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A SIX DIMENSIONAL COMPACT SYMPLECTIC SOLVMANIFOLD WITHOUT KÄHLER STRUCTURES

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1. Introduction

There are strong topological conditions for a compact manifold M of dimension $2n$ to admit a Kähler structure [20, 10] :

(i) the Betti numbers $b_{2i}(M)$ are non-zero for $1 \leq i \leq n$;
(ii) the Betti numbers $b_{2i-1}(M)$ are even ;
(iii) $b_i(M) \geq b_{i-2}(M)$ for $1 \leq i \leq n$;
(iv) the Hard Lefschetz Theorem holds for M ;
(v) the minimal model of M is formal (so in particular all Massey products of M vanish).

Gordon and Benson have proved that if a compact nilmanifold admits a Kähler structure then it is a torus [5] ; more precisely they proved that the condition (iv) fails for any symplectic structure on a non-toral nilmanifold M. This result was independently proved by Hasegawa [12] by showing that (v) fails for M.

For a compact solvmanifold M of dimension 4 it is known that M has a Kähler structure if and only if it is a complex torus or a hyperelliptic surface. In fact, Auslander and Szczarba in [4] proved that if the first Betti number $b_1(M)$ of M is 2, M is a fiber bundle over T^2 with fiber T^2. Then by Ue [19] M has a complex structure only if it is a hyperelliptic surface or a primary Kodaira surface which is a compact nilmanifold. Thus, if M is a Kähler manifold, it must be a hyperelliptic surface. Since $1 \leq b_1(M) \leq 4$, M can be a Kähler manifold only if it is a complex torus or a hyperelliptic surface. The fact that a hyperelliptic surface is a solvmanifold follows from Auslander [3]. The above result may be generalized as the following conjecture : A compact solvmanifold has a Kähler structure if and only if it is a finite quotient of a complex torus.

In contrast to the case of compact nilmanifolds there are compact symplectic

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solvmanifolds which are not nilmanifolds that satisfy both conditions (iv) and (v) [6, 11, 9, 2]. More precisely:

(1) There is a family of 4-dimensional compact solvmanifolds $M^4(k)$ satisfying (i)-(v) which do not admit Kähler structures [11, 9]. In fact, $M^4(k)$ does not admit complex structures. By using these manifolds it is possible to construct new examples of higher dimension, but we do not know whether any of these examples admit complex or Kähler structures. The problem is that the above results depend strongly on Kodaira's classification of surfaces.

(2) There is a family of six-dimensional compact symplectic non-nilpotent solvmanifolds $M^6(k)$ satisfying (i)-(v). These manifolds are the natural generalization to dimension 6 of the manifolds $M^4(k)$ of Fernández and Gray [11]. Unfortunately we do not know whether any of these admit Kähler structures [2]. But it is amazing that the manifolds $M^6(k)$ have complex structures.

(3) In [6] Benson and Gordon have constructed two examples of non-nilpotent solvable Lie groups of dimension 8, each one of those satisfies one of the conditions (iv) or (v), but not the other.

The purpose of this paper is to construct a compact symplectic (non-nilpotent) solvmanifold $M^6=\Gamma/G$ of dimension 6 which does not satisfy either (iv) or (v) and, hence, does not admit Kähler structures. We show that the minimal model of M^6 is not formal by proving that there are non-trivial (quadruple) Massey products, however we remark that all the (triple) Massey products of M^6 vanish. Then the approach used in [7] to show that non-abelian compact nilmanifolds are non formal fails for the solvable case.

Another problem related with the one considered above is the following. Samelson [17] proved that every compact even dimensional Lie group possesses a left invariant complex structure. But the same is not true for non-compact Lie groups. In fact, since the manifold $M^4(k)$ does not admit complex structures then the corresponding Lie group $G^4(k)$ does not admit left invariant complex structures (see Cordero, Fernández and Gray [8]). In the same paper they have constructed a 6-dimensional nilpotent Lie group with no left invariant complex structure. Since we do not know whether the manifold M^6 admits complex structures or not, we can not use this method to decide whether G admits a left invariant complex structure. But from direct computations we prove, in the last section, that G has no left invariant complex structures.

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2. The Lie Group G

Let G be the connected and solvable Lie group of dimension 6 consisting of matrices of the form

$$A=\begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & z_1 \\ 0 & 0 & 0 & e^{-t} & 0 & z_2 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $t, x, y_i, z_i \in \boldsymbol{R}$, $1 \leq i \leq 2$. Then, a global system of coordinates $\{t, x, y_1, y_2, z_1, z_2\}$ for G is given by

$$t(A)=t,\ x(A)=x,\ y_i(A)=y_i,\ z_i(A)=z_i,\ 1 \leq i \leq 2\,;$$

and a standard computation shows that a basis for the left invariant 1-forms on G consists of

$$\begin{aligned} &\alpha=dt,\ \beta=dx,\ \gamma_1=e^{-t}dy_1-xe^{-t}dz_1, \\ &\gamma^2=e^t dy_2-xe^t dz_1,\ \delta_1=e^{-t}dz_1,\ \delta_2=e^t dz_2. \end{aligned}$$

Then we have

$$\begin{aligned} &d\alpha=d\beta=0,\ d\gamma_1=-\alpha\wedge\gamma_1-\beta\wedge\delta_1, \\ &d\gamma_2=\alpha\wedge\gamma_2-\beta\wedge\delta_2,\ d\delta_1=-\alpha\wedge\delta_1,\ d\delta_2=\alpha\wedge\delta_2. \end{aligned} \tag{2.1}$$

We denote by $\{T, X, Y_1, Y_2, Z_1, Z_2\}$ the dual basis of left invariant vector fields. From (2.1) we obtain

$$\begin{aligned} &[T, Y_1]=Y_1,\ [T, Y_2]=-Y_2,\ [T, Z_1]=Z_1, \\ &[T, Z_2]=-Z_2,\ [X, Z_1]=Y_1,\ [X, Z_2]=Y_2, \end{aligned} \tag{2.2}$$

and the other brackets being zero.

