\tau}=X^2+\left(a+\dfrac{c}{2}\right)(X-Y)^2+\left(b+\dfrac{c}{2}\right)(X-Z)^2+\dfrac{c}{2}(Y-Z)^2, \\ \dfrac{dY}{d\tau}=Y^2+\left(a+\dfrac{c}{2}\right)(X-Y)^2+\left(b+\dfrac{c}{2}\right)(X-Z)^2+\dfrac{c}{2}(Y-Z)^2, \\ \dfrac{dZ}{d\tau}=Z^2+\left(a+\dfrac{c}{2}\right)(X-Y)^2+\left(b+\dfrac{c}{2}\right)(X-Z)^2+\dfrac{c}{2}(Y-Z)^2. \end{cases}$$

This system (3.6) is different from the equation obtained by Halphen ([8]). But by a suitable linear transformation between X, Y and Z, we can transform (3.6) into Halphen's equations.

EXAMPLE 3: Chazy's case ([3])

J. Chazy studied some class of third order equations in his study of Painlevé property. As the staring linear equations, he took

$$\text{(3.7)}\qquad x(1-x)\frac{d^2y}{dx^2}+\left(\frac{1}{2}-\frac{7}{6}x\right)\frac{dy}{dx}-\frac{1}{4}\left(\frac{1}{36}-\frac{1}{n^2}\right)y=0,$$

in the case n is an integer larger than six, or $n=\infty$. This is a special case of EXAMPLE 2. We have

$$\alpha=\frac{1}{2}\left(\frac{1}{6}+\frac{1}{n}\right),\quad \beta=\frac{1}{2}\left(\frac{1}{6}-\frac{1}{n}\right),\quad \gamma=\frac{1}{2}.$$

Hence we have

$$a=-\frac{3}{16},\quad b=-\frac{2}{9},\quad c=\frac{36+23n^2}{144n^2}.$$

We can rewrite (3.6) into a single equation. In order to simplify coefficients we take $W=6X$. If $n=\infty$, we have

$$\frac{d^3W}{d\tau^3}=2W\frac{d^2W}{d\tau^2}-3\left(\frac{dW}{d\tau}\right)^2.$$

And if $n>6$, we have

$$\frac{d^3W}{d\tau^3}=2W\frac{d^2W}{d\tau^2}-3\left(\frac{dW}{d\tau}\right)^2+\frac{4}{36-n^2}\left(6\frac{dW}{d\tau}-W^2\right)^2.$$

EXAMPLE 4: Jacobi-Halphen's case ([9], [7])

Halphen's equation (0.3) is the starting point of our investigation ([11]). Halphen deduced (0.3) by using an addition formula of Weierstrass' ζ-function ([7]). In [11], the author deduced (0.2) by the same method as Halphen using more familiar notation. We will show that (0.3) can be deduced as a special case of EXAMPLE 2. This method is essentially the same as Jacobi's method ([9]). If we set

$$\alpha=\frac{1}{2},\quad \beta=\frac{1}{2},\quad \gamma=1$$

in EXAMPLE 2, we have

$$a=-\frac{1}{4},\quad b=-\frac{1}{4},\quad c=\frac{1}{4}.$$

Therefore in this case (3.6) is as follows:

$$\text{(3.8)}\qquad \begin{cases} \dfrac{dX}{d\tau}=X^2-\dfrac{1}{8}(X-Y)^2-\dfrac{1}{8}(X-Z)^2-\dfrac{1}{8}(Y-Z)^2, \\ \dfrac{dY}{d\tau}=Y^2-\dfrac{1}{8}(X-Y)^2-\dfrac{1}{8}(X-Z)^2-\dfrac{1}{8}(Y-Z)^2, \\ \dfrac{dZ}{d\tau}=Z^2-\dfrac{1}{8}(X-Y)^2-\dfrac{1}{8}(X-Z)^2-\dfrac{1}{8}(Y-Z)^2. \end{cases}$$

If we set

$$2X_1=Y+Z,$$
$$2X_2=X+Z,$$
$$2X_3=X+Y,$$

we obtain

$$\text{(3.9)}\qquad \begin{cases} \dfrac{d}{d\tau}(X_2+X_3)=2X_2X_3, \\ \dfrac{d}{d\tau}(X_3+X_1)=2X_3X_1, \\ \dfrac{d}{d\tau}(X_1+X_2)=2X_1X_2, \end{cases}$$

which is the same as Halphen's equation (0.3).

