Hano, J.
Osaka J. Math.
33 (1996), 1-17

ON É. CARTAN'S SPINOR THEORY

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(Received July 1, 1994)

Introduction

This paper consists of expository remarks on the relevance of the manifold of maximal isotropic subspaces in C^{2n+1}, with respect to a nondegenerate symmetric biliear form, to the spinors of $\mathfrak{so}(2n+1, C)$, introduced in Élie Cartan's lecture notes "*Leçons sur la théorie des spineurs* I, II (1938)" ([3]), Chapitre V.

Let us denote by G^* the complex Lie group Spin $(2n+1, C)$, the universal covering group of the complex special orthogonal group $SO(2n+1, C)$, and consider the spin representation of G^*. The dimension of the representation space is 2^n. We denot by P the complex projective space of all complex lines through the origin in the representation space, and by V the G^*-orbit in P through the point determined by the highest weight vectors. Since the center of G^* leaves every point on P fixed, V is a quotient space of $SO(2n+1, C)$.

Making use of the Clifford algebra ([1]), one can study the spin representation in detail and identify V with the space of all maximal isotropic subspaces in C^{2n+1} with respect to a non-degenerate symmetric biliear form (cf. [6], Chap. IV, §9). Further, a concrete description of this projective imbedding $V \to P$ in terms of a suitable coordinate system of V can be obtained ([5], Lemma 2.1).

On the other hand, in his book, Élie Cartan introduces the above projective imbedding $V \to P$ in an explicit form without ado ([3] Chap. V, 92), and takes this setting as the starting point of his spinor theory. In this article, we attempt to shed light on this Cartan's approach.

We show first how this projective imbedding arises naturally within the context of the space V of all maxmal isotropic subspaces (§1), The process of determining coordinate transformations associated to a suitable coordinate chart covering of V leads to a holomorphic line bundle F over V with the property that, the square $F \otimes F$ is the n-th exterior product of the vector bundle whose fibre over a point V is the vector space V itself. The projective imbedding in question is determined by a vector space of holomorphic sections of the line bundle F^{-1}. In this section, a certain determinant (Lemma in **1.5**) plays a crucial role.

Next, we observe that the Lie algebra of holomorphic vector fields on $\boldsymbol{V}$ induced by the $SO(2n+1, \boldsymbol{C})$-action is the restriction to $\boldsymbol{V}$ of a Lie algebra of infinitesimal projective transformations on the complex projective space $\boldsymbol{P}$ (§2). Thus, we have an isomorphism of $\mathfrak{so}(2n+1, \boldsymbol{C})$ into $\mathfrak{sl}(2^n, \boldsymbol{C})$, which is the spin representation. This is one way to reach É. Cartan's original description of the spin representation ([2], XV. 37).

1. The manifold of maximal isotropic subspaces

1.1. The real cartesian space $\boldsymbol{R}^m$ is contained in the complex cartesian space $\boldsymbol{C}^m$ canonically and its standard inner product extends to a complex symmetric bilinear form on $\boldsymbol{C}^m$, which will be denoted by S.

A vector v in $\boldsymbol{C}^m$ is said to be *isotropic* if $S(v, v)=0$, and a complex subspace V of $\boldsymbol{C}^m$ is said to be *isotropic* if the restriction of S to V is identically zero. Suppose that a complex subspace V is isotropic, then its complex conjugate subspace $\overline{V}$ is isotropic, and $V \cap \overline{V}=\{0\}$. Moreover, there is a unique real subspace U in $\boldsymbol{R}^m$ whose complexfication U^c, i. e., the complex subspace spanned by U, is $V+\overline{V}$. Hence, $\dim_{\boldsymbol{R}} U$ is even. As a special case of Witt's Theorem, it is known that every maximal isotropic subspace in $\boldsymbol{C}^m$ is of the same dimension k, where k is the maximum integer such that $2k \leq m$.

Let $\boldsymbol{V}$ be the set of all maximal isotropic subspaces in $\boldsymbol{C}_m$ with rspect to the bilinear form S, ([3], [6] Chap. IV, 9). Depending on whether m is even or odd, the description of the set is slightly different. In this papaer, we are concerned only the case where m is odd.

Suppose that m is odd and $m=2n+1$. Using the standard basis $\{\varepsilon_\lambda;\ \lambda=0, 1, \dots, n, 1', \dots, n'\}$ of $\boldsymbol{R}^{2n+1}$, we put

$$e_0=\varepsilon_0,\ e_i=(1/\sqrt{2})(\varepsilon_i-\sqrt{-1}\varepsilon_{i'}),\ e_{i'}=(1/\sqrt{2})(\varepsilon_i+\sqrt{-1}\varepsilon_{i'}).$$

Then, $\{e_\lambda;\ \lambda=0, 1, \dots, n, 1', \dots, n'\}$ is a basis of $\boldsymbol{C}^{2n+1}$, and

$$S(\textstyle\sum_\lambda a_\lambda e_\lambda,\ \sum_\lambda b_\lambda e_\lambda)=a_0 b_0+\sum_i a_i b_{i'}+\sum_i a_{i'} b_i.$$

With this basis, the standard hermitian form is given by

$$H(\textstyle\sum_\lambda a_\lambda e_\lambda,\ \sum_\lambda b_\lambda e_\lambda)=\sum_\lambda a_\lambda \overline{b}_\lambda.$$

If $v=a_0 e_0+\sum a_i e_i+\sum a_{i'} e_{i'}$, then $\overline{v}=a_0 e_0+\sum \overline{a}_{i'} e_i+\sum \overline{a}_i e_{i'}$. Obviously, $H(u, v)=S(u, \overline{v})$, $u, v \in \boldsymbol{C}^{2n+1}$.

1.2. We shall prove that $\boldsymbol{V}$ is a complex manifold in a primitive way; we first choose an open covering of $\boldsymbol{V}$ consisting of particular 2^n open subsets with complex coordinates, and then show that the coordnate transformations are holomorphic.

Let us denote by N the set of integers $\{1, \ldots, n\}$ and by $\boldsymbol{N}$ the collection of all subsets in N, consisting of 2^n subsets including the empty set ϕ. For $A, B \in \boldsymbol{N}$, $A + B$ is the subset of those integers which belong to $A \cup B$ but not to $A \cap B$ ([4] Ch. II, §XI). This addition is not only commutative but also associative;

$$(A+B)+C=A+(B+C) \text{ for } A, B, C \in \boldsymbol{N}.$$

It is useful to notice that $A+B=C$ implies $A+C=B$.

Given $A, B \in \boldsymbol{N}$, we denote by $p(A, B)$ the number of pairs (i, j) such that $i \in A$, $j \in B$ and $i \geq j$, and put $\varepsilon(A, B)=(-1)^{p(A,B)}$. For $A, B, C \in \boldsymbol{N}$,

$$\varepsilon(A, B+C)=\varepsilon(A, B)\, \varepsilon(A, C) \text{ and } \varepsilon(A+B, C)=\varepsilon(A, C)\, \varepsilon(B, C).$$

Let us denote by $\#(A)$ the nunber of integers in A. Later we need the equality

$$(-1)^{\#(A+C)}=(-1)^{\#(A+B)}(-1)^{\#(B+C)}.$$

Now, we return to $\boldsymbol{V}$. Let $x_0, x_1, \ldots, x_n, x_{1'}, \ldots, x_{n'}$ be the complex coordinates of $\boldsymbol{C}^{2n+1}$ with respect to the basis $\{e_\lambda\}$. These coordinate functions form the dual basis of $\{e_\lambda\}$.

