

IDEAL TETRAHEDRAL DECOMPOSITIONS OF HYPERBOLIC 3-MANIFOLDS

HAN YOSHIDA

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1. Introduction

Since W. Thurston investigated cusped hyperbolic 3-manifolds by decomposing them into ideal tetrahedra [2], the method has become an indispensable tool for the researchers of hyperbolic 3-manifolds. Although it is not known whether a noncompact hyperbolic 3-manifold of finite volume always admits a decomposition into ideal tetrahedra, most of us believe the following:

Conjecture. *Every noncompact hyperbolic 3-manifold of finite volume admits a decomposition into convex ideal tetrahedra.*

In [1], Epstein and Penner have shown that every noncompact hyperbolic 3-manifold of finite volume has a canonical decomposition into convex ideal polyhedra. Therefore, in order to prove the above conjecture it suffices to show that every hyperbolic 3-manifold obtained by glueing ideal polyhedra admits a decomposition into ideal tetrahedra. Wada recently proved that if a noncompact hyperbolic 3-manifold M consists of one convex ideal polyhedron then M can be decomposed into ideal tetrahedra [3].

In this paper, we show the following theorem.

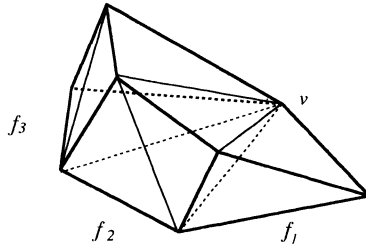
Main Theorem. *Suppose that a noncompact hyperbolic 3-manifold M is obtained by glueing two convex ideal polyhedra P_1 and P_2 in such a way that every face of P_1 is pasted with a face of P_2 . Then M can be decomposed into ideal tetrahedra.*

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2. Cone Decomposition

Let P be a polyhedron and v a vertex of P . Let f_1, \dots, f_r denote the faces of P not incident with the vertex v . Now take a triangulation τ of $\bigcup_{i=1}^r f_i$ without adding a new vertex. Then a triangulation $K(\tau)$ of P can be defined as follows. The sets $K^i(\tau)$ of i -simplices of $K(\tau)$ are given by

$$\begin{aligned} K^3(\tau) &= \{\langle v, \sigma_2 \rangle \mid \sigma_2 \text{ is a 2-simplex of } \tau\}, \\ K^2(\tau) &= \{\langle v, \sigma_1 \rangle \mid \sigma_1 \text{ is a 1-simplex of } \tau\} \cup \{2\text{-simplices of } \tau\}, \\ K^1(\tau) &= \{\langle v, \sigma_0 \rangle \mid \sigma_0 \text{ is a 0-simplex of } \tau\} \cup \{1\text{-simplices of } \tau\}, \\ K^0(\tau) &= \{\text{vertices of } P\}. \end{aligned}$$



This polyhedron is divided into 5 tetrahedra.

figure 1

We call such a triangulation $K(\tau)$ a *cone decomposition* of P from the vertex v .

We say that M admits a decomposition into ideal polyhedra (tetrahedra) if M is obtained by glueing together the faces of ideal polyhedra (tetrahedra). Note that taking cone decompositions of P_1 and P_2 from arbitrary vertices dose not necessary give a decomposition of M , since the triangulations induced by those of P_1 and P_2 on identified faces might not agree.

3. Good Vertex

By V_i , E_i , and F_i , we denote the sets of vertices, edges and faces of P_i respectively. We decompose $F_i = \bigcup_{n \geq 3} F_{n,i}$ where $F_{n,i}$ is the set of n -gons of F_i ($i=1, 2$). Furthermore we decompose $F_{n,2} = F'_{n,2} \cup F''_{n,2}$ where $F'_{n,2}$ is the set of n -gons glued to faces of P_1 and $F''_{n,2}$ is the set of n -gons glued to faces of P_2 .

