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KURANISHI FAMILY OF STRONGLY PSEUDO-CONVEX DOMAINS

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Introduction

The purpose of this paper is to complete T. Akahori's construction of the semi-universal family of strongly pseudo-convex (s.p.c. for abbreviation) domains of $\dim_C X \geq 4$ (cf. [1], [2]), in the general deformation theoretic context. In [6], J. Bingener and S. Kosarew considered deformations of s.p.c. complex spaces, and they proved the existence of the semi-universal family for deformations as germs along a compact subspace with a certain property (in the case of a non-singular complex space, along the exceptional subset) and conjectured the exstence of a formally semi-universal convergent family for deformations as whole complex spaces. In this paper, we will consider deformations of s.p.c. manifolds and will show that the Akahori's canonical family of complex structures on a compact level subset of a strictly pluri-subharmonic exhaustion function (cf. [1], [2]) induces both the semi-universal family and the formally semi-universal converent family as above, if $\dim_C X \geq 4$.

The key step is to construct the semi-universal family for deformations as germs along a compact level subset from the Akahori's canonical family of complex structures. Though the correspondence between families of complex structures and of complex manifolds is not direct, it is rather simple if we restrict ourselves in the classical deformation theory. Let r be a strictly pluri-subharmonic exhaustion function on a s.p.c. complex manifold X with $\dim_C X \geq 4$. We denote $\Omega_\varepsilon = \{x \in X \mid r(x) < \varepsilon\}$ and suppose $K = \cap_{\alpha < \varepsilon} \Omega_\varepsilon$ for some $\inf_X r \leq \alpha < \sup_X r$. Consider a fibred groupoid $p: F_K \to \mathscr{C}$ (the category of germs of complex spaces) of deformations of X as germs along K and its restriction over $\mathscr{C}_{\text{red}}$ (the category of germs of reduced complex spaces) $p_{\text{red}}: (F_K)_{\text{red}} \to \mathscr{C}_{\text{red}}$. In [1] and [2], Akahori constructed a family of complex structures over each $\bar{\Omega}_\varepsilon$ ($\alpha < \varepsilon$) which is effective and complete in the sense that any family of deformations of a neighbourhood of $\bar{\Omega}_\varepsilon$ is induced from that family of complex structures over $\bar{\Omega}_\varepsilon$ (what is called "*versal in the sence of Kuranishi*" in [1]). This implies that, for each $\varepsilon > \alpha$, there exists a family $\mathscr{X}_\varepsilon \to T_\varepsilon$

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in F_K which is effective, complete in the sense that any family of deformations of a neighbourhood of $\bar{\Omega}_\varepsilon$ is induced from $\mathscr{X}_\varepsilon \to T_\varepsilon$ in $(F_K)_{\text{red}}$, and moreover, by the same argument of [16, §2], is formally versal in F_K. Hence the families $\mathscr{X}_\varepsilon \to T_\varepsilon$ are all isomorphic to each other in $(F_K)_{\text{red}}$ and induce an effective family in $(F_K)_{\text{red}}$. The completeness of that effective family in $(F_K)_{\text{red}}$ immediately follows from the above completeness of each $\mathscr{X}_\varepsilon \to T_\varepsilon$. In order to argue in the general deformation theoretic context, we need to remove the restriction on $(F_K)_{\text{red}}$ and consider the *versality* (i.e. having the lifting property) instead of the *completeness*. We can generalize *compleness* in the Akahori's result to *having the lifting property* by modifying the argument of [3, §3] (cf. Proposition 5.2). The main technical improvement is the adaptation of the division theorem by a submodule (cf. [14], [5]) instead of the small trick used in [3, Proposition 3.1]. The generalization from $(F_K)_{\text{red}}$ to F_K is done by the following criterion for versality which is an easy consequence of an argument in [7, pp.415–416]: *Let $p: F \to \mathscr{C}$ be a fibred groupoid satisfying the Schlessinger's conditions* (S1′) *and* (S2), *and $w \in F$ a formally versal element. Then, w is versal if and only if w has the lifting property for any extension $S \to S' = S'_{\text{red}}$.* (In our case, (S2) is due to the coherency theorem for direct images under 1-convex maps (cf. [18]).) Hence we have

Theorem 1. *There exists a semi-universal family in F_K.*

If we set K = the exceptional subset, Theorem 1 asserts the J. Bingener-S. Kosarew's existence theorem in the case of deformations of s.p.c. manifolds of $\dim_{\mathbf{C}} X \geq 4$.

It is a simple matter to obtain a formally semi-universal convergent family in F_X from Theorem 1. Fix $\varepsilon_1 < \varepsilon_2 < \cdots \to \sup_X r$ such that $dr \neq 0$ on $X \backslash \bar{\Omega}_{\varepsilon_1}$. Since $\mathscr{X}_{\varepsilon_i} \to T_{\varepsilon_i}$ and $\mathscr{X}_{\varepsilon_{i+1}} \to T_{\varepsilon_{i+1}}$ are isomorphic to each other in $F_{\bar{\Omega}_{\varepsilon_{i-1}}}$, we have a family $\mathscr{X} \to T$ in F_X by patching all $\mathscr{X}_{\varepsilon_i} \to T_{\varepsilon_i}$ together. From the versality of each $\mathscr{X}_{\varepsilon_i} \to T_{\varepsilon_i}$ and the Schlessinger's formal existence theorem (cf. [17]), we infer that $\mathscr{X} \to T$ has the property conjectured in [6, Bemarkungen (5.10)]. Hence we have

Theorem 2. *There exists a formally semi-universal convergent family in F_X which is a versal element in F_K for any strongly pseudo-convex compact subset*[1] *K in X.*

The argument of this paper has a further generality. Let (V,o) be an isolated singularity with $\dim_{\mathbf{C}}(V,o) \geq 4$ and r a strictly pluri-subharmonic exhaustion function on V which is C^∞ and $dr \neq 0$ on $V \backslash o$. We denote $\Omega_{\delta,\varepsilon} = \{x \in V \backslash o \mid \delta < r(x) < \varepsilon\}$ $(\inf_X r < \delta < \varepsilon < \sup_X r)$ and let $K = \cap_{\delta < \alpha, \beta < \varepsilon} \Omega_{\delta,\varepsilon}$ for some fixed $\inf_X r < \alpha \leq \beta < \sup_X r$. If

1) A compact subset in a s.p.c. manifold X is called a strongly pseudo-conex compact subset if it is the inverse image of a Stein compact subset in the Remmert quotient of X.

we consider the fibred groupoid F_K of deformations of the regular part along K, T. Akahori's construction is still workding on $\bar{\Omega}_{\delta,\varepsilon}$ and the argument of this paper works without any change. If moreover depth$(V,o)\geq 3$, (S2) is satisfied due to the coherency theorem for direct images in the case of a (1.1)-complete map (cf. [12, Proposition 4.4] combined with [13]). Hence we have

Theorem 3. *If* $\dim_C(V,o)\geq 4$ *and* $\mathrm{depth}(V,o)\geq 3$ *then there exists a formally semi-universal convergent family of deformations of* $V\backslash o$ *which is semi-universal in* F_K.

In particular, if $\alpha=\beta$, then we have the versality of the formally semi-universal convergent family of deformations of $V\backslash o$ near a compact real hyper-surface, which was obtained in [16, Theorem 1].

The arrangement of this paper is as follows. In §1, we will obtain the above criterion for versality. In §§2 and 3, we summarize the property of the Akahori's canonical family of complex structures and the division theorem by a submodule, in the form needed in the argument of §§4 and 5. In §§4 and 5, we will prove Theorem 1. Though the argument is a modification of [3, §3] by means of the division theorem by a submodule, we will describe it explicitly because we need extra care other than in [3, §3], in order for the formal solution to be convergent. We will prove Theorem 2 in §6.

1. A criterion for versality

Let $\mathscr{C}$ be the category of all germs of complex spaces. A fibred groupoid over $\mathscr{C}$ is a category F with a covariant functor $p:F\to\mathscr{C}$, such that the following holds:

(1.1) (Existence of basechange) For any morphism $\phi:S\to S'$ and any $a'\in F$ with $p(a')=S'$ there exists a lifting $a\to a'$ of ϕ to F.

(1.2) (Uniqueness) Let $\begin{array}{ccc} S' & \to & S'' \\ \uparrow & \nearrow & \\ S & & \end{array}$ be a commutative diagram in $\mathscr{C}$, then any partial lifting to F of solid arrows $\begin{array}{ccc} a' & \to & a'' \\ \vdots & \nearrow & \\ a & & \end{array}$ can be completed by a unique dotted arrow.