Let $\mathcal{G}$ be the Lie algebra of G. From (2.2) we compute the derived series of $\mathcal{G}$:

$$D^0\mathcal{G}=\mathcal{G},\ D^1\mathcal{G}=[\mathcal{G}, \mathcal{G}]=\langle Y_1, Y_2, Z_1, Z_2\rangle,\ D^r\mathcal{G}=0,\ 2 \leq r$$

and the descending central series of $\mathcal{G}$:

$$C^0\mathcal{G}=\mathcal{G},\ C^1\mathcal{G}=[\mathcal{G}, \mathcal{G}]=\langle Y_1, Y_2, Z_1, Z_2\rangle,\ C^r\mathcal{G}=[\mathcal{G}, C^1\mathcal{G}]=C^1\mathcal{G},\ 2 \leq r.$$

Then G is *a non-nilpotent solvable Lie group*. One says that a Lie group G with Lie algebra $\mathcal{G}$ is completely solvable if $ad(X)\colon \mathcal{G} \longrightarrow \mathcal{G}$ has only real eigenvalues for each $X \in \mathcal{G}$. Equivalently, $\mathcal{G}$ is isomorphic to a Lie subalgebra of the real upper triangular matrices in $gl(n, \boldsymbol{R})$ for some n. A simple inspection shows that G is completely solvable.

Alternatively, G may be described as a semi-direct product $G=\boldsymbol{R}^2 \propto_\phi \boldsymbol{R}^4$, where $\phi(t, x)$ is the linear transformation of $\boldsymbol{R}^4$ given by the matrix

$$\begin{pmatrix} e^t & 0 & xe^t & 0 \\ 0 & e^{-t} & 0 & xe^{-t} \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}$$

We notice that

$$\begin{pmatrix} e^t & 0 & xe^t & 0 \\ 0 & e^{-t} & 0 & xe^{-t} \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus the operation group in G is given by

$$\begin{aligned} &(t, x, y_1, y_2, z_1, z_2)(t', x', y_1', y_2', z_1', z_2') \\ &=(t+t', x+x', y_1'e^t+xz_1'e^t+y_1, y_2'e^{-t}+xz_2'e^{-t}+y_2, z_1'e^t+z_1, z_2'e^{-t}+z_2) \end{aligned}$$

Then $G=\boldsymbol{R}^2\propto_\phi \boldsymbol{R}^4$, where $\boldsymbol{R}^2$ is a connected abelian subgroup and $\boldsymbol{R}^4$ is the nilpotent commutator subgroup.

REMARK 1. Let H be the connected Lie group of dimension 7 consisting of matrices of the form

$$A=\begin{pmatrix} e^t & 0 & x_1e^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & x_2e^{-t} & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & z_1 \\ 0 & 0 & 0 & e^{-t} & 0 & z_2 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where t, x_i, y_i, $z_i\in\boldsymbol{R}$, $1\leq i\leq 2$. We notice that G is a closed subgroup of H. In fact, G is the Lie subgroup of the matrices $A\in H$ such that $x_1=x_2$. As above H may be described as a semi-direct product $H=\boldsymbol{R}^3\propto_{\tilde\phi}\boldsymbol{R}^4$, where $\tilde\phi(t, x_1, x_2)$ is the linear transformation of $\boldsymbol{R}^4$ given by the matrix

$$\begin{pmatrix} e^t & 0 & x_1e^t & 0 \\ 0 & e^{-t} & 0 & x_2e^{-t} \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}$$

A direct computation shows that a basis for the left invariant 1-forms on H consists of

$$\begin{aligned} &\alpha=dt,\ \beta_1=dx_1, \beta_2=dx_2,\ \gamma_1=e^{-t}dy_1-x_1e^{-t}dz_1, \\ &\gamma_2=e^tdy_2-x_2e^tdz_1,\ \delta_1=e^{-t}dz_1,\ \delta_2=e^tdz_2. \end{aligned}$$

Then we have

$$d\alpha=0,\ d\beta_1=-\alpha\wedge\beta_1,\ d\beta_2=-\alpha\wedge\beta_2,\ d\gamma_1=-\alpha\wedge\gamma_1-\beta\wedge\delta_1,$$
$$d\gamma_2=\alpha\wedge\gamma_2-\beta\wedge\delta_2,\ d\delta_1=-\alpha\wedge\delta_1,\ d\delta_2=\alpha\wedge\delta_2.$$

If we put $G_2=H\times S^1$, then G_2 is the Lie group considered by Benson and Gordon [6].

3. The solvmanifold M^6

We shall construct a cocompact discrete subgroup Γ of G.

Let $B\in SL(2, Z)$ be a unimodular matrix with integer entries and with distinct real eigenvalues, say λ and λ^{-1}. Take $a_0=\log\lambda$, i.e. $e^{a_0}=\lambda$. Then there exists a matrix $P\in Gl(2, \boldsymbol{R})$ such that

$$PBP^{-1}=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$$

Consider the subgroup $\Gamma_0=(a_0Z)\times Z$ of $\boldsymbol{R}^2$. We can easily check that the lattice L on $\boldsymbol{R}^4$ defined by

$$L=((m_1,\ m_2)P^t,\ (n_1,\ n_2)P^t),$$

where $m_1,\ m_2,\ n_1,\ n_2\in Z$ and P^t is the transpose of P, is invariant under the subgroup Γ_0. Thus, $\Gamma=\Gamma_0\propto_{\phi}L$ is a cocompact subgroup of G.