Definition. *Given $A=\{j_1, \ldots, j_v\} \in \boldsymbol{N}$, $j_1 < \ldots < j_v$, we denote by A^c the complement of A in the set N. Put $A^C=\{i_1, \ldots, i_\mu\}$, $i_1 < \ldots < i_k$. Let $\boldsymbol{V}_A$ be the set of all those maximal isotropic subspaces in $\boldsymbol{C}^{2n+1}$, on each of which the restrictions of n linear functions*

$$x_{i_1}, \ldots, x_{i_\mu}, x_{j_{1'}}, \ldots, x_{j_{v'}}$$

are linearly independent. The subset $\boldsymbol{V}_A$ will turn out to be open and dense in $\boldsymbol{V}$.

The set $\boldsymbol{V}$ is the union of these 2^n subsets $\boldsymbol{V}_A$, $A \in \boldsymbol{N}$.

Proof. Take an arbitrary $V \in \boldsymbol{V}$. We use the same notation for the restriction of x_λ to the subspace V. We will show that we can choose a basis E of the dual space of V consisting of n restrictions from $\{x_\lambda\}$ with the properties that x_0 does not belong to E and that for each $i (1 \leq i \leq n)$, either x_i or $x_{i'}$ belong to E, but not both. This implies that V belongs one of $\boldsymbol{V}_A$.

Let E' be a basis of the dual of V consisting of n restrictions from $\{x_\lambda\}$. Suppose that x_0 is a member of the basis E'. We express each $x_\lambda \notin E'$ as a linear combination of these n members in E', and obtain $n+1$ homogeneous linear equations whose solution space is V.

If x_0 does not appear in the above $n+1$ linear equations, then the vector e_0 is a solution of the these equations, and belongs to the isotropic subspace V. This is a contradibtion. Thus, the coefficient of x_0 is not zero in at least one linear equation, say $x_\lambda = \ldots$, $x_\lambda \notin E'$ We replace x_0 by x_λ, obtaining a new basis E not containing x_0.

Suppose that both x_k and $x_{k'}$ belong to E for some k. Then, for some l, both x_l and $x_{l'}$ do not belong to the basis. As before, write each non-member x_λ as a linear combination of members in the base E. Suppose that in the linear equations for all pairs x_l and $x_{l'}$ not belonging to E, both x_k and $x_{k'}$ do not appear. Then, we can find two vectors

$$a_0e_0+e_k+\sum a_\lambda e_\lambda \text{ and } b_0e_0+e_{k'}+\sum b_\lambda e_\lambda$$

satisfying the system of equations, where the index λ in the summations runs through the collection of λ such that $x_\lambda \notin \boldsymbol{E}$ and that if $\lambda=m$ then $x_{m'}\in\boldsymbol{E}$, and if $\lambda=m'$ then $x_m\in\boldsymbol{E}$. Since they are isotropic, $a_0=b_0=0$. We have

$$S(e_k+\sum a_\lambda e_\lambda,\ e_{k'}+\sum b_\lambda e_\lambda)=1$$

This is a contradiction, as these vectors are in the isotropic subspace V.

Thus, for some l, both x_l and $x_{l'}$ do not belong to E and at least in one of the two linear equations $x_l=\ldots$ and $x_{l'}\ldots$ in the system, either x_k or $x_{k'}$ appears with non-zero coefficient. We can replace one of x_k and $x_{k'}$ by one of x_l and $x_{l'}$. Repeating this process, we obtain a desired basis. We have finished the proof.

1.3. The next step is to introduce a system of complex coordinates on each V_A. Let us begin with the simplest case. By definition, V_ϕ is the set of all maximal isotropic subspaces in $\boldsymbol{C}^{2n+1}$, on each of which the restrictions of $x_1, \ldots, x_n$ are linearly independent. A complex n-dimensional subspace V of $\boldsymbol{C}^{2n+1}$ belongs to $\boldsymbol{V}_\phi$ if and only if V is the space of solutions of the following $n+1$ linear equations ([3], Ch. V, 92):

$$(1)\qquad \begin{aligned} &x_0-(\sqrt{2})\textstyle\sum_{i=1}^n \xi_i x_i=0,\\ &x_{j'}+(1/\sqrt{2})\xi_j x_0-\textstyle\sum_{i=1}^n \xi_{ji}x_i=0 \qquad (1\leq j\leq n),\end{aligned}$$

for some (ξ_i, ξ_{ji}) in $\boldsymbol{C}^{n+n^2}$ satisfying the condition that

$$\xi_{ij}+\xi_{ji}=0,\ (i,\ j=1,\ \ldots,\ n).$$

We regard $\xi=(\xi_i, \xi_{ij})$, $i<j$, as the coordinates of V. The map $V\mapsto\xi$ is a homeomorphism from $\boldsymbol{V}_\phi$ onto $\boldsymbol{C}^{n(n+1)/2}$.

Take an arbitrary $V\in\boldsymbol{V}_\phi$ and let (ξ_i, ξ_{ij}) be the coordinates of V determined by the Cartan equations (1). For each $i(i=1, \ldots, n)$, there is a unique solution v_i of the Cartan equations such that $x_j(v_j)=\delta_{ij}$. The n column vectors $v_1, \ldots, v_n$ form a basis of V dual to the basis $\{x_1, \ldots, x_n\}$. The $(2n+1, n)$ matrix $\boldsymbol{M}_\phi(V)=(v_1, \ldots, v_n)$ is given by

$$\begin{bmatrix} \sqrt{2}\xi_1 & \cdots & \sqrt{2}\xi_n \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ -\xi_1^2 & & -\xi_1\xi_n+\xi_{1n} \\ & -\xi_j\xi_i+\xi_{ji} & \\ -\xi_n\xi_1+\xi_{n1} & & -\xi_n^2 \end{bmatrix}$$

We will define coordinate functions on $\boldsymbol{V}_A$ for an arbitrary $A\in\boldsymbol{N}$ in a similar way. Put $A=\{j_1, \cdots, j_\nu\}$, $j_1<\cdots<j_\nu$ and $A^c=\{i_1, \cdots, i_\mu\}$, $i_1<\cdots<i_\mu$. By definition, the restrictions of n linear functions

$$x_{i_1}, \cdots, x_{i_\mu}, x_{j_1'}, \cdots, x_{j_\nu'}$$

are linearly independent on each $V\in \boldsymbol{V}_A$. As in the case of $\boldsymbol{V}_\Phi$, an n dimensional subspace V belongs to $\boldsymbol{V}_A$ if and only if V is the solution space of the following $n+1$ linear equations:

$$x_0-(\sqrt{2})(\textstyle\sum_a \xi_{i_a}^A x_{i_a}+\sum_b \xi_{j_b}^A x_{j_b'})=0,$$
$$x_{j_b}+(1/\sqrt{2})\xi_{j_b}^A x_0-(\textstyle\sum_a \xi_{j_b i_a}^A x_{i_a}+\sum_d \xi_{j_b j_d}^A x_{i_d'})=0 \quad (j_b\in A),$$
$$x_{i_{a'}}+(1/\sqrt{2})\xi_{i_a}^A x_0-(\textstyle\sum_c \xi_{i_a i_c}^A x_{i_c}+\sum_b \xi_{i_a j_b}^A x_{j_b'})=0 \quad (i_a\in A^c),$$

for some constants ξ_i^A $(1\leq i\leq n)$, ξ_{jk}^A with $\xi_{jk}^A+\xi_{kj}^A=0$ $(1\leq j,\ k\leq n)$. The map $V \mapsto(\cdots, \xi_i^A, \cdots, \xi_{ij}^A, \cdots)$ defines the coordinates of V, and the open subset $\boldsymbol{V}_A$ is homeomorphic to $\boldsymbol{C}^{n(n+1)/2}$.