We will denote by $F(S)$ $(S\in\mathscr{C})$ a subcategory of F with $\mathrm{Obj}\,F(S):=\{a\in F\,|\,p(a)=S\}$ and $\mathrm{Hom}_{F(S)}(a',a''):=\{(a'\to a'')\in\mathrm{Hom}_F(a',a'')\,|\,p(a'\to a'')=id_S\}$. For $a\in F(S)$, we denote by F_a the category consisting of all morphisms $a\to a'$ and $\mathrm{Hom}_{F_a}(a',a'')$

$:= \{(a' \to a'') \in \mathrm{Hom}_F(a', a'')$ satisfying the commutative diagram $\begin{matrix} a' & \to & a'' \\ \uparrow & \nearrow & \\ a & & \end{matrix}$ $\}$, and denote by $\bar{F}_a$ the set of all isomorphism classes.

Now we recall the Shlessinger's conditions.

(S1) Let $\begin{matrix} S'' & & \\ \uparrow & & \\ S & \xrightarrow{\alpha} & S' \end{matrix}$ be a diagram in $\mathscr{C}$ such that α is an infinitesimal extension (i.e. $\mathcal{O}_{S'} \xrightarrow{\alpha^*} \mathcal{O}_S$ is surjective with nilpotent kernel) and $S_{\mathrm{red}} \to S''$ is a closed embedding. Then for any $a \in F(S)$, the canonical map $F_a(S' \amalg_S S'') \to F_a(S') \times F_a(S'')$ is surjective.

(S1′) In the above situation, the canonical map $F_a(S' \amalg_S S'') \to F_a(S') \times F_a(S'')$ is an equivalence of categories.

(S2) Let $\mathscr{M}$ be a coherent $\mathcal{O}_{S_{\mathrm{red}}}$-module. Then $\bar{F}_a(S[\mathscr{M}])$ is a finitely generated $\mathcal{O}_{S_{\mathrm{red}}}$-module.

EXAMPLE. Let X be a complex manifold and K a compact subset of X. We consider deformations of X as germs along K. Precisely, $F_K = \varinjlim_{U \supset K} F_U$ where U is an open domain in X and F_U denotes the fibred groupoid of deformations of U. Clearly F_K is a fibred groupoid over $\mathscr{C}$ and satisfies (S1′) because each F_U does. If $a \in F_K(S)$ is represented by $\pi: \mathscr{X} \to S$ and $\mathscr{M}$ is a coherent $\mathcal{O}_{S_{\mathrm{red}}}$-module, then $(\bar{F}_K)_a(S[\mathscr{M}]) = \varinjlim_{U \supset K} H^1(\mathscr{U}, \Theta_{\mathscr{U}/S} \otimes_{\mathcal{O}_{S_{\mathrm{red}}}} \mathscr{M})$.

Let $p: F \to \mathscr{C}$ be a fibred groupoid over $\mathscr{C}$ and $w \in F(T)$. We say that w *has the lifting property for an extension* $S \to S'$ in $\mathscr{C}$ if any diagram $\begin{matrix} w & & \\ \uparrow & \nwarrow & \\ a & \to & a' \end{matrix}$ of solid arrows with $p(a \to a') = S \to S'$ is completed by a dotted arrow.

DEFINITION 1.1. Let $w \in F$.

(1) w is versal if w has the lifting property for any extension $S \to S'$ in $\mathscr{C}$.

(2) w is formally versal if w has the lifting property for any extension $S \to S'$ in $\mathscr{C}_0$ where $\mathscr{C}_0$ denotes the subcategory of $\mathscr{C}$ consisting of all artinian complex spaces.

DEFINITION 1.2. Let $w \in F$. w is effective if $dp(\alpha)_0 = dp(\beta)_0$ holds for any two morphisms α, $\beta: a \to w$ in F.

DEFINITION 1.3. Let $w \in F$. w is semi-universal (resp. formally semi-universal) if w is versal (resp. formally versal) and effective.

In this section, we will obtain the following criterion for a formally versal element to be versal.

Theorem 1.1. *Let $p: F \to \mathscr{C}$ be a fibred groupoid satisfying* (S1′) *and* (S2), *and $w \in F$ a formally versal element. Then w is versal if and only if w has the lifting property for any extension $S \to S'$ with $S' = S'_{\mathrm{red}}$.*

The following is the key proposition.

Proposition 1.2. *Suppose that a fibred groupoid $p: F \to \mathscr{C}$ satisfies* (S1) *and* (S2). *Let $w \in F$ be a formally versal element, then w has the lifting property for any infinitesimal extension $S \to S'$ in $\mathscr{C}$.*

A proof using [8, Satz 3.2] is found in [7, pp.415–416].

Proof of Theorem 1.1. We will prove if-part. (Only if-part is trivial.) Let a diagram
$$\begin{tikzcd} & w \\ a \arrow[r] \arrow[ur, phantom, ""] & a' \end{tikzcd}$$
$\begin{array}{ccc} w & & \\ \uparrow & & \\ a & \to & a' \end{array}$ over $\begin{array}{ccc} T & & \\ \uparrow & & \\ S & \to & S' \end{array}$ be given. We may assume that the morphism $S \to T$ is an embedding.

Let $S_0 := S \cap S'_{\mathrm{red}}$, $a_0 := a \times_S S_0$ and $a'_{\mathrm{red}} := a' \times_{S'} S'_{\mathrm{red}}$. Then we have a diagram $\begin{array}{ccc} w & & \\ \uparrow & & \\ a_0 & \to & a'_{\mathrm{red}} \end{array}$ over $\begin{array}{ccc} T & & \\ \uparrow & & \\ S_0 & \to & S'_{\mathrm{red}} \end{array}$ By the assumption, we have a lifting $a'_{\mathrm{red}} \to w$ over $S'_{\mathrm{red}} \to T$. Hence we have a morphism $S \amalg_{S_0} S'_{\mathrm{red}} \xrightarrow{\phi} T$ such that $\phi^* w \times_{S\amalg_{S_0} S'_{\mathrm{red}}} S = a$ and $\phi^* w \times_{S\amalg_{S_0} S'_{\mathrm{red}}} S'_{\mathrm{red}} = a'_{\mathrm{red}}$. Since (S1′) holds for F, we have $\phi^* w = a'_1$ in $F_a(S \amalg_{S_0} S'_{\mathrm{red}})$ where $a'_1 := a' \times_{S'} (S \amalg_{S_0} S'_{\mathrm{red}})$ (note that $S_0 \to S$ is an infinitesimal embedding). Since $S \amalg_{S_0} S'_{\mathrm{red}} \to S'$ is an infinitesimal embedding, by Proposition 1.2, the diagram of solid arrows $\begin{tikzcd} w & \\ a'_1 \arrow[u] \arrow[r] & a' \arrow[ul, dotted] \end{tikzcd}$ over $\begin{tikzcd} T & \\ S \amalg_{S_0} S'_{\mathrm{red}} \arrow[u] \arrow[r] & S' \arrow[ul, dotted] \end{tikzcd}$ can be completed by dotted arrows. Q.E.D.

2. Deformations along a s.p.c. compact level subset-construction of the Kuranishi family

Let X be a s.p.c. complex manifold with $\dim_C X \geq 4$ and with an exhaustion function r which is strictly plurisubharmonic of C^∞-class and $dr\neq 0$ outside a compact subset K. We will denote $\Omega_\varepsilon=\{x\in X\,|\,r(x)<\varepsilon\}$ and suppose $K=\cap_{\alpha<\varepsilon}\Omega_\varepsilon$ for a fixed $\inf_X r\leq\alpha<\sup_X r$. We will consider the fibred groupoid $p:F_K\to\mathscr{C}$ given by $\mathrm{Obj}(F_K)=\{$a smooth morphism $\pi:\mathscr{X}\to S$ as a germ along $K\}$, $\mathrm{Hom}_{F_K}(a,a'):=\{$a pair of morphisms $\sigma:T\to T'$ and $\Phi:\mathscr{X}\to\mathscr{X}'$ such that $\begin{array}{ccc}\mathscr{X} & \to & \mathscr{X}' \\ \downarrow & & \downarrow \\ T & \to & T'\end{array}$ is commutative and $\mathscr{X}=\mathscr{X}'\times_{T'}T\}$ and $p(\pi:\mathscr{X}\to S)=S$. Then F_K satisfies (S1′) and (S2) by the coherency theorem for direct images in the case of 1-convex maps (cf. §1, Example and [18, Main Theorem]).