We denote by $M^6=G/\Gamma$ the compact quotient manifold. Then M^6 is a six dimensional non-nilpotent completely solvable manifold.

REMARK 2. Alternatively, the manifold M^6 may be viewed as the total space of a T^4-bundle over T^2. In fact, let $T^4=\boldsymbol{R}^4/L$ be the 4-dimensional torus and $\rho:\ Z^2\longrightarrow$ Diff (T^4) the representation defined as follows: $\rho(p,\ q)$ represents the transformation of T^4 covered by the linear transformation of $\boldsymbol{R}^4$ given by the matrix

$$\begin{pmatrix}e^{pa_0} & 0 & qe^{pa_0} & 0\\ 0 & e^{-pa_0} & 0 & qe^{-pa_0}\\ 0 & 0 & e^{pa_0} & 0\\ 0 & 0 & 0 & e^{-pa_0}\end{pmatrix}$$

This representation determines an action $A:\ Z^2\times(T^4\times\boldsymbol{R}^2)\longrightarrow T^4\times\boldsymbol{R}^2$ defined by

$$A((p,\ q),\ [y_1,\ y_2,\ z_1,\ z_2],\ (r_1,\ r_2))=(\rho(p,\ q)([y_1,\ y_2,\ z_1,\ z_2]),\ (r_1+p,\ r_2+q)).$$

Then $\pi:\ T^4\times_{Z^2}\boldsymbol{R}^2\longrightarrow T^2$ is a T^4-bundle where the projection π is given by

$$\pi[[y_1,\ y_2,\ z_1,\ z_2],\ (r_1,\ r_2)]=[(r_1,\ r_2)].$$

In fact, this bunble is the suspension of the representation ρ(see [14]). Then it is clear that $T^4\times_{Z^2}\boldsymbol{R}^2$ may be canonically identified with M^6.

Next, we shall compute the real cohomology of M^6. Since M^6 is completely solvable we can use a theorem of Hattori [13] which asserts that the de Rham cohomology ring $H^*(M^6, \boldsymbol{R})$ is isomorphic with the cohomology ring $H^*(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G. For simplicity we denote the left invariant forms $\{\alpha, \beta, \gamma_1, \gamma_2, \delta_1, \delta_2\}$ and their projections onto M^6 by the same symbols. Thus, we obtain :

$H^0(M^6, \boldsymbol{R})=\{1\}$,
$H^1(M^6, \boldsymbol{R})=\{[\alpha], [\beta]\}$,
$H^2(M^6, \boldsymbol{R})=\{[\alpha\wedge\beta], [\delta_1\wedge\delta_2], [\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1]\}$,
$H^3(M^6, \boldsymbol{R})=\{[\alpha\wedge\delta_1\wedge\delta_2], [\beta\wedge\gamma_1\wedge\gamma_2], [\beta\wedge\gamma_1\wedge\delta_2], [\alpha\wedge\gamma_1\wedge\delta_2+\alpha\wedge\gamma_2\wedge\delta_1]\}$,
$H^4(M^6, \boldsymbol{R})=\{[\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2], [\alpha\wedge\beta\wedge\gamma_1\wedge\delta_2], [\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]\}$,
$H^5(M^6, \boldsymbol{R})=\{[\alpha\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2], [\beta\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]\}$,
$H^6(M^6, \boldsymbol{R})=\{[\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]\}$.

Thus,

$$\begin{aligned}
&b_0(M^6)=b_6(M^6)=1,\\
&b_1(M^6)=b_5(M^6)=2,\\
&b_2(M^6)=b_4(M^6)=3,\\
&b_3(M^6)=4.
\end{aligned}$$

Hence M^6 satisfies conditions (i)-(iii).

Now let ω be the 2-form on M^6 given by

$$\omega=a(\alpha\wedge\beta)+b(\delta_1\wedge\delta_2)+c(\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1), \tag{3.1}$$

where a, b, $c\in\boldsymbol{R}$. A simple computation shows that $d\omega=0$ and that

$$\omega^3=4ac^2(\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2).$$

Hence $\omega^3\neq 0$ if and only if $a\neq 0$, $c\neq 0$. This proves the following

Proposition 3.1. *M^6 is a compact symplectic manifold. Let ω be a 2-form on M^6 given by* (3.1), *where a, b, $c\in\boldsymbol{R}$ and $a\neq 0$, $c\neq 0$. Then ω is a symplectic form.*

A compact Kähler manifold satisfies the Hard Lefschetz Theorem. In order to continue the analysis of the manifold M^6 we introduce the following

DEFINITION. *Let (M^{2n}, ω) be a compact symplectic manifold. We say that (M^{2n}, ω)* **satisfies the Hard Lefschetz Theorem** *if the mappings*

$$\wedge\,\omega^{n-p}:\ H^p(M^{2n}, \boldsymbol{R})\longrightarrow H^{2n-p}(M^{2n}, \boldsymbol{R})$$

are all isomorphisms, $0 \leq p \leq n$.

(We notice that in [16], McDuff calls (M^{2n}, ω) a **Lefschetz manifold** when the mapping $\wedge \omega^{n-1}: H^1(M^{2n}, \boldsymbol{R}) \longrightarrow H^{2n-1}(M^{2n}, \boldsymbol{R})$ is an isomorphism. Our present definition is more restrictive.)