Take an arbitrary $V\in \boldsymbol{V}_A$. For each k $(k=1, \cdots, n)$, let v_k^A be a unique vector in V satisfies the condition that

$$x_\lambda(v_k^A)=\begin{cases}1, \text{ if } k\in A^c \text{ and } \lambda=k \text{ or if } k\in A \text{ and } \lambda=k', \\ \qquad 0, \text{ otherwise.}\end{cases}$$

We denote by $\boldsymbol{M}_A(V)$ the matrix $(v_1^A, \cdots, v_n^A)$; the λ-th row of the matrix is $(\delta_{k1}, \cdots, \delta_{kn})$, if $\lambda=k\in A^c$ or if $\lambda=k'$ and $k\in A$ (δ is the Kronecker's delta), and $(-\xi_k^A\xi_1^A+\xi_{k1}^A, \cdots, -\xi_k^A\xi_n^A+\xi_{kn}^A)$, if $\lambda=k\in A$ or if $\lambda=k'$ and $k\in A^c$.

1.4. Suppose that $V\in \boldsymbol{V}_A\cap \boldsymbol{V}_B$. There exists a unique non-singular (n, n)-matrix $T_{AB}(V)$ such that

(2) $$\boldsymbol{M}_B(V)=\boldsymbol{M}_A(V)T_{AB}(V).$$

Obviously, $T_{BA}(V)=T_{AB}(V)^{-1}$, and if $V\in \boldsymbol{V}_A\cap \boldsymbol{V}_B\cap \boldsymbol{V}_C$, A, B, $C\in\boldsymbol{N}$, then

(3) $$T_{AC}(V)=T_{AB}(V).T_{BC}(V).$$

The complex vector bundle associated to the family of transition functions $(\boldsymbol{V}_A \cap \boldsymbol{V}_B,\ T_{AB}(V))$, $\boldsymbol{A}$, $\boldsymbol{B}\in\boldsymbol{N}$, is the vector bundle over $\boldsymbol{V}$ whose fibre over V is the vector space V isself.

Using the equality (2), we determine the inverse of $T_{AB}(V)$. For this purpose, we put $B=\{l_1, \cdots, l_\rho\}$, $l_1<\cdots<l_\rho$ and $B^c=\{k_1, \cdots, k_\sigma\}$, $k_1<\cdots<k_\sigma$. If λ is either k_a or $l_{b'}$, $\boldsymbol{M}_B(V)_{\lambda m}=\delta_{|\lambda| m}$, $m=1, \cdots, n$, where $|\lambda|=k_a$ if $\lambda=k_a$, and $|\lambda|=l_b$ if $\lambda=l_{b'}$. From (2), it follows that

$$(T_{AB}(V)^{-1})_{|\lambda| m}=\boldsymbol{M}_A(V)_{\lambda m}.$$

Therefore,

$$(T_{AB}(V)^{-1})_{tm}=\begin{cases} -\xi_t^A\delta_m^A+\xi_{tm}^A, & \text{if } t\in A+B, \\ \delta_{tm}, & \text{otherwise,} \end{cases} \tag{4}$$

for $m=1, \cdots, n$.

Thus, the entries of $T_{AB}(V)$ are rational functions of the ξ_i^A's and ξ_{ij}^A's. From the equality (2), we obtain the following proposition :

On $V_A\cap V_B$, the coordinate functions ξ_i^B's and ξ_{ij}^B's are rational functions of the coordinate functions ξ_i^A's and ξ_{ij}^A's. Thus, V is a complex manifold of dimension $n(n+1)/2$.

We will determine explicitly these rational functions.

1.5. For the purpose we need some definitions ([3] Chap. IV, 92). Given a positive integer $m\geq 2$, let x_i $(1\leq i\leq m)$ and x_{ij} $(1\leq i, j\leq m)$ be variables such that $x_{ij}+x_{ji}=0$, $x_{ii}=0$. For $1\leq i_1, \cdots, i_{2k}\leq m$, $k\geq 2$, we put

$$x_{i_1\cdots i_{2k}}=(1/2^k k!)\sum \varepsilon(j_1\cdots j_{2k})(x_{j_1j_2})\cdots(x_{j_{2k-1}j_{2k}}),$$

where in the summation $\{j_1, \cdots, j_{2k}\}$ runs over all permutations of $i_1, \cdots, i_{2k}$, and $\varepsilon(j_1, \cdots, j_{2k})$ denotes the sign of the permutation $j_1, \cdots, j_{2k}$. Obviously, $x_{i_1\cdots i_{2k}}$ is skew-symmetric with respect to the indecies, i.e., $x_{i_1\cdots i_a\cdots i_b\cdots i_k}=(-1)x_{i_1\cdots i_b\cdots i_a\cdots i_k}$. Another expression of $x_{i_1\cdots i_{2k}}$ is

$$\sum_{j_{2a-1}<j_{2a}\,;\ j_2<\cdots<j_{2k}} \varepsilon(j_1\cdots j_{2k})(x_{j_1j_2})\cdots(x_{j_{2k-1}j_{2k}}).$$

From this, it follows that

$$x_{i_1\cdots i_{2k}}=\sum_{a=1}^{2k-1}(-1)^{a-1}x_{i_a i_{2k}}x_{i_1\cdots \hat{i}_a\cdots i_{2k-1}}. \tag{5}$$

When the number of indeces is odd and equal to $2k+1$, we put

$$x_{i_1\cdots i_{2k+1}}=(1/2^k k!)\sum \varepsilon(j_1\cdots j_{2k+1})(x_{j_1})(x_{j_2j_3})\cdots(x_{j_{2k}j_{2k+1}}).$$

Then,

$$x_{i_1\cdots i_{2k+1}}=\sum_{a=1}^{2k+1}(-1)^{a-1}x_{i_a i_1\cdots \hat{i}_a\cdots i_{2k+1}}. \tag{6}$$

Again, $x_{i_1\cdots i_{2k+1}}$ is skew-symmetric in indeces.

If $A=\{i_1, \cdots, i_k\}$, $1\leq i_1<\cdots<i_k\leq m$, we also denote by x_A the function $x_{i_1\cdots i_k}$,

and *if* $A=\emptyset$, we put $x_{\emptyset}=1$.

The following lemma is crucial.

Lemma. *Suppose that* $1\leq i_1, \cdots, i_k\leq m$. *Form a* (k, k)*-matrix whose* (a, b)*-entry is* $-x_{i_a}x_{i_b}+x_{i_a i_b}$. *Then,*

$$\det(-x_{i_a}x_{i_b}+x_{i_a i_b})=(-1)^k(x_{i_1\cdots i_k})^2.$$

We will carry out the computation of the above determinant at the end of this section.

1.6. We return to the manifold $\boldsymbol{V}$. On the open subset $\boldsymbol{V}_A$, the function ξ^A_B is defined in terms of the coordinate functions ξ^A_{ij}'s by (5) or by (6) in the same way as x_B is defined, for each $B\in\boldsymbol{N}$.