We recall the property of the canonical family of deformations of complex structures on $\bar\Omega_\varepsilon$ in [1]. Fix an $\varepsilon>\alpha$ and let $r:=\dim_{\boldsymbol{C}}H^1_{\bar\partial}(\bar\Omega_\varepsilon,T'X)$. Let $\|\ \|'_{(0,k)}$ be the norm introduced in [1, §4] and denote by $\mathscr{A}^{0,q}_k$ the completion of $A^{0,q}(\bar\Omega_\varepsilon,T'X)$ with respect to $\|\ \|'_{(0,k)}$. In [1], we obtained a powerseries $\phi(t)\in\mathscr{E}^1[[t_1,\cdots,t_r]]$ satisfying

(2.1) $\phi(0)=0=$ and $\phi(t)\in\mathscr{A}^{0,1}_k\{t_1,\cdots,t_r\}$ for all $k\geq n+2$,

(2.2) if we denote by $\phi_1(t)$ the linear term of $\phi(t)$, then $\bar\partial\phi_1(t)=0$ and $\phi_1(t)$ spans $H^1_{\bar\partial}(\bar\Omega_\varepsilon,T'X)$,

(2.3) $\phi(t)$ is real-analytic on a neighbourhood of $\bar\Omega_{\varepsilon_*}\times D$, where $\varepsilon_*:=\frac{\varepsilon+\alpha}{2}$ and D is a neighbourhood of the origin of $\boldsymbol{C}^r$,

(2.4) $\bar\partial\phi(t)-\frac{1}{2}[\phi(t),\phi(t)]\in\mathfrak{J}_{T_\varepsilon}\mathscr{A}^{0,2}_{(0,k)}\{t^1,\cdots,t^r\}$ for all $k\geq n+2$, where $\mathfrak{J}_{T_\varepsilon}$ denotes the defining ideal of the analytic subspace $T_\varepsilon:=h^{-1}(0)$ of D by denoting $h(t):=\boldsymbol{H}[\phi(t),\phi(t)]$ and $\mathscr{A}^{0,2}_{(0,k)}$ denotes the completion of $A^{(0,2)}(\bar\Omega_\varepsilon,T'X)$ by the tangential Sobolev (0,k)-norm.

Here $\mathscr{E}^1$ denotes a subspace of $A^{0,1}(\bar\Omega_\varepsilon,T'X)$ induced in [1]. We will need the property $\|\phi f\|_{(0,k)}\leq c_k\|\phi\|'_{(0,k)}\|f\|'_{(0,k)}$ for $\phi\in\mathscr{E}^1$ and $f\in\Gamma(\bar\Omega_\varepsilon,1)$ where c_k is a constant independent of ϕ and f.

By an ideal theoretic improvement of the argument of [15, Ch.7], from (2.3) and (2.4), we have a smooth morphism $\pi_\varepsilon:\mathscr{X}_\varepsilon\to T_\varepsilon$ such that $\pi_\varepsilon^{-1}(0)=\Omega_{\varepsilon_*}$. By the same argument as [16, §2], we have

Proposition 2.1. *$\pi_\varepsilon:\mathscr{X}_\varepsilon\to T_\varepsilon$ is formally semi-universal in F_K.*

3. The division theorem by a submodule

In this section, we recall a division theorem with estimate which will be a basic tool in the argument of §§4 and 5.

(I) Pseudo-norm. Let $\sum_{\nu\in\boldsymbol{N}^q}\gamma_\nu s^\nu$ be the Taylor series expansion of $M(s)=\prod_{i=1}^{q}$

$(\frac{1}{2}-\frac{1}{4}\sqrt{1-s_i})$. Let K be a Banach space with a norm $|\ |_K$. For $\rho=(\rho_1,\cdots,\rho_q)\in(R_+^*)^q$ and $f(s)\in K[[s]]:=K[[s_1,\cdots,s_q]]$, we define a pseudo-norm $\|f(s)\|_\rho:=\sup_\nu\{|f_\nu|_K\rho^\nu/\gamma_\nu\}$ and a pseudo-norm on $K[[s]]^m$ by $\|(f_{(1)}(s),\cdots,f_{(m)}(s))\|_\rho:=\max_{1\le\kappa\le m}\{\|f_{(\kappa)}(s)\|_\rho\}$. Refer to [10] for the properties of the pseudo-norm $\|\ \|_\rho$.

(II) Division Theorem. We introduce a total ordering in $N^q\times\{1,\cdots,m\}$. For $(\nu,i)=(\nu_1,\cdots,\nu_m;i)$ and $(\mu,j)=(\mu_1,\cdots,\mu_m;j)$ in $N^q\times\{1,\cdots,m\}$, we set $(\nu,i)>(\mu,j)$ if $|\nu|>|\mu|$, if $|\nu|=|\mu|$ and $i>j$, or if $|\nu|=|\mu|$, $i=j$ and $\nu<\mu$ with respect to the lexicographical ordering in N^q. A subset E of $N^q\times\{1,\cdots,m\}$ is called a monoideal if $E+N^q=E$ holds. $(\nu,i)\in E$ is an *extremal point* if $(\nu,i)\notin\cup_{(\nu,i)\neq(\mu,j)\in E}(\mu,j)+N^q$. In a monoideal, there exist at most a finite number of extremal points (cf. [14, Lemma (1.1.8)]). Let $\{(\mu_k,j_k)\}_{1\le k<p}$ be the set of all extremal points of a monoideal E. Then we have a partition of $N^q\times\{1,\cdots,m\};\Delta_k:=((\mu_k,j_k)+N^q)\backslash(\cup_{k'<k}\Delta_{k'})$ $(1\le k\le p)$ and $\Delta:=(N^q\times\{1,\cdots,m\})\backslash E$.

Let K be a Banach C-space (resp. C-vector space) and denote by $K\{s\}$ (resp. $K[[s]]$) the convergent powerseries ring (resp. the formal powerseries ring) in $s:=(s_1,\cdots,s_q)$ with coefficients in K. $f\in K[[s]]^m$ is expressed as $f=(\Sigma_\nu f_{(1)\nu}s^\nu,\cdots,\Sigma_\nu f_{(m)\nu}s^\nu)$. f is called to be *reduced with respect to* Δ if $f_{(i)\nu}=0$ for $(\nu,i)\notin\Delta$. For $f\in K[[s]]^m$, we denote $\exp(f)=\inf\{(\nu,i)\,|\,f_{(i),\nu}\neq 0\}$.

Now let $\mathscr{M}\subset C\{s\}^m$ be a submodule, then $E(\mathscr{M}):=\{\exp(f)\,|\,f\in\mathscr{M}\}$ is a mono-ideal of $N^q\times\{1,\cdots,m\}$. Let $\{(\mu_k,j_k)\}_{1\le k\le p}$ be the set of all extremal poits of $E(\mathscr{M})$ and $(\Delta_1,\cdots,\Delta_p,\Delta)$ the associated partition of $N^q\times\{1,\cdots,m\}$. The division theorem by a submodule $\mathscr{M}$ is as follows.

Theorem 3.1. ([14, Theorem (1.2.2)], [5,Theorem 6.2]) *There exists a system of generators $f^1,\cdots,f^p$ of $\mathscr{M}$ having the following property: Let K be an arbitrary C-vector space.*

(1) $\exp(f^k)=(\mu_k,j_k)$ $(1\le k\le p)$.

(2) *$f\in K[[s]]^m$ is expressed uniquely as*

$$f=\sum_{k=1}^p g_kf^k+h$$

where $g_k=\Sigma_{(\nu,j_k)\in\Delta_k}(g_k)_\nu s^{\nu-\mu_k}\in K[[s]]$ $(1\le k\le p)$, $h\in K[[s]]^m$ and h is reduced with respect to Δ.

(3) *If K is a Banach C-space and f is in $K\{s\}^m$ then $g_k\in K\{s\}$ $(1\le k\le p)$ and $h\in K\{s\}^m$.*

(4) *For any $a\in(R_+^*)^q$, $a\in(R_+^*)^q\cup\{(0,\cdots,0)\}$ if $m=1$, there exists an open set of poly-radious $V_{\beta,C}=\{\rho=(\eta^{1+\delta_1},\cdots,\eta^{1+\delta_q})\in(R_+^*)^q\,|\,(\delta_1,\cdots,\delta_q)\in\Lambda_\beta,\ 0<\eta<C\ (\delta_1,\cdots,\delta_q)\}$ such that*

$$\|g_k\|_\rho \le 2m\rho^{-\mu_k - a}\|f\|_\rho (k=1,\cdots,p) \text{ and}$$

$$\|h\|_\rho \le 2m\rho^{-a}\|f\|_\rho \text{ hold for all } \rho \in \Lambda_{\beta,C}$$

hold where $\Lambda_\beta = \{(\delta_1,\cdots,\delta_q) \in (\boldsymbol{R}_+^*)^q \mid 0 < \delta_1 < \delta_2\beta_1 < \cdots < \delta_n\beta_{n-1} < \beta_n\}$ *for some* $0 < \beta_1 < \cdots < \beta_n$ *and* $C: \Lambda_\beta \to \boldsymbol{R}_+^*$ *is a continuous function.*

REMARK 3.1. Though the above (4) is obtained with respect to another norm in [14], we can obtain this estimate also by a similar argument.

From now on, we denote by $\mathrm{red}_{\mathscr{M}} f$ the h in Theorem 3.1(2).