To prove the main theorem, it suffices to show that we can so triangulate polyhedra P_1 and P_2 without adding a new vertex that triangulations induced on each pair of faces identified under the glueing map agree. We say that $(v_1, v_2) \in$

$V_1 \times V_2$ is a *good pair* if some cone decomposition of P_1 from v_1 and some cone decomposition of P_2 from v_2 agree on each pair of faces identified under the glueing map. We call $v_1 \in V_1$ a *good vertex* if there exists a vertex $v_2 \in V_2$ such that (v_1, v_2) is a good pair. Otherwise, we say that v_1 is a *bad vertex*. Therefore it suffices to show the existence of a good vertex.

In the following we will derive a sufficient condition for the existence of a good vertex.

Fix a vertex $v \in V_1$. Let $A(v)$ be the set of vertices $v_2 \in V_2$ such that for any cone decomposition of P_1 from v and any cone decomposition of P_2 from v_2 , the triangulations induced by those of P_1 and P_2 do not agree on some faces $f \in F_1$ and $f' \in F_2$ when glueing P_1 and P_2 . Then for $w \in A(v)$, (v, w) is not a good pair. Let B be the set of vertices $v_2 \in V_2$ such that for any cone decomposition of P_2 from v_2 , the triangulations induced by that of P_2 do not agree on some faces $f_2, f'_2 \in F_2''$ when pasted. Thus if $v_2 \in B$, (v, v_2) is not a good pair for every $v \in V_1$.

Let v be a bad vertex. By the definition, $(v, v_2) \in V_1 \times V_2$ is not a good pair for all $v_2 \in V_2$. Then we have

$$(3.1) \quad |A(v)| + |B| \geq |V_2|.$$

First we consider about $A(v)$. Suppose $w \in A(v)$ and take cone decompositions of P_1 and P_2 from the vertices v and w respectively. Then there exist faces $f_1 \in F_1$ and $f_2 \in F_2$ on which the triangulations induced by those of P_1 and P_2 do not agree. If a face f_1 (resp. f_2) is not incident with a vertex v (resp. w), we can replace the triangulation of f_1 (resp. f_2) so that it matches that of f_2 (resp. f_1) under the glueing map. The number of vertices of a face f is called *degree of f* , and denoted by $\deg f$. We may assume that $\deg f_1 = \deg f_2 \geq 4$ and that the faces f_1 and f_2 are incident with the vertices v and w respectively. Since the triangulations do not agree on f_1 and f_2 , if $\deg f_1 = \deg f_2 \geq 5$, w dose not correspond to v when glueing f_1 and f_2 . Suppose that the faces f_1 and f_2 are quadrilaterals. Let u be the vertex of f_1 which is not adjacent to v . The vertex w does not correspond to neither v nor u when glueing f_1 and f_2 . (See figure 2)

Then we have

$$(3.2) \quad |A(v)| \leq \sum_{v \in f} d(f)$$

where

$$d(f) = \begin{cases} \deg f - 1 & \text{if } \deg f \geq 5, \\ 2 & \text{if } \deg f = 4, \\ 0 & \text{if } \deg f = 3. \end{cases}$$

Next we consider about B . Suppose that we take a cone decomposition of P_2

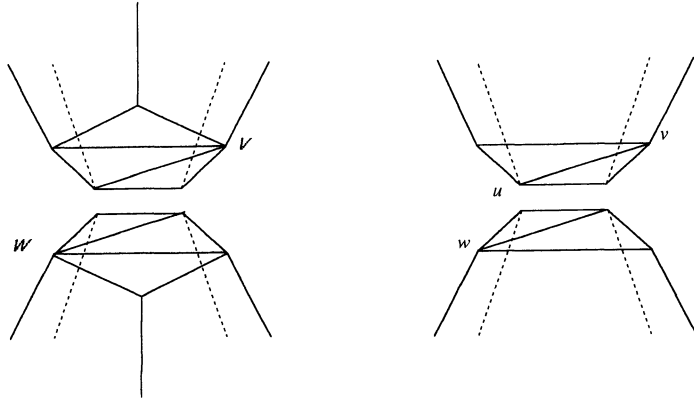


figure 2

from a vertex $w \in B$. Then there exist a pair of faces $f_2, f'_2 \in F''_2$ on which the triangulations induced by that of P_2 do not agree. We can arbitrarily choose the triangulations on the faces which are not incident with w . Hence we may assume that a pair of faces $f_2, f'_2 \in F''_2$ satisfy the following conditions ;

- (1) $w \in f_2 \cap f'_2$,
- (2) f_2 is glued to f'_2 ,
- (3) $\deg f_2 = \deg f'_2 \geq 4$.