Applying the above lemma on the matrix $(T_{AB})^{-1}$, whose form is given by (4), we have

$$\det\,(T_{AB})^{-1}=(-1)^{\#(A+B)}(\xi^A_{A+B})^2 \text{ on } \boldsymbol{V}_A\cap\boldsymbol{V}_B. \tag{7}$$

One implication of the above equality is that $\boldsymbol{V}_A\cap\boldsymbol{V}_B$ is the subset of $\boldsymbol{V}_A$ where the function ξ^A_{A+B} does not vanish, and is connected and dense in $\boldsymbol{V}$. This follows from the facts that ξ^A_{A+B} is not identically zero and the subset of $\boldsymbol{V}_A$ consisting of the points where the function does not vanish is connected and dense, and that the matrix $T_{AB}(V)$ is non-singular if and only if $V\in\boldsymbol{V}_A\cap\boldsymbol{V}_B$. Now, it is clear that the intersection of any number of open subsets $\boldsymbol{V}_A$'s is connected and dense in $\boldsymbol{V}$.

To state another implication of (7), we remark that from $T_{BA}=(T_{AB})^{-1}$, it follows that

$$(\xi^B_{B+A})^{-2}=(\xi^A_{A+B})^2 \text{ on } \boldsymbol{V}_A\cap\boldsymbol{V}_B. \tag{8}$$

By the equalities (3), (7) and

$$(-1)^{\#(A+C)}=(-1)^{\#(A+B)}(-1)^{\#(B+C)},$$

we have

$$(\xi^A_{A+C})^{-2}=(\xi_{A+B}{}^A)^{-2}(\xi^B_{B+C})^{-2} \text{ on } \boldsymbol{V}_A\cap\boldsymbol{V}_B\cap\boldsymbol{V}_C, \tag{9}$$

The equalities (8) and (9) yield the following:

The family of the transition functions $(\boldsymbol{V}_A\cap\boldsymbol{V}_B, (\xi^A_{A+B})^{-2})$, $A,B\in N$, *defines the holomorphic line bundle* $\boldsymbol{L}$ *over* $\boldsymbol{V}$ *whose fibre over a point* $V\in\boldsymbol{V}$ *is the n-th exterior product* $\wedge^n(V)$ *of the vecor space* V. (In fact, the set of transition functions $(\boldsymbol{V}_A\cap\boldsymbol{V}_B, (-1)^{\#(A+B)})$ defines the trivial line bundle.)

1.7. The purpose of this section is to prove the following

Theorem 1. *The set*

$$\{V_A \cap V_B,\ \varepsilon(A, A+B)(\xi^A_{A+B})^{-1}),\ A, B \in N\}$$

is a collection of transition functions and defines a holomorphic line bundle F *over* V *such that* $F^2 = L$.

Proof. It suffices to show that

$$\varepsilon(B, B+A)(\xi^B_{B+A})^{-1} = \varepsilon(A, A+B)(\xi^A_{A+B}) \text{ on } V_A \cap V_B,$$

and

$$\varepsilon(A, A+C)(\xi^A_{A+C})^{-1} = \varepsilon(A, A+B)(\xi^A_{A+B})^{-1}\varepsilon(B, B+C)(\xi^B_{B+C})^{-1} \tag{10}$$

on $V_A \cap V_B \cap V_C$, for $A, B, C \in N$.

The first equality is a special case of the second one (10) where $C=A$, as $A+C=\phi$ and $\xi^A_\phi = \varepsilon(A, \phi) = 1$. In the second equality, we put $B+C=D(C=B+D)$. We will verify the equality rewritten in the following form:

$$\varepsilon(B, D)\xi^B_D = \varepsilon(A, A+B+D)\xi^A_{A+B+D}(\varepsilon(A, A+B)\xi^A_{A+B})^{-1} \tag{11}$$

on $V_A \cap V_B$, for any $A, B \in N$.

Furthermore, it is sufficient to prove the equality (10) for the case where $A=\phi$. In fact, we can easily derive (11) from the equality

$$\varepsilon(B, D)\xi^B_D = \xi^\phi_{B+D}(\xi^\phi_B)^{-1} \text{ on } V_\phi \cap V_B, \tag{12}$$

for arbitrary $B, D \in N$. In the rest of the proof, we drop the superscript ϕ from the functions ξ^ϕ_B.

By (9),

$$(\xi^B_D)^2 = (\xi_{B+D})^2(\xi_B)^{-2} \text{ on } V_\phi \cap V_B, \text{ for any } D \in N.$$

Thus, the squares of both sides of the equality (12) are equal. Since the open set $V_\phi \cap V_B$ is connected, it suffices to verify (11) at one convenient point in $V_\phi \cap V_B$ where both sides of the equality do not vanish. Let us choose a point V_1 in V_ϕ given by $\xi_i = 1$ $(1 \le i \le n)$, $\xi_{jk} = 1$ $(1 \le j < k < n)$. Then, $\xi_D(V_1) = 1$ for every $D \in N$. Hence $V_1 \in V_\phi \cap V_B$ by the remark made right after (7). What we need is to show

$$\xi^B_D(V_1) = \varepsilon(B, D) \text{ for every } D \in N. \tag{13}$$

In order to prove the equality (13) for $D=\{i\}$, we look at the first rows of both side of the equality (2) $M_B(V_1) = M_\phi(V_1)T_{\phi B}(V_1)$. Then,

$$(\xi^B_1(V_1), \cdots, \xi^B_n(V_1))(T_{AB}(V_1))^{-1} = (\xi_1(V_1), \ldots, \xi_n(V_1)) = (1, \ldots, 1). \tag{14}$$

Using the expression of the matrix $(T_{AB})^{-1}$ given by (4) and the equalities

$$(\xi_i\xi_j+\xi_{ij})(V_1)=\begin{cases} 0, \text{ if } i<j, \\ -1, \text{ if } i=j, \\ -2, \text{ if } i>j, \end{cases}$$

we can easily show that $\xi_i^B(V_1)=\xi(B, \{i\})$, $1\leq i\leq n$, satisfy (14). Thus, the equality (13) is verified for $D=\{i\}$, $1\leq i\leq n$.

Using (2) in a similar way, we prove the equality (13) at the point V_1 for $D=\{i, j\}$, $1\leq i<j\leq n$. For an arbitrary D with $\#(D)>2$, we verify (13) easily by making use of the definitions (5) and (6) for ξ_D^B in **1.5** and a remark on $\varepsilon(B, *)$ in **1.2**. We have finished the proof of the theorem.

Corollary 1. *The cooordinate transformation on* $\boldsymbol{V}_A\cap\boldsymbol{V}_B$ *is given by*

$$\xi_i^B=\varepsilon(A+B, \{i\})\xi_{A+B+\{i\}}^A(\xi_{A+B}^A)^{-1}, \ (i=1, \ldots, n),$$
$$\xi_{ij}^B=\varepsilon(A+B, \{i, j\})\xi_{A+B+\{ij\}}^A(\xi_{A+B}^A)^{-1}, \ (1\leq i<j\leq n).$$

Proof. By putting $D=\{i\}$ and $\{i, j\}$ in (11), we obtain the above results.