Let $\mathfrak{J} \subset \boldsymbol{C}\{s\}$ be an ideal and $\{\lambda_1,\cdots,\lambda_m\}$ the set of all extremal points of $E(\mathfrak{J})$. Then, by Theorem 3.1, we have a unique system of generators $\{\omega_1,\cdots,\omega_m\}$ of an ideal $\mathfrak{J}$ such that $\omega_\kappa = s^{\lambda_\kappa} + \alpha_\kappa$ with α_κ reduced with respect to Δ and $\exp(\alpha_\kappa) > \lambda_\kappa$. We call such a system of generators of $\mathfrak{J}$ *the Weierstraß family* of $\mathfrak{J}$.

Now we consider $\mathfrak{J}$, $\mathfrak{J}'$ ideals of $\boldsymbol{C}\{s\}$. Let $\{\omega_1,\cdots,\omega_m\}$ be the Weierstraß family of $\mathfrak{J}$. Let $\bar{\chi}: \boldsymbol{C}\{s\}^m \to \mathfrak{J}$ be a homomorphism defined by $\bar{\chi}(g_{(1)},\cdots,g_{(m)}) = \Sigma_{\kappa=1}^m g_{(\kappa)}\omega_\kappa$. Set $\mathscr{M} := \bar{\chi}^{-1}(\mathfrak{J} \cap \mathfrak{J}')$. For any Banach $\boldsymbol{C}$-space $\boldsymbol{K}$ we define a homomorphism $\chi: \boldsymbol{K}\{s\}^m \to \mathfrak{J}\boldsymbol{K}\{s\}$ by the same way. From Theorem 3.1, by the Hahn-Banach extension theorem, we infer

Lemma 3.2.

(1) $\chi(g_{(1)},\cdots,g_{(m)}) \in \mathfrak{J}'\boldsymbol{K}\{s\}$ *if and only if* $(g_{(1)},\cdots,g_{(m)}) \in \mathscr{M}\boldsymbol{K}\{s\}$.

(2) $\chi(g_{(1)},\cdots,g_{(m)}) \in (\mathfrak{J}' + \mathfrak{m}^\mu\mathfrak{J})\boldsymbol{K}\{s\}$ *if and only if* $(g_{(1)},\cdots,g_{(m)}) \in \mathscr{M}\boldsymbol{K}\{s\} + \mathfrak{m}^\mu\boldsymbol{K}\{s\}^m$, *where* $\mathfrak{m}$ *denotes the maximal ideal of* $\boldsymbol{C}\{s\}$.

Then we define a homomorphism $Q: \mathfrak{J}\boldsymbol{K}\{s\} \to \boldsymbol{K}\{s\}^m$ by $Q(f) := (g_{(1)},\cdots,g_{(m)})$ for $f = \Sigma_{\kappa=1}^m g_{(\kappa)}\omega_\kappa$.

Let $\mu_\kappa := \exp(\omega_\kappa)$ $(\kappa=1,\cdots,m)$. By Theorem 3.1 (4), We have

Lemma 3.3. *Suppose that* a, $\rho \in (\boldsymbol{R}_+^*)^q$ *satisfy* Theorem 3.1 (4), *then we have*

(1) $\|\chi(g_{(1)},\cdots,g_{(m)})\|_\rho \le 2m(\max_{1\le\kappa\le m}\rho^{\mu_\kappa})\|(g_{(1)},\cdots,g_{(m)})\|_\rho$,

(2) $\|Q(f)\|_\rho \le 2(\min_{1\le\kappa\le m}\rho^{\mu_\kappa})^{-1}\|f\|_\rho$,

(3) $\|\mathrm{red}_{\mathscr{M}} \circ Q(f)\|_\rho \le 4m(\min_{1\le\kappa\le m}\rho^{\mu_\kappa})^{-1}\rho^{-a}\|f\|_\rho$.

Lemma 3.4. *Suppose that* $|\mu_\kappa| = n_o$ $(\kappa=1,\cdots,m)$. *Let* $V := \{\rho = (\eta^{1+\delta_1},\cdots,\eta^{1+\delta_q}) \mid (\delta_1,\cdots,\delta_q) \in \Lambda,\ 0<\eta<1\}$ *be a set of poly-radious with* Λ *an open set of* $(\boldsymbol{R}_+^*)^q$. *For any* $a \in (\boldsymbol{R}_+^*)^q$ *with* $\max_{1\le\kappa\le m} <\mu_\kappa, a> \ne 0$, *there exists an* $\varepsilon := (\varepsilon_1,\cdots,\varepsilon_q) \in (\boldsymbol{R}_+^*)^q$ *such that*

$$\frac{\max_{1\le\kappa\le m}\rho^{\mu_\kappa}}{\min_{1\le\kappa\le m}\rho^{\mu_\kappa}}<\rho^{-a}$$

holds for $\rho\in V$ *defined by* $(\delta_1,\cdots,\delta_q)\in\Lambda$ *with* $0<\delta_j<\varepsilon_j$ $(j=1,\cdots,q)$.

Proof. Let $a:=(a_1,\cdots,a_q)$ and suppose $0<\delta_j<(\Sigma_{k=1}^q a_k)a_j/\max_{1\le\kappa\le m}<\mu_\kappa,a>)$ $(j=1,\cdots,q)$. Then $(\max_{1\le\kappa\le m}\rho^{\mu_\kappa})$ / $(\min_{1\le\kappa\le m}\rho^{\mu_\kappa})$ $=$ $(\max_{1\le\kappa\le m}\eta^{n_o+<\mu_\kappa,\delta>})$ / $(\min_{1\le\kappa\le m}\eta^{n_o+<\mu_\kappa,\delta>})$. Since $<\mu_\kappa,\delta><(\Sigma_{k=1}^q a_k)$ $<\mu_\kappa,a>$ / $(\max_{1\le\kappa\le m}$ $<\mu_\kappa,a>)$, we have $(\max_{1\le\kappa\le m}\rho^{\mu_\kappa})/(\min_{1\le\kappa\le m}\rho^{\mu_\kappa})<\eta^{-\Sigma_{k=1}^q a_k}<\eta^{-\Sigma_{k=1}^q a_k-<\delta,a>}=\rho^{-a}$.

Q.E.D.

4. Deformations along a s.p.c. compact level subset - an infinitesimal lifting

In this section, we will get a canonical way constructing a lifting along the infinitesimal extension $S\cup S'_{\mu-1}\to S\cup S'_\mu$, where S is a subspace of S' and S'_μ denotes μ-th infinitesimal neighbourhood of $o\in S'$.

Let $\phi(t):=\phi_\varepsilon(t)$ $(t\in T_\varepsilon)$ be the family of complex structures on $\bar\Omega_\varepsilon$ obtained in §2. Let $\varpi:\mathscr{Y}\to S'$ be a smooth morphism with $\bar\Omega_\varepsilon\subset\varpi^{-1}(0)$ given by a locally finite system of local charts $\{\mathscr{Y}_i,\ (w_i,s):=(w_i^1,\cdots,w_i^n,s^1,\cdots,s^q)\}_{i\in I}$ with transition functions $w_i^\alpha=\tilde h_{ij}^\alpha(w_j,s)$ $(\alpha=1,\cdots,n)$ on $\mathscr{Y}_i\cap\mathscr{Y}_j$. Let $U_i:=\mathscr{Y}_i\cap\varpi^{-1}(o)$, $z_i^\alpha=w_{i|U_i}^\alpha$ $(\alpha=1,\cdots,n)$. If $I'=\{i\,|\,U_i\cap\Omega_\varepsilon\neq\phi\}$ then $\sharp I'<+\infty$. In §§4 and 5, we choose $\{U_i,z_i:=(z_i^1,\cdots,z_i^n)\}_{i\in I'}$ as a system of local charts of a neighbourhood of $\bar\Omega_\varepsilon$.

Suppose that S and S' are analytic subspaces of $\boldsymbol{C}^q$ defined by ideals $\mathfrak{J}$ and $\mathfrak{J}'$ with $\mathfrak{J}'\subset\mathfrak{J}$ respectively. We may assume that $\tilde h_{ij}^\alpha(w_j,s)$ is represented by a holomorphic function $\tilde h_{ij}^\alpha(w_j,s)$ over $(W_i\cap W_j)\times D$ satisfying $h_{ij}^\alpha(h_{jk}(w_k,s),s)-h_{ik}^\alpha(w_k,s)\equiv 0 \bmod \mathfrak{J}'$.