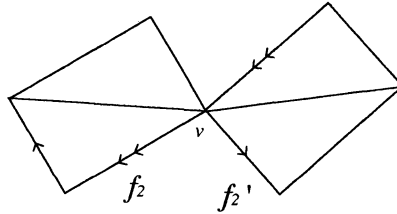


figure 3

Since the faces f_2 and f'_2 share at most two vertices, we get

$$(3.3) \quad |B| \leq \sum_{n \geq 4} |F''_{n,2}|.$$

Therefore by (3.1), (3.2) and (3.3), we obtain

$$(3.4) \quad \sum_{v \in f} d(f) + \sum_{n \geq 4} |F''_{n,2}| \geq |V_2|$$

for every bad vertex v .

Using this result, we have the following lemma.

Lemma 1. *If v is a bad vertex, then*

$$\sum_{v \in f} d(f) \geq |V_1|.$$

Proof. We have the following formulas,

$$\begin{aligned} |V_2| - |E_2| + |F_2| &= 2, \\ |E_2| &= \frac{1}{2} \sum_{n \geq 3} F_{n,2}, \\ |F'_{n,2}| &= |F_{n,1}|. \end{aligned}$$

As we put $F_{n,2} = F'_{n,2} \cup F''_{n,2}$, we have

$$|F_{n,2}| = |F'_{n,2}| + |F''_{n,2}|.$$

From the above formulas, we get

$$(3.5) \quad |V_2| = 2 + \sum_{n \geq 3} \left(\frac{n}{2} - 1 \right) |F_{n,1}| + \sum_{n \geq 3} \left(\frac{n}{2} - 1 \right) |F''_{n,2}|.$$

Similarly we can get

$$(3.6) \quad |V_1| = 2 + \sum_{n \geq 3} \left(\frac{n}{2} - 1 \right) |F_{n,1}|.$$

From (3.4) and (3.5), we obtain

$$(3.7) \quad \begin{aligned} \sum_{v \in f} d(f) &\geq 2 + \sum_{n \geq 3} \left(\frac{n}{2} - 1 \right) |F_{n,1}| + \sum_{n \geq 4} \left(\frac{n}{2} - 2 \right) |F''_{n,2}| + \frac{1}{2} |F''_{3,2}| \\ &\geq 2 + \sum_{n \geq 3} \left(\frac{n}{2} - 1 \right) |F_{n,1}| \end{aligned}$$

for every bad vertex v . The inequality of Lemma 1 follows from (3.6) and (3.7).

The following lemma gives the sufficient condition for the existence of a good vertex, which we have been seeking.

Lemma 2. *If*

$$|V_1|^2 - \sum_{f \in F_1} d(f) \deg f > 0,$$

there exists a good vertex.

Proof. If no vertex of P_1 is a good vertex, i.e. if all vertices are bad vertices, then we obtain

$$|V_1|^2 - \sum_{v \in V_1} \sum_{v \in f} d(f) \leq 0$$

by summing the inequality of Lemma 1 over all vertices $v \in V_1$. Since

$$\sum_{v \in V_1} \sum_{v \in f} d(f) = \sum_{f \in F_1} d(f) \deg f,$$

we have Lemma 2.

We prepare the following lemma in order to prove the main theorem. Let $M = \max\{n \in \mathcal{N} ; |F_{n,1}| \neq 0\}$.

Lemma 3. *If one of the following conditions holds,*

- (1) $M=4$ and $|V_1| \geq 6$,
- (2) $M=5$ and $|V_1| \geq 12$,
- (3) $M \geq 6$ and $|V_1| \geq 2M+1$,

then there exists a good vertex.