Corollary 2. *For each* $C\in\boldsymbol{N}$,

$$\{(\boldsymbol{V}_A, \varepsilon(A, A+C)\xi_{A+C}^A), A\in\boldsymbol{N}\}$$

defines a holomorphic section s_C *of the holomorphic line bundle* $\boldsymbol{F}^{-1}$. *The set of holomorphic sections* $\{s_C, C\in\boldsymbol{N}\}$ *determines a map* ι *from* $\boldsymbol{V}$ *into the complex projective space* $\boldsymbol{P}^{2^n-1}$. *To be precise, if* $V\in\boldsymbol{V}_A$, *the image* $\iota(V)\in\boldsymbol{P}^{2^n-1}$ *is the point with homogeneous coordinates*

$$[\varepsilon(A, A)\xi_A^A(V), \ldots, \varepsilon(A, A+C)\xi_{A+C}^A(V), \ldots, \varepsilon(A, A^C)\xi_{A^C}^A(V)].$$

The map ι *is a holomorphic imbedding of the complex manifold* $\boldsymbol{V}$ *into the complex projective space* $\boldsymbol{P}^{2^n-1}$.

The restriction of ι *to* $\boldsymbol{V}_\phi$ *is given by*

$$V \mapsto [1, \ldots, \xi_C^\phi(V), \ldots, \xi_M^\phi(V)]. \tag{15}$$

The expression (15) is found in Cartan's Lecture notes [3].

Proof. The equalities (10) shows that indeed s_C is a secton of the line bundle $\boldsymbol{F}^{-1}$. Suppose that for $V\in\boldsymbol{V}_A$ and $V'\in\boldsymbol{V}_B$, $\iota(V)=\iota(V')$. As $V\in\boldsymbol{V}_A$, the A-th homogeneous coordinateof $\iota(V)$ is $\xi_\phi^A(V)=1$. As $V'\in\boldsymbol{V}_B$, the A-th homogeneous coordinate of the same point is $c\xi_{B+A}^B(V')$ with a non-zero constant c. Thus, $\xi_{B+A}^B(V')\neq 0$, and hence $V'\in\boldsymbol{V}_A$ by the remark following (7). The map ι is obviously injective on $\boldsymbol{V}_A$ by definition. We have shown that $V=V'$. The holomorphic map ι is imbedding.

1.8. Proof of the lemma (1.5). We use the same notations as in **1.5.** In order to prove the Lemma, it suffices to verify the equality

$$\det\begin{vmatrix} x_1x_1 & & x_1x_k+x_{1k} \\ & x_ix_j+x_{ij} & \\ x_kx_1+x_{k1} & & x_kx_k \end{vmatrix}=(x_{1\dots k})^2.$$

We sketch the computation of the determinant. We begin with the case where $x_1=\dots=x_k=0$, and the matrix is skew-symmetric. It is known that,

$$\det(x_{ij})=\begin{cases}(x_{1\cdots k})^2, & \text{if } k \text{ is even,}\\ 0, & \text{if } k \text{ is odd.}\end{cases}$$

Further, we need the determinant of the minor matrix Δ_{ab} obtained by deleting the a-th row and the b-th column of the matrix (x_{ij}).

$$\det\Delta_{ab}=\begin{cases}0, & \text{if } k \text{ is even,}\\ x_{1\dots\hat{a}\dots k}x_{1\dots\hat{b}\dots k}, & \text{if } k \text{ is odd.}\end{cases}$$

One verifies the result by induction on k. Applying elementary column operations on the matrix Δ_{ab}, one get a matrix of the form

$$\left[\begin{array}{cc|ccc} 0 & x_{12} & * & \dots & * \\ x_{21} & 0 & * & \dots & * \\ \hline 0 & 0 & & & \\ & & & (y_{ij}) & \\ 0 & 0 & & & \end{array}\right]$$

where $y_{ij}=(x_{12}x_{ij}-x_{1j}x_{i2}-x_{1i}x_{j2})/x_{12}$. Since $y_{ij}+y_{ji}=0$, one can use the induction hypothesis and obtains

$$\det\ \Delta_{ab}=(x_{12})^2y_{3\dots\hat{a}\dots k}y_{3\dots\hat{b}\dots k}.$$

Making use of the equality

$$x_{12}y_{3\dots\hat{a}\dots k}=x_{1\dots\hat{a}\dots k}$$

([7], p. 95), one finishes the computation.

Retern to the general case. Consider the determinamt in question as a polynomial of the x_{ij}'s. The degree of the polynomial is at most k. We denote by Pv its homogeneous component of degree v. $P_0=\det(x_ix_j)=0$. Further, one verifies easily that $P_1=\dots=P_{k-2}=0$,

$$P_{k-1}=\textstyle\sum_{a,b}(-1)^{a+b}x_ax_b\Delta_{ab}=\begin{cases}0, & \text{if } n \text{ is even,}\\ (x_{1\cdots k})^2, & \text{if } n \text{ is odd, and}\end{cases}$$

$$P_k=\begin{cases}(x_{1\cdots k})^2, \text{ if } n \text{ is even,}\\ 0, \text{ if } n \text{ is odd.}\end{cases}$$

We have finished the proof of the lemma.

2. The spin representation

2.1. The purpose of this section is to construct the spin representation of the complex orthogonal Lie algebra $\mathfrak{so}(2n+1, \boldsymbol{C})$ by making use of the projective immersion $\iota: \boldsymbol{V}\rightarrow \boldsymbol{P}^{2^n-1}$ given in Corollary 2.

We donote by G_c and by G the matrix representations of the complex special orthogonal group $SO(2n+1, C)$ and the special orthogonal group $SO(2n+1)$ respectively, with respect to the basis $\{e_\lambda\}$ of $\boldsymbol{C}^{2n+1}$ defined in **1.1.** The group G_c is the set of all complex $(2n+1, 2n+1)$ matrices with determinant equal to 1 and leaving the symmetric bilinear form S invariant. Obviously, a matrix belonging to G_c maps a maximal isotropic subspace with respect to S onto a maximal isotropic subspace. The group G is the intersection of G_c and the unitary group $U(2n+1)$.

As in the previous section, we donote $\boldsymbol{V}$ the set of all maximal isotropic subspaces in $\boldsymbol{C}^{2n+1}$. The group G acts on $\boldsymbol{V}$ transitively and hence so does the group G_c. We present a proof.

The subspace V_0 spanned by $e_1, \ldots, e_n$ is a maximal isotropic subspace. Take an maximal isotropic subspace V, and choose an orthonormal basis $\{f_1, \ldots, f_n\}$ of V with respect to the hermitian form H (**1.1**). Let $\overline{f}_i$ denote the complex conjugate of f_i with respect to $\boldsymbol{R}^{2n+1}$. Then there exists one and only one unit vector f_0 which is orthogonal to $f_1, \ldots, f_n, \overline{f}_1, \ldots, \overline{f}_n$ and satisfies the equality

$$e_0\wedge e_1\wedge\ldots\wedge e_n\wedge \overline{e}_1\wedge\ldots\wedge \overline{e}_n=f_0\wedge f_1\wedge\ldots\wedge f_n\wedge \overline{f}_1\wedge\ldots\wedge \overline{f}_n.$$

Arranging these $2n+1$ column vectors $f_0, f_1, \ldots, f_n, \overline{f}_1, \ldots, \overline{f}_n$, we obtain a matrix belonging to G, which maps e_i to f_i and $\overline{e}_i$ to $\overline{f}_i$ $(1\leq i\leq n)$, and hence V_0 to V. Thus, G acts transitively on $\boldsymbol{V}$.