We will consider a holomorphic map $\tau:S\cup S'_\mu\to T_\varepsilon$ and a map $g:\bar\Omega_\varepsilon\times(S\cup S'_\mu)\to\mathscr{Y}$ which is holomorphic with respect to the family of compelx structures $\phi(\tau(s))$ $(s\in S\cup S'_\mu)$. They are represented by $w_i^\alpha=g_i^\alpha(z_i,s)$ with $g_i^\alpha(s)\in\Gamma'_k(U_i\cap\bar\Omega_\varepsilon,1)\{s\}$ $(\alpha=1,\cdots,n)$ and $t^\sigma=\tau^\sigma(s)$ with $\tau^\sigma(s)\in\boldsymbol{C}\{s\}$ $(\sigma=1,\cdots,r)$ with satisfying

(4.1) $g_i^\alpha(z_i,0)=z_i^\alpha$ $(\alpha=1,\cdots,n)$ and $\tau^\sigma(0)=0$ $(\sigma=1,\cdots,r)$,

$(4.2)_\mu$ $g_i^\alpha(s)-h_{ij}^\alpha(g_j(s),s)\in(\mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J})\Gamma'_k(U_i\cap U_j\cap\bar\Omega_\varepsilon,1)\{s\}$ $(\alpha=1,\cdots,n)$,

$(4.3)_\mu$ $(\bar\partial-\phi(\tau(s)))g_i^\alpha(s)\in(\mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J})\Gamma_k(U_i\cap\bar\Omega_\varepsilon,(T''X)^*)\{s\}$ $(\alpha=1,\cdots,n)$,

$$\text{where } (\bar\partial-\phi(\tau(s)))g_i^\alpha(s):=\sum_{\beta=1}^n\left(\frac{\partial}{\partial\bar z_i^\beta}-\sum_{\lambda=1}^n\phi_\beta^\lambda(\tau(s))\frac{\partial}{\partial z_i^\lambda}\right)g_i^\alpha(s)d\bar z_i^\beta\ (\alpha=1,\cdots,n),$$

$(4.4)_\mu$ $h_\rho(\tau(s))\in(\mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J})$ $(\rho=1,\cdots,l)$, where $h_1(t),\cdots,h_l(t)$ are generators of $\mathfrak{J}_{T_\varepsilon}$,

where we denote by $\Gamma_k(U\cap\bar\Omega_\varepsilon,V)$ (resp. $\Gamma'_k(U\cap\bar\Omega_\varepsilon,V)$) the completion of $\Gamma(U\cap\bar\Omega_\varepsilon,V)$ with respect to $\|\ \|_{(0,k)}$ (resp. $\|\ \|'_{(0,k)}$).

Proposition 4.1. *Let $\bar{g}_i^\alpha(s)\in\Gamma'_k(U_i\cap\bar{\Omega}_\varepsilon,1)\{s\}$ $(\alpha=1,\cdots,n)$ and $\bar{\tau}^\sigma(s)\in\boldsymbol{C}\{s\}$ $(\sigma=1,\cdots,r)$ satisfying (4.1), $(4.2)_{-1}\sim(4.4)_{-1}$ are given. Then, for $\mu\geq-1$, there is a canonical way lifting $g_i^{(\mu-1)\alpha}(s)\in\Gamma'_k(U_i\cap\bar{\Omega}_\varepsilon,1)\{s\}$ $(\alpha=1,\cdots,n)$ and $\tau^{(\mu-1)\sigma}(s)\in\boldsymbol{C}\{s\}$ $(\sigma=1,\cdots,r)$ satisfying $(4.2)_{\mu-1}\sim(4.4)_{\mu-1}$ and*

$(4.1)'_{\mu-1}$ $\quad g_i^{(\mu-1)\alpha}(s)-\bar{g}_i^\alpha(s)\in\mathfrak{J}\Gamma'_k(U_i\cap\bar{\Omega}_\varepsilon,1)\{s\}$ $(\alpha=1,\cdots,n)$ *and*

$$\tau^{(\mu-1)\alpha}(s)-\bar{\tau}^\sigma(s)\in\mathfrak{J}\ (\sigma=1,\cdots,r)$$

to $g_i^{(\mu)\alpha}(s)\in\Gamma'_k(U_i\cap\bar{\Omega}_\varepsilon,1)\{s\}$ $(\alpha=1,\cdots,n)$ and $\tau^{(\mu)\sigma}(s)\in\boldsymbol{C}\{s\}$ $(\sigma=1,\cdots,r)$ satisfying $(4.2)_\mu\sim(4.4)_\mu$ and

$(4.1)''_\mu$ $\quad g_i^{(\mu)\alpha}(s)-g_i^{(\mu-1)\alpha}(s)\in\mathfrak{m}^\mu\mathfrak{J}\Gamma'_k(U_i\cap\bar{\Omega}_\varepsilon,1)\{s\}$ $(\alpha=1,\cdots,n)$ *and*

$$\tau^{(\mu)\sigma}(s)-\tau^{(\mu-1)\sigma}(s)\in\mathfrak{m}^\mu\mathfrak{J}\ (\sigma=1,\cdots,r).$$

Proof. Let $\omega_1(s),\cdots,\omega_m(s)$ be the Weierstraß family of $\mathfrak{J}$ and $\chi:\boldsymbol{K}\{s\}^m\to\mathfrak{J}\boldsymbol{K}\{s\}$ and $Q:\mathfrak{J}\boldsymbol{K}\{s\}\to\boldsymbol{K}\{s\}^m$ be the homomorphisms defined in §3 using this Weietstraß family. We fix the following decompositions by means of $(4.2)_{-1}$ and $(4.3)_{-1}$ respectively

$$\bar{g}_i^\alpha(s)-h_{ij}^\alpha(\bar{g}_j(s),s)=\bar{\sigma}_{ij}^\alpha(s)+\bar{\sigma}'^\alpha_{ij}(s)\ (\alpha=1,\cdots,n),$$

$$(\bar{\partial}-\phi(\bar{\tau}(s)))\bar{g}_i^\alpha(s)=\bar{\xi}_i^\alpha(s)+\bar{\xi}'^\alpha_i(s)\ (\alpha=1,\cdots,n)$$

where $\bar{\sigma}_{ij}^\alpha(s)\in\mathfrak{J}\Gamma'_k(U_i\cap U_j\cap\bar{\Omega}_\varepsilon,1)\{s\}$, $\bar{\sigma}'^\alpha_{ij}(s)\in\mathfrak{J}'\Gamma'_k(U_i\cap U_j\cap\bar{\Omega}_\varepsilon,1)\{s\}$, $\bar{\xi}_i^\alpha(s)\in\mathfrak{J}\Gamma'_k(U_i\cap\bar{\Omega}_\varepsilon,(T''X)^*)\{s\}$, $\bar{\xi}'^\alpha_i(s)\in\mathfrak{J}'\Gamma'_k(U_i\cap\bar{\Omega}_\varepsilon,(T''X)^*)\{s\}$.

In the followings, we fix a partition of unity $\{\rho_j\}$ subordinate to $\{U_j\}$.

The first task is to construct $g'^\alpha_{i|\mu}(s)$ $(\alpha=1,\cdots,n)$ such that

(4.5) $\quad g_i^{(\mu-1)\alpha}(s)+g'^\alpha_{i|\mu}(s)-h_{ij}^\alpha(g_j^{(\mu-1)}(s)+g'_{j|\mu}(s),s)\ \in\ (\mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J})\Gamma'_k(U_i\cap U_j\cap\bar{\Omega}_\varepsilon,1)\{s\}$ $(\alpha=1,\cdots,n)$ holds.

In the following argument, we identify $\{g'^\alpha_{i|\mu}(s)\}_{\alpha=1,\cdots,n}$ with $g'_{i|\mu}(s)=\Sigma_{\alpha=1}^n g'^\alpha_{i|\mu}(s)\frac{\partial}{\partial z_i^\alpha}$. Let $(\sigma^\alpha_{(1)ij|\mu}(s),\cdots,\sigma^\alpha_{(m)ij|\mu}(s))$ $(\alpha=1,\cdots,n)$ be homogeneous polynomials of s of order μ such that

$$(\sigma^\alpha_{(1)ij|\mu}(s),\cdots,\sigma^\alpha_{(m)ij|\mu}(s))\equiv\mathrm{red}_{\mathscr{M}}\circ Q(g_i^{(\mu-1)\alpha}(s)-h_{ij}^\alpha(g_j^{(\mu-1)}(s),s)-\bar{\sigma}'^\alpha_{ij}(s))$$
$$\mathrm{mod}\,\mathfrak{m}^{\mu+1}$$

holds, and denote

$$(\sigma^\alpha_{(1)ij|\mu}(s),\cdots,\sigma_{(m)ij|\mu}(s))\equiv\left(\sum_{\alpha=1}^n\sigma^\alpha_{(1)ij|\mu}(s)\frac{\partial}{\partial z_i^\alpha},\cdots,\sum_{\alpha=1}^n\sigma^\alpha_{(m)ij|\mu}(s)\frac{\partial}{\partial z_i^\alpha}\right).$$

If we solve

(4.6) $\quad\delta\{(g'_{(1)i|\mu}(s),\cdots,g'_{(m)i|\mu}(s))\}=\{(\sigma_{(1)ki|\mu}(s),\cdots,\sigma_{(m)ki|\mu}(s))\},$

then $g'_{i|\mu}$ is given by

$$g'_{i|\mu}(s):=\chi(g'_{(1)i|\mu}(s),\cdots,g'_{(m)i|\mu}(s)).$$

From the following sublemma, we infer that

$$(g'_{(1)i|\mu}(s),\cdots,g'_{(m)i|\mu}(s)):=\sum_k \rho_k(\sigma_{(1)ki|\mu}(s),\cdots,\sigma_{(m)ki|\mu}(s))$$

gives a solution of (4.6)

Sublemma 4.2. $\delta\{(\sigma_{(1)ij|\mu}(s),\cdots,\sigma_{(m)ij|\mu}(s))\}=(0,\cdots,0)$.