Theorem 3.1. *M^6 does not satisfy the Hard Lefschetz Theorem.*

Proof. Let us compute the morphism

$$\wedge[\omega]: H^2(M^6, \boldsymbol{R}) \longrightarrow H^4(M^6, \boldsymbol{R}).$$

We obtain

$$\wedge[\omega]([\alpha \wedge \beta]) = 2c[\alpha \wedge \beta \wedge \gamma_1 \wedge \delta_2],$$
$$\wedge[\omega]([\delta_1 \wedge \delta_2]) = 0,$$
$$\wedge[\omega]([\gamma_1 \wedge \delta_2 + \gamma_2 \wedge \delta_1]) = 2a[\alpha \wedge \beta \wedge \gamma_1 \wedge \delta_2] + 2c[\gamma_1 \wedge \gamma_2 \wedge \delta_1 \wedge \delta_2].$$

This implies that $\wedge[\omega]: H^2(M^6, \boldsymbol{R}) \longrightarrow H^4(M^6, \boldsymbol{R})$ is not an isomorphism. ■.

Corollary 1. *The compact symplectic solvmanifold M^6 does not admit Kähler structures.*

We note that a straightforward computation shows that all the (triple) Massey products of M^6 vanish. However we have the following

Theorem 3.2. *The minimal model of M^6 is not formal.*

Proof. It is sufficient to exhibit a (quadruple) non-trivial Massey product. For this we recall that if there are cohomology classes $[\lambda_1] \in H^p(M^6, \boldsymbol{R})$, $[\lambda_2] \in H^q(M^6, \boldsymbol{R})$, $[\lambda_3] \in H^r(M^6, \boldsymbol{R})$ and $[\lambda^4] \in H^s(M^6, \boldsymbol{R})$(represented by differential forms λ_1, λ_2, λ_3 and λ_4) such that the (triple) Massey products $\langle [\lambda_1], [\lambda_2], [\lambda_3] \rangle$ and $\langle [\lambda_2], [\lambda_3], [\lambda_4] \rangle$ are zero, then there exists the (quadruple) Massey product $\langle [\lambda_1], [\lambda_2], [\lambda_3], [\lambda_4] \rangle$. Moreover, it is zero if and only if there are differential forms $f_1 \in \Omega^{p+q-1}(M^6)$, $f_2 \in \Omega^{q+r-1}(M^6)$, $f_3 \in \Omega^{r+s-1}(M^6)$, $\mu_1 \in \Omega^{p+q+r-2}(M^6)$ and $\mu_2 \in \Omega^{q+r+s-2}(M^6)$ satisfying:

(1) $\lambda_1 \wedge \lambda_2 = df_1$,
(2) $\lambda_2 \wedge \lambda_3 = df_2$,
(3) $\lambda_3 \wedge \lambda_4 = df_3$,
(4) $\lambda_1 \wedge f_2 + (-1)^{p+1} f_1 \wedge \lambda_3 = d\mu_1$,
(5) $\lambda_2 \wedge f_3 + (-1)^{q+1} f_2 \wedge \lambda_4 = d\mu_2$,
(6) the cohomology class $[(-1)^{p+1} \lambda_1 \wedge \mu_2 + (-1)^{q+1} \mu_1 \wedge \lambda_4 + f_1 \wedge f_3]$ is zero in $H^{p+q+r+s-2}(M^6, \boldsymbol{R})$.

Now, because all the (triple) Massey products on M^6 are zero, it is defined the (quadruple) Massey product $\langle[\delta_1\wedge\delta_2], [\beta], [\beta], [\beta]\rangle$. We shall prove that is non-zero.

Let us suppose that $\langle[\delta_1\wedge\delta_2], [\beta], [\beta], [\beta]\rangle=0$. Then, there exist differential forms $f_2, f_3, \mu_2\in\Omega^1(M^6)$ and $f_1, \mu_1\in\Omega^2(M^6)$ satisfying

$$\begin{aligned}
&(1') && \delta_1\wedge\delta_2\wedge\beta=df_1,\\
&(2') && 0=df_2,\\
&(3') && 0=df_3,\\
&(4') && \delta_1\wedge\delta_2\wedge f_2-f_1\wedge\beta=d\mu_1,\\
&(5') && \beta\wedge f_3+f_2\wedge\beta=d\mu_2,\\
&(6') && [-\delta_1\wedge\delta_2\wedge\mu_2+\mu_1\wedge\beta+f_1\wedge f_3]=0.
\end{aligned}$$

Since $\delta_1\wedge\delta_2\wedge\beta=d(-\gamma_1\wedge\delta_2)$, from (1') we get a differential form f_1' with $df_1'=0$ and such that

$$(7)\quad f_1=-\gamma_1\wedge\delta_2+f_1'.$$

Substituting (7) in (4') we have

$$(8)\quad \delta_1\wedge\delta_2\wedge f_2+\gamma_1\wedge\delta_2\wedge\beta+f_1'\wedge\beta=d\mu_1.$$

From (8) and (2') it follows that the cohomology class $[\delta_1\wedge\delta_2]\wedge[f_2]$ belongs to $[\beta]H^*(M^6, \boldsymbol{R})$; and so

$$(9)\quad f_2=t\beta \text{ for some } t\in\boldsymbol{R}.$$

Moreover, because the cohomology class $[\delta_1\wedge\delta_2\wedge\beta]$ is zero, from (8) and (9) we obtain $[\beta\wedge\gamma_1\wedge\delta_2]=[\beta]\wedge[f_1']$, and then we have

$$(10)\quad f_1'=\frac{1}{2}(\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1)+p\alpha\wedge\beta+q\delta_1\wedge\delta_2,$$

for some $p, q\in\boldsymbol{R}$. Now, from (7) and (10) we get

$$(11)\quad f_1=\frac{1}{2}(\gamma_2\wedge\delta_1-\gamma_1\wedge\delta_2)+p\alpha\wedge\beta+q\delta_1\wedge\delta_2.$$