Let $\mathfrak{g}$ and $\mathfrak{g}_c$ be the Lie algebras of G and G_c respectively. A complex $(2n+1, 2n+1)$ matrix X belongs to $\mathfrak{g}_c$, if and only if its entries $X_{\lambda\mu}$ satisfy the following conditions:

$$X_{00}=0,\ X_{0i'}=-X_{i0},\ X_{0i}=-X_{i'0},\ X_{i'j'}=-X_{ji},$$
$$X_{ij'}=-X_{ji'},\ X_{i'j}=-X_{j'i},\ (i, j=1, \ldots, n),$$

and X belongs to $\mathfrak{g}$, if and only if X is skew-hermitian and belongs to $\mathfrak{g}_c$.

2.2. The action of G (resp. G_c) an $\boldsymbol{V}$ induces the Lie algebras of vector fields anti-isomorphic to $\mathfrak{g}$ (resp. $\mathfrak{g}_c$). Let us express these vector fields on $\boldsymbol{V}_\phi$ in terms of the coordinates introduced in **1.2**. Since we are only concerned with the open subset $\boldsymbol{V}_\phi$ in this section, we delet the superscript ϕ from the notations of coordinate functions.

Take an arbitrary $V\in \boldsymbol{V}_\phi$ and let (ξ_i, ξ_{ij}) be the coordinates of V determined by the Cartan equations (1). As in **1.3**, for each $i(1<i<n)$, v_i is the solution of the Cartan equations such that $x_j(v_i)=\delta_{ij}$, $1<j<n$. The n column vectors $v_1, \cdots, v_n$ form a basis of V.

Take $X\in\mathfrak{g}_c$, and put $\sigma_t=\exp tX$. Given $V\in \boldsymbol{V}_\phi$, if the absolute value of t is sufficiently small, $\sigma_t.V$ is in $\boldsymbol{V}_\phi$, and is spanned by $\sigma_t.v_1, \cdots, \sigma_t.v_n$. The Cartan equations for $\sigma_t.v_l$ are

$$\begin{aligned}&x_0(\sigma_t.v_l)-(1/\sqrt{2})\textstyle\sum_{i=1}^n \xi_i(\sigma_t.V)x_i(\sigma_t.v_l)=0,\\ &x_{j'}(\sigma_t.v_l)+\sqrt{2}\xi_j(\sigma_t.V)x_0(\sigma_t.v_l)-\textstyle\sum_{i=1}^n \xi_{ij}(\sigma_t.V)x_i(\sigma_t.v_l)=0,\\ &(1\leq j\leq n).\end{aligned}$$

We differentiate both sides of the equalities at $t=0$ and obtain

$$\begin{aligned}\dot{\xi}_i(X)\equiv(d/dt_{t=0})\xi_i(\sigma_t.V)=(1/\sqrt{2})(X_{0i}+\textstyle\sum_j X_{0j'}(\xi_{ji}+\xi_i\xi_j))\\ -\textstyle\sum_j X_{ji}\xi_j-\sum_{j<k}X_{jk'}(\xi_j\xi_{hi}-\xi_k\xi_{ji}),\ (1\leq i\leq n),\end{aligned}\tag{1}$$

$$\begin{aligned}\dot{\xi}_{ij}(X)&\equiv(d/dt_{t=0})\xi_{ij}(\sigma_t.V)\\ &=(1/\sqrt{2})\{(X_{0i}\xi_i-X_{0i}\xi_j+\textstyle\sum_k X_{0k'}(\xi_j\xi_{kj}+\xi_j\xi_{ki})\}\\ &\quad+\textstyle\sum_k(X_{ki}\xi_{jk}-X_{kj}\xi_{ik})+X_{i'j}-\sum_{k<l}X_{kl}'(\xi_{ik}\xi_{lj}-\xi_{il}\xi_{kj}),\\ &\quad(1\leq i<j\leq n).\end{aligned}\tag{2}$$

The holomorphic vector field

$$\xi(X)=\textstyle\sum_i\dot{\xi}_i(X)\partial/\partial\xi_i+\sum_{i<j}\dot{\xi}_{ij}(X)\partial/\partial\xi_{ij}\tag{3}$$

is the (1,0)-component of the real vector field on $\boldsymbol{V}_\phi$, induced by the one parameter group $\exp tX$. The correspondence $X\mapsto\xi(X)$ is a Lie algebra anti-isomorphism.

2.3. The next step is to lift the vector field $\xi(X)$ on $V\in\mathfrak{g}_c$, to an infinitesimal projective transformation $u(X)$ on $\boldsymbol{P}^{2^n}-1$ by making use of the projective immersion $\iota:\ \boldsymbol{V}\rightarrow\boldsymbol{P}^{2^n-1}$ defined in Corollary 2.

Let $[z_A]$, $A\in\boldsymbol{N}$, be homogeneous coordinates on $\boldsymbol{P}^{2^n-1}$. We denote by $\boldsymbol{U}_\phi$ the open subset in $\boldsymbol{P}^{2^n-1}$ defined by $z_\phi\neq0$. Putting $u_A=z_A/z_\phi$, for $A\neq\phi$, we have a system of coordinates (u_A) on $\boldsymbol{U}_\phi$. We denote by $\boldsymbol{N}'$ the subset of $\boldsymbol{N}$ omitting the empty set ϕ. It is convenient to stipulate that $\mu_\phi=1$. By (15),

$$u_A(\iota(V))=\xi_A(V),\ A\in N',\ \text{for}\ V\in\boldsymbol{V}_\phi.$$

From the definition of $\xi_{i_1\cdots i_k}$ for $k\geq3$ ((5) and (6) in **1.4**), we have

$$\partial\xi_{i_1\cdots i_k}/\partial\xi_i=\begin{cases}-\varepsilon(\{i\},\{i_1\cdots i_k\})\xi_{i_1\cdots\bar{i}\cdots i_k}, & \text{if } i\in\{i_1\cdots i_k\},\text{ and if } k \text{ is odd},\\ 0,\ \text{otherwise},\end{cases}$$

and

$$\partial\xi_{i_1\cdots i_k}/\partial\xi_{jk}=\begin{cases}-\varepsilon(\{jk\},\{i_1\cdots i_k\})\xi_{i_1\cdots\hat{j}\cdots\hat{k}\cdots i_k}, & \text{if}\{jk\subset\{i_1\cdots i_k\},\\ 0, \text{ otherwise},\end{cases}$$

where $j<k$.

Keeping these in mind, we define the following vector fields on U_ϕ:

(4) $$\partial/\partial u_i+\sum_{\{i_1\cdots i_k\}\supset\{i\},\ k\text{ odd},\ k\geq 3}-\varepsilon(\{i\},\{i_1\cdots i_k\})u_{i_1\cdots\hat{i}\cdots i_k}\partial/\partial u_{i_1\cdots i_k},$$

where $1\leq i_1<\cdots<i_k\leq n$, $(1\leq i\leq n)$, and

(5) $$\partial/\partial u_{jk}+\sum_{\{i_1\cdots i_k\}\supset\{jk\},\ k\geq 3}-\varepsilon(\{jk\},\{i_1\cdots i_k\})u_{i_1\cdots\hat{j}\cdots\hat{k}\cdots i_k}\partial/\partial u_{i_1\cdots i_k},$$

where $1<i_1<\cdots<i_k\leq n$, $(1\leq j<k\leq n)$.

These vector fields are tangent to the image $\iota(V_\phi)$, and the vector field $\partial/\partial\xi_i$ (resp. $\partial/\partial\xi_{jk}$) is ι-related to the vector field (4) (resp. (5)) on U_ϕ. A vector field Y on V_ϕ and a vector field Z on U_ϕ are said to be ι-related, if at each point $p\in V_\phi$, the image of Y_p under the differential ι_* is $Z_{\iota(p)}$ ([4], Chap. III).