Proof. Using the formula (3.6), we have

$$\begin{aligned} |V_1|^2 - \sum_{f \in F_1} d(f) \deg f &= |V_1| \left(2 + \sum_{n \geq 3} \left(\frac{n}{2} - 1 \right) |F_{n,1}| \right) - \sum_{f \in F_1} d(f) \deg f \\ &= 2|V_1| + \sum |V_1| \left(\frac{n}{2} - 1 \right) |F_{n,1}| \\ &\quad - \left(\sum_{n \geq 5} n(n-1) |F_{n,1}| + 8|F_{4,1}| \right) \\ &= 2 \left(2 + \sum_{n \geq 3} \left(\frac{n}{2} - 1 \right) |F_{n,1}| \right) + \sum_{n \geq 3} |V_1| \left(\frac{n}{2} - 1 \right) |F_{n,1}| \\ &\quad - \sum_{n \geq 5} (n^2 - n) |F_{n,1}| - 8|F_{4,1}| \\ &= 4 + \left(1 + \frac{1}{2} |V_1| \right) |F_{3,1}| + (|V_1| - 6) |F_{4,1}| \\ &\quad + \sum_{n \geq 5} \left(|V_1| \left(\frac{n}{2} - 1 \right) - n^2 + 2n - 2 \right) |F_{n,1}|. \end{aligned}$$

By the definition of M , we have $|F_{n,1}| = 0$ for $n > M$. It is not hard to see that the coefficient of $|F_{n,1}|$ in the above formula is nonnegative for $n \leq M$ in each case of the assumption. Hence we obtain

$$|V_1|^2 - \sum_{f \in F_1} d(f) \deg(f) > 0.$$

Therefore there exists a good vertex by Lemma 2.

4. Proof of Main Theorem

We divide the proof into several cases.

Case 1. Suppose that $M=3$. Then all vertices of P_1 are good vertices.

Case 2. Suppose that $M=4$. Clearly $|V_1| \geq 5$. If $|V_1| \geq 6$, there exists a good vertex

by Lemma 3. If $|V_1|=5$, then the faces of P_1 consist of one quadrilateral and four triangles, and the inequality of Lemma 2 holds. Therefore there exists a good vertex.

Case 3. Suppose that $M=5$. Assume that all vertices of P_1 were bad vertices. By Lemma 2, we have

$$(4.1) \quad 20|F_{5,1}| + |F_{4,1}| \geq |V_1|^2.$$

We have the following conditions about P_1 . By the formula (3.6),

$$(4.2) \quad |V_1| = \sum_{n=3}^5 \left(\frac{n}{2} - 1 \right) |F_{n,1}| + 2.$$

Since a pentagon is adjacent to at least five faces,

$$(4.3) \quad \sum_{n=3}^5 |F_{n,1}| \geq 6.$$

Because every vertex is incident with at least three faces,

$$(4.4) \quad \sum_{n=3}^5 n |F_{n,1}| \geq 3 |V_1|.$$

If $|V_1| \geq 12$, there exists a good vertex by Lemma 3. Therefore we may assume $6 \leq |V_1| \leq 11$. Furthermore we may assume that $|F_{4,1}|$ and $|F_{3,1}|$ are nonnegative integers and that $|F_{5,1}|$ is a positive integer. However there exists no possible combination $(|V_1|, |F_{3,1}|, |F_{4,1}|, |F_{5,1}|)$ which satisfies (4.1) (4.2) (4.3) and (4.4). Therefore P_1 must have a good vertex.

Case 4. Suppose that $M \geq 6$. Let f_0 be one of the M -gonal faces of P_1 . Assuming that all vertices of f_0 are bad vertices, we will show

$$|V_1| \geq 2M + 1$$

step by step. Then it follows that there exists a good vertex somewhere else by Lemma 3.

Note that, we have

$$(4.5) \quad |V_1| \geq 2 + \left(\frac{\deg f_0}{2} - 1 \right) + \sum_{f \text{ is adjacent to } f_0} \left(\frac{\deg f}{2} - 1 \right)$$

by (3.6). We denote the last term of right side of (4.5) by $V(f_0)$.

Step 1. In this step, we show $|V_1| \geq 5M/4 + 1$.

Let v be a vertex of f_0 . By the assumption, v is a bad vertex and $|V_1| \geq M + 1$. Thus by Lemma 1, we have

$$\sum_{v \in f} d(f) \geq M + 1.$$

We denote by $I(v)$ the set of faces other than f_0 which are incident with v . Since $\sum_{v \in f} d(f) = \sum_{f \in I(v)} d(f) + M - 1$, it follows

$$\sum_{f \in I(v)} d(f) \geq 2.$$

This implies that $I(v)$ contains a face of degree $n (\geq 4)$. Since every vertex is incident with at least three faces, $V(f_0)$ is minimal if a vertex v is incident with a triangle, a quadrilateral and f_0 for all $v \in f_0$.