We will show that logarithmic derivatives of theta constants satisfy (3.9). We will take two independent solutions of (3.5)

$$y=z^{1/2}(1-z)^{1/2}F(\frac{1}{2},\frac{1}{2},1;z),$$

$$w=iz^{1/2}(1-z)^{1/2}F(\frac{1}{2},\frac{1}{2},1;1-z).$$

We have

$$K=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$=F(\frac{1}{2},\frac{1}{2},1;k^2)=\frac{\pi}{2}\theta_3^2,$$

$$K'=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}},$$

by the theory of elliptic integrals (See, e.g., [10]). Hence if we set

$$z^{1/2}=k=\frac{\theta_2^2}{\theta_3^2},\quad (1-z)^{1/2}=k'=\frac{\theta_4^2}{\theta_3^2},$$

and take

$$\tau=i\frac{K'}{K}=\frac{w}{y},$$

we obtain

$$Y+Z=2\frac{d}{d\tau}\log K=2\frac{d}{d\tau}\log\theta_3^2,$$

$$X+Z=2\frac{d}{d\tau}\log kK=2\frac{d}{d\tau}\log\theta_2^2,$$

$$X+Y=2\frac{d}{d\tau}\log k'K=2\frac{d}{d\tau}\log\theta_4^2.$$

Therefore

$$X_1=2\frac{d}{d\tau}\log\theta_3,$$

$$X_2=2\frac{d}{d\tau}\log\theta_2,$$

$$X_3=2\frac{d}{d\tau}\log\theta_4.$$

EXAMPLE 5: modular forms with level three

The solutions of Halphen's equation are written by theta null values, i.e., modular forms with level two. In the same as Halphen's equation, we can obtain nonlinear equations which are satisfied by modular forms with other modular groups. In this example we will show the nonlinear equations whose solutions are written by modular forms with level three. The detail is written in [12].

We take the starting linear equation

$$(1-a^3)\frac{d^2\kappa}{da^2}-3a^2\frac{d\kappa}{da}-a\kappa=0, \tag{3.10}$$

which is a Picard-Fuchs equation of Hasse pencil

$$x^3+y^3+1-3axy=0.$$

The solutions of (3.10) is given by elliptic integrals on Hesse pencil:

$$\kappa=\int_0^{-1}\frac{dx}{y^2-ax},$$

$$\kappa'=\int_0^{-1}\frac{dx}{y^2-a\omega x},$$

$$\kappa''=\int_0^{-1}\frac{dx}{y^2-a\omega^2 x}.$$

Here κ, κ' and κ'' are modular forms with level three and have a linear relation

$$\kappa+\kappa'+\kappa''=0,$$

if a is sufficiently small. Note that a is a modular function of level three.

We take

$$\tau=\frac{\omega\kappa'-\kappa}{\kappa},$$

and

$$W=\frac{d}{d\tau}\log\kappa,$$

$$X=\frac{d}{d\tau}\log\{(a-1)\kappa\},$$

$$Y=\frac{d}{d\tau}\log\{(a-\omega)\kappa\},$$

$$Z=\frac{d}{d\tau}\log\{(a-\omega^2)\kappa\}.$$

Then W, X, Y, Z satisfy the following equations.

$$\begin{cases}\dfrac{d}{d\tau}(W+X+Y)=WX+XY+YW,\\ \dfrac{d}{d\tau}(W+Y+Z)=WY+YZ+ZW,\\ \dfrac{d}{d\tau}(W+X+Z)=WX+XZ+ZW,\\ \dfrac{d}{d\tau}(X+Y+Z)=XY+YZ+ZX,\\ e^{\frac{4}{3}\pi i}(XZ+YW)+e^{\frac{2}{3}\pi i}(XW+YZ)+(XY+ZW)=0.\end{cases}$$

REMARK. We can find nonlinear equations which are satisfied by modular forms with general modular groups by the same method as level two and three. The strategy is as follows:

1) Find elliptic modular surfaces,
2) Calculate Picard-Fuchs equations,
3) Represent elliptic integrals and moduli parameters as modular forms,
4) Apply Jacobi's method.

From now on we will consider non-Fuchsian cases. In the followings, τ is a ratio of two solutions of starting linear equations.