2.4. Given a $(2^n, 2^n)$-complex matrix X^*, the one-parameter group $\exp tX^*$ induces a vector field on C^{2^n} and on $P^{2^n}-1$. If $X^*=(X^*_{AB})$, the (1, 0)-component of the corresponding vector field on C^{2^n} is

$$\textstyle\sum_A(\sum_B X^*_{AB}z_B)\partial/\partial z_A,$$

and that of the vector field induced on U_ϕ is

(6) $$\textstyle\sum_{A\neq\phi}\{\sum_{B\in N}(X^*_{AB}u_B-X^*_{\phi B}\ u_B u_A)\}\partial/\partial u_A.$$

The correspondence which assings to $X^*\in\mathfrak{sl}(2^n, C)$ the holomorphic vectorfield (6) on U_ϕ is an anti-isomorphism.

If the vector field (6) vanishes on the submanifold V_ϕ, then X^* is a scalar matrix and its infinitesimal projective transformation is identically zero. This is an immediate consequence of the following fact:

Given $A\in N'$, the polynomials in the set

$$\{\xi_B;\ B\in N\}\cup\{\xi_B\xi_A;\ B\in N'\}$$

are linearly independent over C on $V_\phi(=C^{2n+1})$.

We prove this proposition by induction on n. When $n=2$, we verify it by inspection. Next, we show that the polynomials ξ_B's $B\in N$, are linearly independent by induction on n. Suppose that $\sum c_B\xi_B=0$. Apply $\partial/\partial\xi_n$ on both sides of the equality and obtain a linear equation involving only those ξ_C's, where $C=B+\{i\}$, $n\in B$, and $\#(B)$ is odd, by the formulas for derivatives in **2.3**. By induction hypothesis, $c_B=0$ if $n\in B$, and $\#(B)$ is odd. Do the same with $\partial/\partial\xi_{jn}$, $1<j<n$. After thses operations, the original equality is reduced to $\sum c_{B'}\xi_{B'}=0$ where all $B'\subset\{1, \ldots, n-1\}$. By induction hypothesis $c_{B'}=0$. Thus, $c_B=0$ for all B.

If a given set A is $\{1, \ldots, n\}$, using degrees of polynomials and the linear independence of $\{\xi_B;\ B\in \boldsymbol{N}\}$, we can prove the claim easily. If $A\neq\{1, \ldots, n\}$, say $A\subset\{1, \ldots, n-1\}$, applying $\partial/\partial\xi_n$ and $\partial/\partial\xi_{jn}$ in the same way as above, we prove the statement.

2.5.

Thorem 2. *Given a matrix $X\in\mathfrak{g}_c$, there corresponds one and only one matrix $X^*\in\mathfrak{sl}(2^n, \boldsymbol{C})$ such that the infinitesimal projective transformation induced by X^* on $\boldsymbol{P}^{2n-1}$ is ι-related to the vector fied $\xi(X)$ induced by X on V_ϕ ((3)). The correspondence $X \mapsto X^*$ is an isomorphism and is the spin representation of the Lie algebra $\mathfrak{so}(2n+1, \boldsymbol{C})$. Under the isomorphism the image of $\mathfrak{g}$ is contained in $\mathfrak{su}(2^n)$.*

Proof. We denote by $E_{\lambda\mu}$ the $(2n+1, 2n+1)$ matrix whose (λ, μ)-entry is 1 and all other entries are zero, for $\lambda, \mu=0, 1, \ldots, 1', \ldots, n'$, and similarly by E^*_{AB} the $(2^n, 2^n)$ matrix whose (A, B)-entries is 1 and all other entries are zero, for A, $B\in\boldsymbol{N}$. The Lie algebra $\mathfrak{g}_c$ is generated by

$$E_{0i}-E_{j'0},\ E_{0j'}-E_{i0},\ 1\leq i\leq n.$$

Let X be one of these generators. First, we choose a vector field $u'(X)$ on $\boldsymbol{U}_\phi$ so that $\xi(X)$ and $u'(X)$ are ι-related. In the expression of $\xi(X)$ given by (1), (2) and (3), we substitute ξ_i, ξ_{jk}, $\partial/\partial\xi_i$ and $\partial/\partial\xi_{jk}$ by u_i, u_{jk}, the vector field (4) and the vector field (5) respectively, and obtain a desired $u'(X)$. Next, adding to $u'(X)$ a suitable vector field on $\boldsymbol{U}_\phi$ vanishing on the submanifold $\boldsymbol{V}_\phi$, we have a vector field $u(X)$, which is not only ι-related to $\xi(X)$ but also an infinitesimal projective transformation induced by some $X^*\in\mathfrak{sl}(2^n, \boldsymbol{C})$. Once such X^* is found, it is unique in virtue of **2.4**.

If $X=E_{0j}-E_{j'0}$,

$$\xi(E_{0i}-E_{i'0})=(1/\sqrt{2})(\partial/\partial\xi_i+\textstyle\sum_{j,j<i}\xi_j\partial/\partial\xi_{ji}-\sum_{j,i<j}\xi_j\partial/\partial\xi_{ji}).$$

On account of the equality for an even k

$$\textstyle\sum_{a=1}^{k}(-1)^{a-1}u_{i_a}u_{i_1\ldots\hat{i}_a\ldots i_k}=0 \text{ on } \boldsymbol{V}_\phi,$$

we put

$$u(E_{0i}-E_{i'0})=$$
$$(1/\sqrt{2})\textstyle\sum_{\{i_1\ldots i_k\}\supset\{i\}}(-1)^k\varepsilon(\{i\}, \{i_1\ldots i_k\})u_{i_1\ldots\hat{i}\ldots i_k}\partial/\partial u_{i_1\cdots i_k}$$
$$(1\leq i_1<\ldots<i_k\leq n).$$

Then, $u(E_{0i}-E_{i'0})$ is the infinitesimal projective transformation associated to the $(2^n, 2^n)$ matrix

$$(1\sqrt{2})\textstyle\sum_{A;\ i\in A}(-1)\varepsilon(A, \{i\})E^*_{AA+\{i\}}, \tag{7}$$

and is ι-related to $\xi(E_{0i}-E_{i'0})$.

If $X=E_{0i'}-E_{i0}$, $\xi(E_{0i'}-E_{i0})=(1/\sqrt{2})\{\sum_j(\xi_{ij}+\xi_j\xi_i)\partial/\partial\xi_j+\sum_{j<k}(\xi_j\xi_{ik}-\xi_k\xi_{ij})\partial/\partial\xi_{jk}\}$.

Using the equality for an even k

$$\sum_{a=1}^{k}(-1)^{a-1}u_{i_a l}u_{i_1\ldots\hat{i}_a\ldots i_k}=u_l u_{i_1\ldots i_k}-u_{i_1\ldots i_k l} \text{ on } V_\phi,$$

we put

$$u(E_{0i'}-E_{i0})=(1\sqrt{2})\sum_{\{i_1\cdots i_k\}}(-u_{i_1\cdots i_k i}+u_i u_{i_1\ldots i_k})\partial/\partial u_{i_1\cdots i_k}(1\leq i_1<\ldots<i_k\leq n),$$

which is ι-related to $\xi(E_{0i'}-E_{i0})$, and is the infinitesimal projective transformation determined by the matrix

$$(8) \qquad (1/\sqrt{2})\sum_{A\in\mathbf{N};\, i\notin A}(-1)\varepsilon(A,\{i\})E^*_{AA+\{i\}}$$

in $\mathfrak{sl}(2^n, \boldsymbol{C})$.