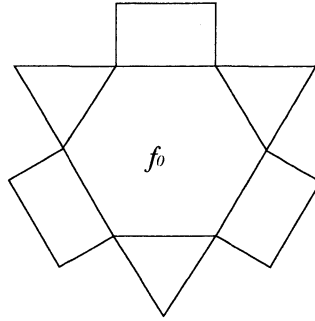


figure 4

Hence by (4.5), we get

$$|V_1| \geq \frac{5}{4}M + 1.$$

Step 2. In this step, we show $|V_1| \geq 3M/2 + 1$.

By the assumption, v is a bad vertex, and $|V_1| \geq 5M/4 + 1$ from Step 1. By Lemma 1, we have

$$\sum_{v \in f} d(f) \geq \frac{5}{4}M + 1.$$

Since we denote by $I(v)$ the set of faces other than f_0 which are incident with v , and $\sum_{v \in f} d(f) = \sum_{f \in I(v)} d(f) + M - 1$, it follows

$$\sum_{f \in I(v)} d(f) \geq \frac{1}{4}M + 2.$$

By the assumption, $M \geq 6$, then

$$\sum_{f \in I(v)} d(f) \geq \frac{5}{2}.$$

This implies that

- (i) $I(v)$ contains a face of degree $n (\geq 5)$, or

(ii) $I(v)$ contains two quadrilaterals,
 for every vertex $v \in f_0$. We will evaluate the minimal value of $V(f_0)$. Then we may assume that every vertex of f_0 is incident with only three faces. Let v_1, \dots, v_k be the vertices of f_0 which satisfy (i) and v_{k+1}, \dots, v_M the vertices which satisfy (ii). Thus we may assume that, for $i=1, \dots, k$, $I(v_i)$ consists of a triangle and a pentagon, or a quadrilateral and a pentagon and for $i=k+1, \dots, M$, $I(v_i)$ consists of two quadrilaterals. If we replace the faces of $I(v_i)$ by quadrilaterals for $i=i, \dots, k$, the value of $V(f_0)$ does not increase. Therefore we have minimal value of $V(f_0)$ when $I(v)$ consists of two quadrilaterals for every vertex $v \in f_0$.

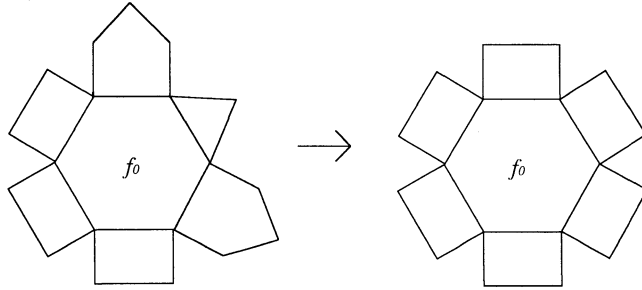


figure 5

Hence by (4.5) we get

$$|V_1| \geq \frac{3}{2}M + 1.$$

Step 3. We repeat the argument, assuming the above inequality. For every vertex $v \in f_0$,

- (i) $I(v)$ contains a face of degree $n (\geq 6)$,
- (ii) $I(v)$ contains a pentagon and a face of degree $n (=4, 5)$ or
- (iii) $I(v)$ contains three quadrilaterals.

We have the minimal value of $V(f_0)$ when $I(v)$ consists of a triangle and a hexagon for every vertex $v \in f_0$. Then we have

$$|V_1| = \frac{7}{4}M + 1.$$

Step 4. We repeat the argument once more, assuming the above inequality. We have the minimal value of $V(f_0)$ when $I(v)$ consists of two pentagons for every vertex $v \in f_0$. Then we obtain the desired inequality

$$|V_1| \geq 2M + 1.$$

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Department of Mathematics,
Nara Women's University,
Kitauoya-Nishimachi,
Nara, 630
Japan