Proof. Since $(4.2)_{\mu-1}$ implies $\delta\{\chi(\sigma_{(1)ij|\mu}(s),\cdots,\sigma_{(m)ij|\mu}(s))\}\equiv 0 \bmod \mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J}$, it follows Lemma 3.2. Q.E.D.

Next task is to construct homogeneous polynomials of s of degree μ, $g''_{\mu}(s)\in\Gamma'_k(\bar{\Omega}_\varepsilon,T'X)[s]$ and $\tau_u(s)\in \boldsymbol{C}^r[s]$ satisfying

$$(4.7)\quad \bar{\partial}g''_{\mu}(s)-\phi_1\tau_{\mu}(s)\equiv-(\bar{\partial}-\phi(\tau^{(\mu-1)}(s)))(g_i^{(\mu-1)}(s)+g'_{i|\mu}(s)) \bmod \mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J},$$

where $\phi_1\tau_{\mu}(s)$ denotes $\Sigma_{\sigma=1}^r\frac{\partial\phi_1}{\partial t^\sigma}(0)\tau_{\mu}^{\sigma}(s)$.

Before proceeding the second task, we remark the following formula: Let $\phi\in A^{0,1}(U,T'X)$, $g\in C^\infty(U)$ and $\theta:=(\bar{\partial}-\phi)g$, then

(4.8) $\bar{\partial}\theta=-(\bar{\partial}\phi-\frac{1}{2}[\phi,\phi])g+\phi\wedge\theta$ holds, where U denotes an open set of $\bar{\Omega}_\varepsilon$.
This formula follows by a standard calculation.

Sublemma 4.3. $h_\rho(\tau^{(\mu-1)}(s))\equiv 0 \bmod \mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J}$ $(\rho=1,\cdots,l)$.

Proof. Since (4.5) implies that $\theta(s):=(\bar{\partial}-\phi(\tau^{(\mu-1)}(s)))(g_i^{(\mu-1)}(s)+g'_{i|\mu}(s))$ is a $T'X$-valued global (0,1)-from modulo $\mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J}$, this follws from the formula (4.8) and the definition of $h(t)$. Q.E.D.

Let $(\xi^\alpha_{(1)i|\mu}(s),\cdots,\xi^\alpha_{(m)i|\mu}(s))$ $(\alpha=1,\cdots,n)$ be homogeneous polynomials of s of order μ such that

$$(\xi^\alpha_{(1)i|\mu}(s),\cdots,\xi^\alpha_{(m)i|\mu}(s))\equiv \mathrm{red}_{\mathcal{M}}\circ Q((\bar{\partial}-\phi(\tau^{(\mu-1)}(s)))(g_i^{(\mu-1)\alpha}(s)+g'^{\alpha}_{i|\mu}(s))-\bar{\xi}'^{\alpha}_i(s))$$
$$\bmod \mathfrak{m}^{\mu+1}$$

holds, and denote

$$(\xi_{(1)i|\mu}(s),\cdots,\xi_{(m)i|\mu}(s))=\left(\sum_{\alpha=1}^n\xi^\alpha_{(1)i|\mu}(s)\frac{\partial}{\partial z_i^\alpha},\cdots,\sum_{\alpha=1}^n\xi^\alpha_{(m)i|\mu}(s)\frac{\partial}{\partial z_i^\alpha}\right)$$

as in the first task. If we solve

(4.9) $\bar{\partial}(g''_{(1)\mu}(s),\cdots,g''_{(m)\mu}(s))-(\phi_1\tau_{(1)\mu}(s),\cdots,\phi_1\tau_{(m)\mu}(s))=-(\xi_{(1)\mu}(s),\cdots,\xi_{(m)\mu}(s))$,

then $g''_\mu(s)$ and $\tau_\mu(s)$ are given by

$$g''_\mu(s):=\chi(g''_{(1)\mu}(s),\cdots,g''_{(m)\mu}(s)),$$
$$\tau_\mu(s):=\chi(\tau_{(1)\mu}(s),\cdots,\tau_{(m)\mu}(s)).$$

Sublemma 4.4.

(1) $\delta\{(\xi_{(1)i|\mu}(s),\cdots,\xi_{(m)i|\mu}(s))\}=(0,\cdots,0)$,

(2) $\bar{\partial}(\xi_{(1)\mu}(s),\cdots,\xi_{(m)\mu}(s))=(0,\cdots,0)$.

Proof. Applying $\bar{\partial}-\phi(\tau^{(\mu-1)}(s))$ to (4.5), we have $\delta\{\chi(\xi_{(1)i|\mu}(s),\cdots,\xi_{(m)i|\mu}(s))\}\equiv 0 \bmod \mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J}$.

(2) Applying the formula (4.8) to $\theta(s):=\chi(\xi_{(1)i|\mu}(s),\cdots,\xi_{(m)i|\mu}(s))$, we have $\bar{\partial}\chi(\xi_{(1)i|\mu}(s),\cdots,\xi_{(m)i|\mu}(s))\equiv 0 \bmod \mathfrak{J}'+\mathfrak{m}^{\mu+1}\mathfrak{J}$. Q.E.D.

Hence, there are uniquely determined homogeneous polynomials $(\tau_{(1)\mu}(s),\cdots,\tau_{(m)\mu}(s))$ of s of order μ such that $(\xi_{(1)\mu}(s),\cdots,\xi_{(m)\mu}(s))-(\phi_1\tau_{(1)\mu}(s),\cdots,\phi_1\tau_{(m)\mu}(s))$ is $\bar{\partial}$-exact where we denote by $\xi_{(\kappa)\mu}(s)$ $(\kappa=1,\cdots,m)$ the $T'X$-valued global (0,1)-form defined by $\{\xi_{(\kappa)i|\mu}(s)\}$. Therefore if we set

$$(g''_{(1)\mu}(s),\cdots,g''_{(m)\mu}(s)):=-\vartheta N((\xi_{(1)\mu}(s),\cdots,\xi_{(m)\mu}(s))-(\phi_1\tau_{(1)\mu}(s),\cdots,\phi_1\tau_{(m)\mu}(s))),$$

where N denotes the standard Neumann operator (cf. [9]), then $(g''_{(1)\mu}(s),\cdots,g''_{(m)\mu}(s))$ and $(\tau_{(1)\mu}(s),\cdots,\tau_{(m)\mu}(s))$ are solutions of (4.9). Thus we get $g_i^{(\mu)\alpha}(s)$ $(\alpha=1,\cdots,n)$ and $\tau^{(\mu)\sigma}(s)$ $(\sigma=1,\cdots,r)$ by

$$g_i^{(\mu)\alpha}(s):=g_i^{(\mu-1)\alpha}(s)+g'^{\alpha}_{i|\mu}(s)+g''^{\alpha}_\mu(s)\ (\alpha=1,\cdots,n),$$
$$\tau^{(\mu)\sigma}(s):=\tau^{(\mu-1)\sigma}(s)+\tau^\sigma_\mu(s)\ (\sigma=1,\cdots,r).$$

This completes the proof of Proposition 4.1.

5. Deformations along a s.p.c. compact level subset - proof of versality

Let $\pi_\varepsilon:\mathscr{X}_\varepsilon\to T_\varepsilon$ be the family in §2. We denote by w_ε the object in F_K corresponding to it. Then the main result of this section is the following theorem implying Theorem 1.

Theorem 5.1. *w_ε has the lifting property for any extension $S\to S'=S'_{\mathrm{red}}$.*

At first, we prove the following.

Proposition 5.2. *Let $\varpi:\mathscr{Y}\to S'$ be a smooth map with $\bar{\Omega}_\varepsilon\subset\varpi^{-1}(0)$ and k a* *positive integer. Suppose that there exist a holomorphic map* $\tau^{(0)}: S' \supset S \to T$ *and a map* $g^{(0)}: \bar{\Omega}_\varepsilon \times S \to \mathscr{Y}$ *represented by* $\tau^{(0)\sigma}(s) \in \boldsymbol{C}\{s\}$ $(\sigma = 1, \cdots, r)$ *and* $g_i^{(0)\alpha}(s) \in \Gamma'_k(U_i \cap \bar{\Omega}_\varepsilon, 1)\{s\}$ $(\alpha = 1, \cdots, n)$ *respectively, with respect to a system of local charts* $\{\mathscr{Y}_i, (w_i, s)\}$ *of* $\mathscr{Y}$ *as in* §4, *satisfying the following*

$(5.1)_0$ $\quad g_i^{(0)\alpha}(z_i, 0) = z_i^\alpha$ $(\alpha = 1, \cdots, n)$.