On the other hand, from (9), (3') and (5') we obtain $f_2-f_3=s\beta$ for some $s\in\boldsymbol{R}$, and so

$$(12)\quad f_3=(t-s)\beta.$$

From (9) and (12), condition (5') becomes:

$$(5'')\quad d\mu_2=0.$$

It is easy to get:

$$(13)\quad f_1\wedge f_3=d((t-s)\gamma_2\wedge(q\delta_1-\frac{1}{2}\gamma_1)).$$

From (9), (11), (5") and (13), conditions (4') and (6') become

$$(4")\quad (t+q)\beta\wedge\delta_1\wedge\delta_2-\frac{1}{2}\beta\wedge(\gamma_2\wedge\delta_1-\gamma_1\wedge\delta_2)=d\mu_1,$$

$$(6")\quad -[\delta_1\wedge\delta_2]\wedge[\mu_2]+[\mu_1\wedge\beta]=0.$$

But, we can check that $\beta\wedge\delta_1\wedge\delta_2=d(-\gamma_1\wedge\delta_2)$ and $\beta\wedge(\gamma_2\wedge\delta_1-\gamma_1\wedge\delta_2)=d(\gamma_1\wedge\gamma_2)$. These equations and (4") imply that there is a closed differential form μ_1' such that $\mu_1=-(t+q)\gamma_1\wedge\delta_2-\frac{1}{2}\gamma_1\wedge\gamma_2+\mu_1'$, and thus $[\mu_1\wedge\beta]=-(t+q)[\gamma_1\wedge\delta_2\wedge\beta]-\frac{1}{2}[\gamma_1\wedge\gamma_2\wedge\beta]+[\mu_1']\wedge[\beta]$. Then, condition (6") becomes:

$-[\delta_1\wedge\delta_2]\wedge[\mu_2]-(t+q)[\gamma_1\wedge\delta_2\wedge\beta]-\frac{1}{2}[\gamma_1\wedge\gamma_2\wedge\beta]+[\mu_1']\wedge[\beta]=0$. So, the cohomology class $[\gamma_1\wedge\gamma_2\wedge\beta]$ belongs to $[\beta]H^2(M^6, \boldsymbol{R})+[\delta_1\wedge\delta_2]H^1[M^6, \boldsymbol{R})$, which is generated by $\{[\beta\wedge\gamma_1\wedge\delta_2], [\alpha\wedge\delta_1\wedge\delta_2]\}$. This is impossible because the family $\{[\gamma_1\wedge\gamma_2\wedge\beta], [\beta\wedge\gamma_1\wedge\delta_2], [\alpha\wedge\delta_1\wedge\delta_2]\}$ is free.

REMARK 3. Theorem 3.2 also proves that M^6 does not admit Kähler structures.

Next, we shall prove that the minimal model of the complex of left invariant differential forms of G_2 ($G_2=H\times S^1$, where H is the Lie group of dimension 7 constructed in Remark 1) is formal, but it does not verify the Hard Lefschetz Theorem (see [6]). Then a compact manifold of the form Γ/G_2 could not be Kähler. Unfortunately we do not know if G_2 admits a cocompact subgroup.

Proposition 3.2. *The complex of the left invariant differential forms of G_2 is formal.*

Proof. We need to show that the d.g.c.a. $(\Lambda\mathcal{G}_2^*, d_2)$, made up of the left invariant differential forms of G_2, and the d.g.c.a. $(H^*(\mathcal{G}_2, \boldsymbol{R}), 0)$ have the same minimal model. We first recall the structure of $(\Lambda\mathcal{G}_2^*, d_2)$ and $H^*(\mathcal{G}_2, \boldsymbol{R})$(cf. [6]). A basis of $\mathcal{G}_2^*$ is the family $\{\alpha, \beta, \mu_1, \nu_1, \xi_1, \mu_2, \nu_2, \xi_2\}$ and the differential operator d is given by

$$\begin{array}{ll} d\alpha=0 & d\beta=0 \\ d\mu_1=-\alpha\wedge\mu_1, & d\mu_2=\alpha\wedge\mu_2, \\ d\nu_1=2\alpha\wedge\nu_1, & d\nu_2=-2\alpha\wedge\nu_2, \\ d\xi=\alpha\wedge\xi_1-\mu_1\wedge\nu_1, & d\xi=-\alpha\wedge\xi_2-\mu_2\wedge\nu_2. \end{array}$$

On the other hand, the cohomology of $\mathcal{G}_2$ can be written as a product $\Lambda\{[\alpha], [\beta]\}\otimes\mathcal{A}$ where $\mathcal{A}$ is the following gca:

$\mathscr{A}^0=\{1\}$,
$\mathscr{A}^2=\{[\mu_1\wedge\mu_2],\ [\nu_1\wedge\nu_2],\ [\mu_1\wedge\xi_1],\ [\mu_2\wedge\xi_2]\}$,
$\mathscr{A}^4=\{[\nu_1\wedge\nu_2\wedge\xi_1\wedge\xi_2],\ [\mu_1\wedge\xi_1\wedge\nu_1\wedge\nu_2],\ [\nu_1\wedge\nu_2\wedge\mu_2\wedge\xi_2],\ [\mu_1\wedge\xi_1\wedge\mu_2\wedge\xi_2]\}$,
$\mathscr{A}^6=\{[\nu_1\wedge\nu_2\wedge\mu_1\wedge\xi_1\wedge\mu_2\wedge\xi_2]\}$,

and $\mathscr{A}^1=\mathscr{A}^3=\mathscr{A}^5=0$. Consider $\varphi:\ (\Lambda Z,\ d)\longrightarrow\mathscr{A}$ the bigraded model of $\mathscr{A}$ (cf. [15]). A straightforward calculation gives