EXAMPLE 6: Airy's equation

We will start from Airy's equations:

$$\frac{d^2y}{dz^2}-zy=0.$$

If we set

$$X=\frac{d}{d\tau}\log y,\quad Y=\frac{d}{d\tau}\log(y\,e^z),\quad Z=\frac{d}{d\tau}\log\left(y\exp\left(\frac{1}{2}z^2\right)\right),$$

we obtain

$$\begin{cases}\dfrac{dX}{d\tau}=X^2+(Y-X)(Z-X),\\[2ex]\dfrac{dY}{d\tau}=Y^2+(Y-X)(Z-Y),\\[2ex]\dfrac{dZ}{d\tau}=Z^2+(Y-X)^2+(Z-X)(Y-Z).\end{cases}$$

EXAMPLE 7: Whittaker's equation

We start from Whittaker's equation:

$$\frac{d^2y}{dz^2}-\left(\frac{1}{4}+\frac{k}{z}+\frac{l}{z^2}\right)y=0.$$

If we set

$$X=\frac{d}{d\tau}\log y,\quad Y=\frac{d}{d\tau}\log\frac{y}{z},\quad Z=\frac{d}{d\tau}\log(y\,e^z),$$

we obtain

$$\begin{cases}\dfrac{dX}{d\tau}=X^2+\dfrac{1}{4}(Z-X)^2+k(Y-X)(Z-X)+l(Y-X)^2,\\[2ex]\dfrac{dY}{d\tau}=Y^2+\dfrac{1}{4}(Z-X)^2+k(Y-X)(Z-X)+l(Y-X)^2,\\[2ex]\dfrac{dZ}{d\tau}=Z^2-\dfrac{3}{4}(Z-X)^2+k(Y-X)(Z-X)+l(Y-X)^2.\end{cases}$$

EXAMPLE 8: Weber's equation

We start from Weber's equations:

$$\frac{d^2y}{dz^2}-(az+bz^2)y=0.$$

If we set

$$X=\frac{d}{d\tau}\log y,\quad Y=\frac{d}{d\tau}\log(y\,e^z),\quad Z=\frac{d}{d\tau}\log\left(y\exp\left(\frac{1}{2}z^2\right)\right),$$

we obtain

$$\begin{cases}\dfrac{dX}{d\tau}=X^2+a(Y-X)^2+b(Y-X)(Z-X),\\[2ex] \dfrac{dY}{d\tau}=Y^2+(a-1)(Y-X)^2+b(Y-X)(Z-X),\\[2ex] \dfrac{dZ}{d\tau}=Z^2+(a+1)(Y-X)^2+b(Y-X)(Z-X)-(Z-X)^2.\end{cases}$$

EXAMPLE 9: Lamé's equations

Finally we start from Lamé's equations:

$$\frac{d^2y}{dz^2}-(A\wp(z)+B)y=0.$$

Here $\wp(z)$ is Weierstrass' $\wp$-function, which satisfies the following equation:

$$\left(\frac{d\wp(z)}{dz}\right)^2=4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3).$$

Lamé's equation can be considered as the equation with four regular singularities, when we take $\wp(z)$ as a new independent variable. If we set

$$X_0=\frac{d}{d\tau}\log y,\quad X_j=\frac{d}{d\tau}\log(y(\wp(z)-e_j)),\quad (j=1,2,3)$$

we obtain

$$\begin{cases}\dfrac{dX_0}{d\tau}=X_0^2+p(X_1-X_0)(X_2-X_0)+q(X_1-X_0)(X_3-X_0),\\[2ex] \dfrac{d}{d\tau}(X_1-X_0)=(X_1-X_0)\left(\dfrac{3}{2}X_0-\dfrac{1}{2}X_1+\dfrac{1}{2}X_2+\dfrac{1}{2}X_3\right),\\[2ex] \dfrac{d}{d\tau}(X_1-X_0)=(X_2-X_0)\left(\dfrac{3}{2}X_0-\dfrac{1}{2}X_2-\dfrac{1}{2}X_2+\dfrac{1}{2}X_3\right),\end{cases}$$

$$\begin{cases} \dfrac{d}{d\tau}(X_1-X_0)=(X_3-X_0)\left(\dfrac{3}{2}X_0+\dfrac{1}{2}X_3+\dfrac{1}{2}X_2-\dfrac{1}{2}X_3\right), \\ \dfrac{(X_2-X_0)(X_1-X_3)}{e_1-e_3}=\dfrac{(X_2-X_3)(X_1-X_0)}{e_2-e_3}. \end{cases}$$

Here p and q are defined by the equations

$$4(p+q)+A=0, \quad 4pe_3+4qe_2-B=0.$$

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