$(5.2)_0$ $\quad g_i^{(0)\alpha}(s) - h_{ij}^\alpha(g_j^{(0)}(s), s) \in \mathfrak{J}_S \Gamma'_k(U_i \cap U_j \cap \bar{\Omega}_\varepsilon, 1)\{s\}$ $(\alpha = 1, \cdots, n)$.

$(5.3)_0$ $\quad (\bar{\partial} - \phi_\varepsilon(\tau^{(0)}(s))) g_i^{(0)\alpha}(s) \in \mathfrak{J}_S \Gamma_k(U_i \cap \bar{\Omega}_\varepsilon, (T''X)^*)\{s\}$ $(\alpha = 1, \cdots, n)$.

$(5.4)_0$ $\quad h_\rho(\tau^{(0)}(s)) \in \mathfrak{J}_S$ $(\rho = 1, \cdots, l)$, *where* $h_1(t), \cdots, h_l(t)$ *are generators of* $\mathfrak{J}_{T_\varepsilon}$.

Then there exist liftings $\tau: S' \to T$ *and* $g: \bar{\Omega}_\varepsilon \times S' \to \mathscr{Y}$ *represented by* $\tau^\sigma(s) \in \boldsymbol{C}\{s\}$ $(\sigma = 1, \cdots, r)$ *and* $g_i^\alpha(s) \in \Gamma'_k(U_i \cap \bar{\Omega}_\varepsilon, 1)\{s\}$ $(\alpha = 1, \cdots, n)$ *respectively, satisfying*

$(5.1)'_\infty$ $\quad g_i^\alpha(s) - g_i^{(0)\alpha}(s) \in \mathfrak{J}_S \Gamma'_k(U_i \cap U_j \cap \bar{\Omega}_\varepsilon, 1)\{s\}$ $(\alpha = 1, \cdots, n)$,

$\qquad \tau^\sigma(s) - \tau^{(0)\sigma}(s) \in \mathfrak{J}_S$ $(\sigma = 1, \cdots, r)$.

$(5.2)_\infty$ $\quad g_i^\alpha(s) - h_{ij}^\alpha(g_j(s), s) \in \mathfrak{J}_{S'} \Gamma'_k(U_i \cap U_j \cap \bar{\Omega}_\varepsilon, 1)\{s\}$ $(\alpha = 1, \cdots, n)$,

$(5.3)_\infty$ $\quad (\bar{\partial} - \phi_\varepsilon(\tau(s))) g_i^\alpha(s) \in \mathfrak{J}_{S'} \Gamma_k(U_i \cap \bar{\Omega}_\varepsilon, (T''X)^*)\{s\}$ $(\alpha = 1, \cdots, n)$,

$(5.4)_\infty$ $\quad h_\rho(\tau(s)) \in \mathfrak{J}_{S'}$ $(\rho = 1, \cdots, l)$.

Proof. Throughout the proof, we will denote $\phi := \phi_\varepsilon$. First set $\mathfrak{J} = \mathfrak{J}_S$, $\mathfrak{J}' = \mathfrak{J}_{S'}$, $\bar{g}_i^\alpha(s) := g_i^{(0)\alpha}(s)$ and $\bar{\tau}^\sigma(s) := \tau^{(0)\sigma}(s)$.

Let n_o be a positive integer such that all extremal points of $E(\mathfrak{m}^{n_o} \cap \mathfrak{J}_S)$ are of total degree n_o. Then by repeating the infinitesimal lifting in Proposition 4.1 we have $g_i^{(n_o+1)\alpha}(s) \in \Gamma'_k(U_i \cap \bar{\Omega}_\varepsilon, 1)\{s\}$ $(\alpha = 1, \cdots, n)$ and $\tau^{(n_o+1)\sigma}(s) \in \boldsymbol{C}\{s\}$ $(\sigma = 1, \cdots, r)$ satisfying $(4.1)''_{n_o+1}$ and $(4.2)_{n_o+1} \sim (4.4)_{n_o+1}$.

Next, set $\mathfrak{J} = \mathfrak{m}^{n_o} \cap \mathfrak{J}_S$, $\mathfrak{J}' = \mathfrak{J}_{S'}$. Remark that if $(\omega_1(s), \cdots, \omega_m(s))$ be the Weierstraß family of $\mathfrak{J}$ then $|\exp(\omega_k(s))| = n_o$ $(\kappa = 1, \cdots, m)$.

Set $\bar{g}_i^\alpha(s) := g_i^{(n_o+1)\alpha}(s)$ $(\alpha = 1, \cdots, n)$ and $\bar{\tau}^\sigma(s) := \tau^{(n_o+1)\sigma}(s)$ $(\sigma = 1, \cdots, r)$, then they satisfy (4.1) and $(4.2)_1 \sim (4.4)_1$.

Again, starting with $g_i^{(1)\alpha}(s) := \bar{g}_i^\alpha(s)$ and $\tau^{(1)\sigma}(s) := \bar{\tau}^\sigma(s)$ and by repeating the infinitesimal lifting of Proposition 4.1, we have a sequences $\{g_i^{(\mu)\alpha}(s)\}_{1 \le \alpha \le n}$ and $\{\tau^{(\mu)\sigma}(s)\}_{1 \le \sigma \le r}$ satisfying $(4.1)''_\mu \sim (4.4)_\mu$ for $\mu \ge 1$. Remark that $g_i^{(\mu)\alpha}(s) \equiv \bar{g}_i^\alpha(s) \bmod \mathfrak{J}$, $\tau^{(1)\sigma}(s) \equiv \bar{\tau}^\sigma(s) \bmod \mathfrak{J}$ and $Q(g_i^{(\mu)\alpha}(s) - \bar{g}_i^\alpha(s))$ and $Q(\tau^{(\mu)\sigma}(s) - \bar{\tau}^\sigma(s))$ are reduced with respect to $\mathscr{M}$.

We will prove the convergence of $\lim_{\mu \to +\infty} g_i^{(\mu)\alpha}(s)$ and $\lim_{\mu \to +\infty} \tau^{(\mu)\sigma}(s)$.

Lemma 5.3. *Let* $U \subset\subset W \subset\subset \boldsymbol{C}^n$ *be open domains,* $h \in \Gamma_b(W, \mathcal{O})\{s\}$ *and* k *a positive integer. Then, for* $g(s) \in \Gamma'_k(U, 1)^n\{s\}$ *with* $g^\alpha(z, 0) = z^\alpha$ $(1 \le \alpha \le n)$, *there exist positive constants* ε *and* $C_1 \sim C_3$ *such that*

$$\|h(g(s)+\psi(s),s)-h(g(s),s)-\frac{\partial h}{\partial z}(z,0)\psi(s)\|_\rho\leq$$

$$C_1\|\psi(s)\|_\rho^2+C_2\max_{\alpha,\beta}\|\frac{\partial h^\alpha}{\partial z^\beta}(\bar{g}(s),s)-\frac{\partial h^\alpha}{\partial z^\beta}(z,0)\|_\rho\|\psi(s)\|_\rho$$

holds for $\psi(s)\in\Gamma'_k(U,1)^n\{s\}$ *with* $\psi(z,0)=0$ *and* $\|\psi\|_\rho<\varepsilon$, *where* $\Gamma_b(W,\mathcal{O})$ *denotes the Banach space of all bounded holomorphic functions on* W *and we consider* $h(g(s)+\psi(s),s)-h(g(s),s)-\frac{\partial h}{\partial z}(z,0)\psi(s)$ *as an element of* $\Gamma_k(U,1)^n\{s\}$.

We can prove Lemma 5.3 along the way of [11, Lemma 2.22].

Lemma 5.4. *Let* $\boldsymbol{K}$ *be a Banach space with its norm* $|\ |_{\boldsymbol{K}}$ *and* $\phi(t)\in\boldsymbol{K}\{t_1,\cdots,t_r\}$. *Then, for* $\tau(s)\in\boldsymbol{C}^r\{s_1,\cdots,s_q\}$ *with* $\tau(0)=0$, *there exist* $\varepsilon>0$, $C_4>0$ *and* $C_5>0$ *such that*

$$\|\phi(\tau(s)+\eta(s))-\phi(\tau(s))-\frac{\partial\phi}{\partial t}(0)\eta(s)\|_\rho\leq C_4\|\eta(s)\|_\rho^2+C_5\|\tau(s)\|_\rho\|\eta(s)\|_\rho$$

holds for $\eta(s)\in\boldsymbol{C}^r\{s_1,\cdots,s_q\}$ *with* $\eta(0)=0$ *and* $\|\eta(s)\|_\rho<\varepsilon$.