$$\begin{array}{llllll}
 & & & & & Z_0^6=\{0\}\\
 & & & & Z_1^5=\{f\} & Z_0^5=\{0\}\\
 & & & Z_2^4=\{x_i,\ y_j,\ z_j\} & Z_1^4=\{h_j\} & Z_0^4=\{e\}\\
 & & Z_3^3=\{0\} & Z_2^3=\{0\} & Z_1^3=\{c_i,\ g_j\} & Z_0^3=\{0\}\\
 & Z_4^2=\{0\} & Z_3^2=\{0\} & Z_2^2=\{0\} & Z_1^2=\{0\} & Z_0^2=\{b_i\}\\
Z_5^1=\{0\} & Z_4^1=\{0\} & Z_3^1=\{0\} & Z_2^1=\{0\} & Z_1^1=\{0\} & Z_0^1=\{0\}
\end{array}$$

with

$$\begin{aligned}
&db_i=de=0,\\
&dc_i=b_i^2,\\
&dg_j=b_1b_j,\\
&dh_j=eb_j,\\
&df=b_2b_3b_4-eb_1,\\
&dx_i=b_1c_i-b_ig_i,\\
&dy_j=b_jc_1-b_1g_j,\\
&dz_j=b_kg_l-b_lg_k,
\end{aligned}$$

$1\le i\le 4$, $(j,\ k,\ l)$ any permutation of $(2,\ 3,\ 4)$, and

$$\begin{aligned}
&\varphi(b_1)=[\mu_1\wedge\mu_2],\ \varphi(b_2)=[\mu_1\wedge\xi_1],\ \varphi(b_3)=[\mu_2\wedge\xi_2],\ \varphi(b_4)=[\nu_1\wedge\nu_2],\\
&\varphi(e)=[\nu_1\wedge\nu_2\wedge\xi_1\wedge\xi_2].
\end{aligned}$$

We construct a d.g.c.a.-morphism $\psi:\ \Lambda(\{\alpha,\ \beta\},\ 0)\otimes(\Lambda Z,\ d)\longrightarrow(\Lambda\mathscr{G}_2^*,\ d)$ inducing an isomorphism in cohomology. This will end the proof. Put

$$\begin{aligned}
&\psi(\alpha)=\alpha,\ \psi(\beta)=\beta,\\
&\psi(b_1)=\mu_1\wedge\mu_2,\ \psi(b_2)=\mu_1\wedge\xi_1,\ \psi(b_3)=\mu_2\wedge\xi_2,\ \psi(b_4)=\nu_1\wedge\nu_2,\\
&\psi(e)=\nu_1\wedge\nu_2\wedge\xi_1\wedge\xi_2,\\
&\psi(g_4)=\mu_1\wedge\nu_1\wedge\xi_2,\\
&\psi(\text{other genarators})=0.
\end{aligned}$$

This map extends naturally to a g.c.a.-morphism $\psi:\ (\Lambda\{\alpha,\ \beta\},\ 0)\otimes(\Lambda Z,\ d)\longrightarrow(\Lambda\mathscr{G}_2^*,\ d)$. It remains to prove that ψ is a differential operator and that ψ^* is an isomorphism.

- A direct calculation shows that $\psi(dx)=d\psi(x)$ for $x=\alpha,\ \beta$ and for each generator $x\in\bigcup_{p+q\le 6}(\Lambda Z)_q^p$. Consider $x\in Z_q^p$ a generator with $p+q>6$; by

definition of ψ we have $\psi(x)=0$. Since $dZ_q^p \subset (\Lambda Z)_{q-1}^{p+1}$ the writing of dx does not contain any of the following monomials :
b_i, e, g_4, b_1e, b_1b_4, b_2b_4, b_3b_4. By construction the operator ψ vanishes when is evaluated on the other monomials of Z. We conclude that $\psi(dx)$ is 0.

- ψ^* is an isomorphism because
$\psi^*[\alpha]=[\alpha]$,
$\psi^*[\beta]=[\beta]$,
$\psi^*[b_1]=[\mu_1\wedge\mu_2]$,
$\psi^*[b_2]=[\mu_1\wedge\xi_1]$,
$\psi^*[b_3]=[\mu_2\wedge\xi_2]$,
$\psi^*[b_4]=[\nu_1\wedge\nu_2]$,
$\psi^*[e]=[\nu_1\wedge\nu_2\wedge\xi_1\wedge\xi_2]$. ■

4. The moduli space of complex structures on M^6

First, let us recall the following lemma [1] :

Lemma 4.1. *Let G be a (real) Lie group of (real) dimension $2n$. Then the space of left invariant almost complex structures on G has dimension $2n^2$.*

In our case the space of left invariant almost complex structures on G has dimension 18. This lemma gives no information about left invariant complex structures, but since G has a canonical parallelization, it is extremely easy to determine when a left invariant almost complex structure is integrable.

We set

$$E_1=T,\ E_2=X,\ E_3=Y_1,\ E_4=Y_2,\ E_5=Z_1,\ E_6=Z_2.$$

Let J be a left invariant almost complex structure on G. Then J has constant coefficients with respect to the basis $\{E_1, E_2, E_3, E_4, E_5, E_6\}$. Write

$$JE_j=\sum_{k=1}^{6} a_{jk}E_k,$$

where the a_{jk} are constants. The Nijenhuis tensor N_J of J is defined by

$$N_J(U,V)=[JU,JV]-J[JU,V]-J[U,JV]-[U,V],$$

for all vector fields U, V on G.