Since the ι-relatedness between $\xi(X)$ and $u(X)$ preserves the bracket product, it is obvious that there is a unique isomorphism s from $\mathfrak{g}_c$ into $\mathfrak{sl}(2^n, \boldsymbol{C})$ which maps $E_{0i}-E_{i'0}$ to the matrix (8) and $E_{0i'}.-E_{i0}$ to the matrix (9), $1\leq i\leq n$.

In order to show that s is the spin representaton, let

$$H=\sqrt{-1}\sum_i\lambda_i(E_{ii}-E_{i'i'}),\ \lambda_i\in\boldsymbol{R}.$$

The subspace spanned by these diagonal matrics is a Cartan subalgebra in $\mathfrak{g}_c$. From the equalitiy

$$E_{ii}-E_{i'i'}=[E_{0i}-E_{i'0},\ E_{0i'}-E_{i0}],\ 1\leq i\leq n,$$

in follows that

$$(9) \qquad s(H)=\sqrt{-1}\sum_{\{i_1\cdots i_k\}}(1/2\sum_{i=1}^{n}\lambda_i-\sum_{a=1}{}^{k}\lambda_{i_a})E_{\{i_1\cdots i_k\},\{i_1\cdots i_k\}}.$$

Thus, the highest weight of the representation s is $(1/2)\sum_i\lambda_i$, and s is the spin representation.

The compact form $\mathfrak{g}$ is generated by

$$-(1/\sqrt{2})(E_{0i}-E_{i'0}+E_{0i'}-E_{i0}),$$
$$(\sqrt{-1}/\sqrt{-2})(E_{0i}-E_{i'0}+E_{0i'}-E_{i0}),\ 1\leq i\leq n,$$

and their images under s are

$$(1/2)\sum_{A\in\mathbf{N}}\varepsilon(A,\{i\})E^*_{AA+\{i\}},$$
$$(\sqrt{-1}/2)\sum_{A\in\mathbf{N}}\varepsilon(A,\{i\})\delta(A,\{i\})E^*_{AA+\{i\}},$$

where $\delta(A,\{i\})=-1$ if $i\in A$, $=1$ if $i\notin A$, respectively, $1\leq i\leq n$. These matrices are in $\mathfrak{su}(2^n)$ and hence $s(\mathfrak{g})\subset\mathfrak{su}(2^n)$. We have finished the proof.

2.6. Under the spin representation $s: \mathfrak{so}(2n+1, \boldsymbol{C}) \rightarrow \mathfrak{sl}(2^n, \boldsymbol{C})$,

$$s(E_{0i}-E_{i'0})=(1/\sqrt{2})\sum_{A;\, i\in A}(-1)\varepsilon(A, \{i\})E^{*}_{AA+\{i\}},\ 1\leq i\leq n;$$

$$s(E_{0i'}-E_{i0})=(1/\sqrt{2})\sum_{A\in \mathbf{N};\, i\notin A}(-1)\varepsilon(A, \{i\})E^{*}_{AA+\{i\}},\ 1\leq i\leq n;$$

$$s(E_{ii}-E_{i'i'})=(1/2)\{-Z_{A\in N;\, i\in A}E^{*}_{AA}+Z_{A\in N;\, i\notin A}E^{*}_{AA}\},\ 1\leq i\leq n;$$

$$s(E_{jk}-E_{k'j'})=Z_{A;\, A\supset\{jk\}}\varepsilon(A, \{jk\})(E^{*}_{A+\{j\}A+\{k\}}-E^{*}_{A+\{k\}A+\{j\}}),\ 1\leq j,\ k\leq n;$$

$$s(E_{j'k}-E_{k'j})=Z_{A;\, A\supset\{jk\}}(-1)\varepsilon(A, \{jk\})E^{*}_{AA+\{jk\}},\ 1\leq j<k\leq n;$$

$$s(E_{jk'}-E_{kj'})=Z_{A;\, A\supset\{jk\}}\varepsilon(A, \{jk\})E^{*}_{A+\{jk\}A},\ 1\leq j<k\leq n.$$

2.7. The Clifford algebra $\mathfrak{a}$ over the vector space $\boldsymbol{R}^{2n+1}$ equipped with the standard inner product is the quotient algebra of the tensor algebra over $\boldsymbol{R}^{2n+1}$ modulo the ideal generated by $v\otimes v+(v, v)1$, $v\in\boldsymbol{R}^{2n+1}$ ([1], [3] and [6]). It is known that the spin representation is associated to a representation of the Clifford algebra $\mathfrak{a}$ on the same representation module $\boldsymbol{C}^{2^n}$. Here, we give its description. We use the same notation s for the representation.

Let $\{\varepsilon_\lambda\}$ be the standard basis of $\boldsymbol{R}^{2n+1}$ as in **1.1**. The algebra $\mathfrak{a}$ is generated by 1 and the ε_λ's. We set

$$s(\varepsilon_0)=\sqrt{-1}_{A\in N}(-1)^{\#(A)}E^{*}_{AA},$$

$$s(\varepsilon_i)=\sqrt{-1}Z_{A\in N}(-1)^{\#(A)}\varepsilon(A, \{i\})E^{*}_{AA+\{i\}},$$

$$s(\varepsilon_{i'})=\sqrt{-1}Z_{A\in N}(-1)^{\#(A)}\varepsilon(A, \{i\})\delta(A, \{i\})E^{*}_{AA+\{i\}},$$

where $\delta(A, \{i\})=-1$ if $i\in A$, $=1$ if $i\notin A (1\leq i\leq n)$. Then, for $v=Z_\lambda v_\lambda\varepsilon_\lambda\in\boldsymbol{R}^{2n+1}$, $s(v)^2+(v, v)\boldsymbol{I}=0$. Thus, s is a representation of the algebra $\mathfrak{a}$. Moreover,

$$s((1/2)\varepsilon_0 e_i)=-s(E_{0i}-E_{i'0}),\quad\text{and}\quad s((1/2)\varepsilon_0 e_{i'})=-s(E_{0i'}-E_{i0})(1\leq i\leq n).$$

Hence, the spin representation is determined by the above representation of the algebra $\mathfrak{a}$.

2.8. Let $\mathfrak{g}_1$ be the subalgebra in $\mathfrak{g}$ consisting of matrices X with

$$X_{00}=X_{0i'}=X_{i0}=X_{0i}=X_{i'0}=0 (i=1, \cdots, n).$$

The subalgebra is isomorphic to the orthogonal Lie algebra $\mathfrak{o}(2n)$. From the result **2.5**, it is easy to show that the restriction to $\mathfrak{g}_1$ of the spinor representation of $\mathfrak{o}(2n+1)$ on $\boldsymbol{C}^{2^n}$ is the direct sum of the two inequivalent irreducible representations. The subspace in $\boldsymbol{C}^{2^n}$ of vectors z whose components $z_A=0$ if $\#(A)$ is odd, is an irreducible $\mathfrak{g}_1$-module whose highest weight is $(1/2)(\lambda_1+\cdots+\lambda_n)$; the subspace of vectors z with $z_A=0$ if $\#(A)$ is even, is also irreducible and its highest weight is $(1/2)(\lambda_1+\cdots+\lambda_{n-1}-\lambda_n)$.

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