Proposition 4.1. *G has no left invariant complex structures. Equivalently, the manifold M^6 has no complex structures with constant coefficients with respect to the canonical parallelization $\{E_1, E_2, E_3, E_4, E_5, E_6\}$.*

Proof. Let J be a left invariant integrable almost complex structure on G. Below, we shall prove that the matrix of J with respect to the basis $\{E_i, 1\leq i\leq 6\}$ must have the form

$$J=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix} \tag{4.1}$$

Let us suppose (4.1). Since $J^2=-I$, we obtain

$$a_{11}^2+a_{12}a_{21}=a_{22}^2+a_{12}a_{21}=-1, \tag{4.2}$$

$$a_{12}(a_{11}+a_{22})=a_{21}(a_{11}+a_{22})=0, \tag{4.3}$$

and so a_{12}, a_{21} are non-zero, and $a_{11}+a_{22}=0$.

Moreover, we have

$$\begin{aligned} 0&=N_J(E_2, E_3) \\ &=(a_{22}a_{35}-a_{33}a_{35}-a_{36}a_{43})E_3+(-2a_{21}a_{34}+a_{22}a_{36}-a_{34}a_{35}-a_{36}a_{44})E_4 \\ &\quad -(a_{35}^2+a_{36}a_{45})E_5-a_{36}(2a_{21}+a_{35}+a_{46})E_6, \end{aligned}$$

$$\begin{aligned} 0&=N_J(E_2, E_4) \\ &=(2a_{21}a_{43}+a_{22}a_{45}-a_{33}a_{45}-a_{43}a_{46})E_3+(a_{22}a_{46}-a_{34}a_{45}-a_{44}a_{46})E_4 \\ &\quad +a_{45}(2a_{21}-a_{35}-a_{46})E_5-(a_{46}^2+a_{36}a_{45})E_6. \end{aligned}$$

These equations imply

$$a_{35}^2=-a_{36}a_{45}, \tag{4.4}$$

$$0=a_{36}(2a_{21}+a_{35}+a_{46}), \tag{4.5}$$

$$a_{45}(a_{22}-a_{33})=a_{43}(a_{46}-2a_{21}), \tag{4.6}$$

$$0=a_{45}(2a_{21}-a_{35}-a_{46}), \tag{4.7}$$

$$a_{46}^2=-a_{36}a_{45}. \tag{4.8}$$

From (4.4) and (4.8) we obtain

$$a_{46}=\pm a_{35}. \tag{4.9}$$

Assume that $a_{46}=-a_{35}$. Since $a_{21}\neq 0$, then (4.5) and (4.7) imply $a_{36}=a_{45}=0$. Again (4.4) and (4.8) imply $a_{35}=a_{46}=0$. Now, from (4.6) it follows that $a_{43}=0$; and so $JE_4=a_{44}E_4$. This implies $a_{44}^2=-1$, which is a contradiction. Thus, (4.9) must be

$$a_{46}=a_{35}. \tag{4.10}$$

From (4.10), (4.5) and (4.7) we have

$$a_{36}(a_{21}+a_{35})=0, \quad a_{45}(a_{21}-a_{35})=0, \tag{4.11}$$

or equivalently

$$a_{36}a_{21}=-a_{36}a_{35}, \quad a_{45}a_{21}=a_{35}a_{45}. \tag{4.12}$$

Then, from (4.4) and (4.12) we obtain

$$a_{35}^3=-a_{36}a_{45}a_{35}=a_{21}a_{36}a_{45}=a_{35}a_{36}a_{45}=-a_{35}^3,$$

that is $a_{35}=0$. Thus $a_{46}=0$. But $a_{21}\neq 0$ and (4.12) imply $a_{36}=a_{45}=0$. Now, $a_{21}\neq 0$ and (4.6) imply $a_{43}=0$, and hence $JE_4=a_{44}E_4$, which is not possible. Therefore, G carries no left invariant integrable almost complex structures.

Finally we shall give a proof of (4.1): For this we shall show: (I) $a_{41}=0$, (II) $a_{31}=0$, (III) $a_{61}=0$, (IV) $a_{42}=0$, (V) $a_{51}=0$, (VI) $a_{32}=0$, (VII) $a_{52}=0$, and (VIII) $a_{62}=0$.

From (2.2) we have

$$\begin{aligned}0&=N_J(E_3, E_4)\\ &=2a_{31}a_{41}E_1+(a_{31}a_{42}+a_{32}a_{41})E_2+(2a_{31}a_{43}+a_{32}a_{45}\\ &\quad -a_{35}a_{42})E_3+(2a_{34}a_{41}+a_{32}a_{46}-a_{36}a_{42})E_4+2a_{31}a_{45}E_5+2a_{36}a_{41}E_6.\end{aligned} \tag{4.13}$$

To prove (I), let us suppose that $a_{41}\neq 0$. Then from (4.13), equating to zero the coefficients of E_1, E_2, E_6 and E_4, it follows

$$a_{31}=a_{32}=a_{34}=a_{36}=0. \tag{4.14}$$

Also, from (4.13) we have $a_{35}a_{42}=0$. If $a_{35}=0$, then $JE_3=a_{33}E_3$, and so $a_{33}^2=-1$, which is a contradiction. Thus, we get

$$a_{35}\neq 0 \text{ and } a_{42}=0. \tag{4.15}$$

Since $J^2=-I$ and $JE_3=a_{33}E_3+a_{35}E_5$ we have

$$a_{51}=a_{52}=a_{54}=a_{56}=a_{12}=a_{33}+a_{55}=0. \tag{4.16}$$

From (4.14) and (4.16) we obtain

$$\begin{aligned}0&=N_J(E_2, E_5)\\ &=\{a_{22}(a_{55}-a_{33})-a_{33}a_{55}-1\}E_3-a_{35}(a_{22}+a_{55})E_5.\end{aligned} \tag{4.17}$$

Since $a_{35}\neq 0$, from (4.16) and (4.17) we get

$$a_{22}+a_{55}=0 \text{ and } 2a_{22}a_{55}+a_{55}^2-1=0.$$

These equations imply $a_{55}^2=-1$, which a contradiction. This proves (I).

(II) Let us suppose that $a_{31}\neq 0$. Then from (4.13) and (I) we obtain

$$a_{42}=a_{43}=a_{45}=0. \tag{4.18}$$

From (I), (2.2) and (4.18) it follows that

$$0=N_J(E_2,\ E_4)=a_{46}(a_{22}-a_{44})E_4-a_{46}^2E_6.$$

This identity implies that $a_{46}=0$. Then, from (I) and (4.18) we obtain $JE_4=a_{44}E_4$ which is not possible.

To prove (III), we suppose that $a_{61}\neq 0$. Then, from (I) and (II) we obtain that the coefficient of E_1 in $N_J(E_5,\ E_6)$ is $2a_{51}a_{61}$, and hence, we get

$$a_{51}=0. \tag{4.19}$$

From (I), (II) and (4.19) we deduce that the coefficient of E_1 in $N_J(E_2,\ E_6)$ and in $N_J(E_1,\ E_3)$ is $a_{21}a_{61}$ and $-a_{36}a_{61}$, respectively. Thus, we have

$$a_{21}=a_{36}=0. \tag{4.20}$$

From (4.20) we conclude that the coefficient of E_5 in $N_J(E_2,\ E_3)$ is $-a_{35}^2$. Thus, we get

$$a_{35}=0. \tag{4.21}$$

From (I), (II) and (4.21) we deduce that the coefficient of E_2 in $N_J(E_3,\ E_5)$ is $-a_{32}^2$. Thus, we get

$$a_{32}=0. \tag{4.22}$$

From (4.22) we have

$$0=N_J(E_3,\ E_6)=2a_{34}a_{61}E_4.$$

This equation implies that

$$a_{34}=0. \tag{4.23}$$

Now, from (I), (II) and (4.20)-(4.23) we have $JE_3=a_{33}E_3$, which is not possible; and we obtain (III).

To prove (IV), we compute the coefficient of E_2 in $N_J(E_4,\ E_6)$ and we obtain $-a_{42}^2E_2=0$, from which we deduce (IV).

To prove (V), let us suppose that $a_{51}\neq 0$. Then from (IV) we deduce that the coefficient of E_1 in $N_J(E_2,\ E_5)$ is $-a_{21}a_{51}$, from which we deduce

$$a_{21}=0. \tag{4.24}$$

Moreover we have that the coefficient of E_1 in $N_J(E_1,\ E_6)$ and in $N_J(E_1,\ E_3)$ is $-a_{51}a_{65}$ and $-a_{35}a_{51}$, respectively, from which we get

$$a_{35}=a_{65}=0. \tag{4.25}$$

Also we have that the coefficient of E_1 in $N_J(E_1,\ E_2)$ is $-a_{25}a_{51}$, and that the coefficient of E_5 in $N_J(E_4,\ E_5)$ is $-2a_{45}a_{51}$, from which we obtain

$$a_{25}=a_{45}=0. \tag{4.26}$$

Now, from (I), (II) and (4.26), we obtain

$$0=N_J(E_3, E_4)=a_{32}a_{46}E_4,$$

and so $a_{32}a_{46}=0$. Suppose that $a_{32}\neq 0$ and $a_{46}=0$. Then $JE_4=a_{43}E_3+a_{44}E_4$, and since $J_2=-I$, we find

$$\begin{aligned} &a_{43}a_{32}=0,\\ &a_{43}a_{34}+a_{44}^2=-1. \end{aligned} \tag{4.27}$$

Because $a_{32}\neq 0$, (4.27) implies $a_{43}=0$ and $a_{44}^2=-1$. But it is not possible. Thus, it must be $a_{32}=0$. From this identity, (4.26), and equating to zero the coefficient of E_2 in $N_J(E_1, E_2)$, we get

$$a_{26}a_{62}=0. \tag{4.28}$$

Thus, if $a_{51}\neq 0$, from (I)-(IV), (4.24), (4.25), (4.26) and $a_{32}=0$, it follows that the matrix of J is of the form

$$J=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16}\\ 0 & a_{22} & a_{23} & a_{24} & 0 & a_{26}\\ 0 & 0 & a_{33} & a_{34} & 0 & a_{36}\\ 0 & 0 & a_{43} & a_{44} & 0 & a_{46}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56}\\ 0 & a_{62} & a_{63} & a_{64} & 0 & a_{66} \end{pmatrix}$$

From (4.28) and $J^2=-I$ we obtain $a_{22}^2=-1$, which is not possible. Thus, we have (V).

To prove (VI) we compute the coefficients of E_2 and E_5 in $N_J(E_3, E_5)$. They are $-(a_{32}^2+a_{35}a_{52})$ and $-a_{32}a_{35}$, respectively. Then we deduce (VI).

To prove (VII), let us suppose that $a_{52}\neq 0$. If we compute the coefficient of E_2 in $N_J(E_2, E_5)$, then we obtain $-a_{21}a_{52}=0$, which implies that $a_{21}=0$. Thus, if $a_{52}\neq 0$, according to (I)-(VI), the matrix of J would be:

$$J=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16}\\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26}\\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36}\\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46}\\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56}\\ 0 & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix}, \tag{4.29}$$

and, since $J^2=-I$, we obtain $a_{11}^2=-1$, which is a contradiction. This proves (VII).

Finally, to prove (VIII), let us suppose that $a_{62}\neq 0$. If we compute the coefficient of E_2 in $N_J(E_2, E_6)$, then we obtain $a_{21}a_{62}=0$, which implies that $a_{21}=0$. Then, if $a_{62}\neq 0$, the matrix of J is of the form (4.29). Again we have a contradiction. This proves (VIII), and the proof of (4.1) is completed.

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