Proof is a direct calculation.

The following is well known, where constants c and d are independent of g or ξ:

(5.5) $\|\bar{\partial}g\|_{(0,k)}\leq c\|g\|'_{(0,k)}$ for all $g\in C^\infty(U_i\cap\bar{\Omega}_\varepsilon,1)$,

(5.6) $\|\vartheta N\xi\|'_{(0,k)}\leq d\|\xi\|_{(0,k)}$ for all $\xi\in A^{0,1}(\bar{\Omega}_\varepsilon,\mathrm{T}'X)$.

In the followings, we fix an $a=(a_1,\cdots,a_q)\in(\boldsymbol{R}_+^*)^q$ with $a_1+\cdots+a_q<1/8$ and satisfying $<\exp(\omega_\kappa),a>\neq 0$ for some $1\leq\kappa\leq m$. Let $\rho_o\in(\boldsymbol{R}_+^*)^q$ be such that $\max_i(\rho_o)_i<1$ and $\|\bar{g}_i^\alpha(s)\|_{\rho_o}$ $(\alpha=1,\cdots,n)$ and $\|\bar{\tau}^\sigma(s)\|_{\rho_o}$ $(\sigma=1,\cdots,r)$ are all finite. Suppose that

$\|\bar{g}_i^\alpha(s)-z_i^\alpha\|_{\rho_o}\leq\sigma_o$ $(\alpha=1,\cdots,n)$,

$\|\bar{\tau}^\sigma(s)\|_{\rho_o}\leq\sigma_o$ $(\sigma=1,\cdots,r)$,

$\|\phi(\bar{\tau}(s))\|_{\rho_o}\leq\sigma_o$,

$\max_{\alpha,\beta}\|\frac{\partial h^\alpha}{\partial z^\beta}(\bar{g}(s),s)-\frac{\partial h^\alpha}{\partial z^\beta}(z,0)\|_{\rho_o}\leq\sigma_o$,

$\|\mathrm{red}_{\mathcal{M}}\circ Q(\bar{\sigma}'^{\alpha}_{ij}(s))\|_{\rho_o}\leq\sigma_o$ $(\alpha=1,\cdots,n)$,

$\|\mathrm{red}_{\mathcal{M}}\circ Q(\bar{\xi}'^{\alpha}_i(s))\|_{\rho_o}\leq\sigma_o$ $(\alpha=1,\cdots,n)$.

Since $\bar{g}_i^\alpha(s)\equiv z_i^d \bmod \mathfrak{m}$ and $\bar{\tau}^\sigma(s)\equiv 0 \bmod \mathfrak{m}$, we infer from (3.3) that, for $\rho\in(\boldsymbol{R}_+^*)^q$ with $\rho\leq\rho_o$, (5.7) $\|\bar{g}_i^\alpha(s)-z_i^\alpha\|_\rho \leq (\frac{\rho}{\rho_o})\sigma_o(\alpha=1,\cdots,n)$,

$\|\bar{\tau}^\sigma(s)\|_\rho \leq (\frac{\rho}{\rho_o})\sigma_o\ (\sigma=1,\cdots,r)$,

$\|\phi(\bar{\tau})(s)\|_\rho \leq (\frac{\rho}{\rho_o})\sigma_o$,

$\max_{\alpha,\beta}\|\frac{\partial h^\alpha}{\partial z^\beta}(\bar{g}(s),s)-\frac{\partial h^\alpha}{\partial z^\beta}(z,0)\|_\rho \leq (\frac{\rho}{\rho_o})\sigma_o$.

Since $\mathrm{red}_{\mathscr{M}}\circ Q(\bar{\sigma}_{ij}'^\alpha(s))\equiv 0 \bmod \mathfrak{m}^2$ and $\mathrm{red}_{\mathscr{M}}\circ Q(\bar{\xi}_i'^\alpha(s))\equiv 0 \bmod \mathfrak{m}^2$, we have

(5.8) $\|\mathrm{red}_{\mathscr{M}}\circ Q(\bar{\sigma}_{ij}'^\alpha(s))\|_\rho \leq (\frac{\rho}{\rho_o})^2\sigma_o\ (\alpha=1,\cdots,n)$,

(5.9) $\|\mathrm{red}_{\mathscr{M}}\circ Q(\bar{\xi}_i'^\alpha(s))\|_\rho \leq (\frac{\rho}{\rho_o})^2\sigma_o\ (\alpha=1,\cdots,n)$.

It is enough to show that the following estimates hold for some $\rho\in(\boldsymbol{R}_+^*)^q$, $\sigma>0$ and for all $\mu\geq 1$:

$(5.10)_\mu$ $\|(g_{(1)i|\mu}'^\alpha(s)+g_{(1)\mu}''^\alpha(s),\cdots,g_{(m)i|\mu}'^\alpha(s)+g_{(m)\mu}''^\alpha(s))\|_\rho \leq \sigma\}\ (\alpha=1,\cdots,n)$,

$(5.11)_\mu$ $\|(\tau_{(1)\mu}^\sigma(s),\cdots,\tau_{(m)\mu}^\sigma(s))\|_\rho \leq \sigma\ (\sigma=1,\cdots,r)$.

Since $g_{(\kappa)i|1}^\alpha(s)=0\ (\alpha=1,\cdots,n,\ \kappa=1,\cdots,m)$ and $\tau_{(\kappa)1}^\sigma(s)=0\ (\sigma=1,\cdots,r,\ \kappa=1,\cdots,m)$, $(5.10)_1$ and $(5.11)_1$ hold for any $\rho\in(\boldsymbol{R}_+^*)^q$ and $\sigma>0$.

Proposition 5.5. *Suppose that all inequalities in* Lemmas 3.3 *and* 3.4 *hold for* $\rho\in(\boldsymbol{R}_+^*)^q$ *with* $\rho\leq\rho_o$. *If* $(5.10)_\nu$ *and* $(5.11)_\nu$ *with* $0<\sigma<min\{1,\varepsilon\}$ *hold for* $\nu=1,2,\cdots,\mu-1$ $(\mu\geq 2)$, *then the followings holds*

(1) $\|(g_{(\kappa)i|\mu}'^\alpha(s)+g_{(\kappa)\mu}''^\alpha(s)\|_\rho \leq C_6(\frac{\rho}{\rho_o})^2\sigma_o+C_7\rho^{-2a}\sigma^2+C_8\rho^{-2a}(\frac{\rho}{\rho_o})\sigma\ (\alpha=1,\cdots,n,\ \kappa=1,\cdots,m)$,

(2) $\|\tau_{(\kappa)\mu}^\sigma(s)\|_\rho \leq C_6(\frac{\rho}{\rho_o})^2\sigma_o+C_7\rho^{-4a}\sigma^2+C_8\rho^{-2a}(\frac{\rho}{\rho_o})\sigma\ (\sigma=1,\cdots,r),\ \kappa=1,\cdots,m)$,

where $C_6\sim C_8$ *are constants independent of* ρ, μ *and* σ *and* ε *is the constant in* Lemmas 5.3 *and* 5.4.

The following lemma implies Proposition 5.5.

Lemma 5.6. *Under the assumption of* Proposition 5.5,

(1) $\|(\sigma_{ij(1)|\mu}(s),\cdots,\sigma_{ij(m)|\mu}(s))\|_\rho \leq C_9(\frac{\rho}{\rho_o})^2\sigma_o+C_{10}\rho^{-2a}\sigma^2+C_{11}\rho^{-2a}(\frac{\rho}{\rho_o})\sigma$,

(2) $\|(g_{(1)i|\mu}'(s),\cdots,g_{(m)i|\mu}'(s))\|_\rho \leq C_{12}(\frac{\rho}{\rho_o})^2\sigma_o+C_{13}\rho^{-2a}\sigma^2+C_{14}\rho^{-2a}(\frac{\rho}{\rho_o})\sigma$,

(3) $\|(\xi_{(1)\mu}(s),\cdots,\xi_{(m)\mu}(s))\|_\rho \leq C_{15}(\frac{\rho}{\rho_o})^2\sigma_o+C_{16}\rho^{-2a}\sigma^2+C_{17}\rho^{-2a}(\frac{\rho}{\rho_o})\sigma$,

(4) $\|(\tau_{(1)\mu}(s),\cdots,\tau_{(m)\mu}(s))\|_\rho \leq C_{18}(\frac{\rho}{\rho_o})^2\sigma_o+C_{19}\rho^{-2a}\sigma^2+C_{20}\rho^{-2a}(\frac{\rho}{\rho_o})\sigma$,

(5) $\|(g_{(1)\mu}''(s),\cdots,g_{(m)\mu}''(s))\|_\rho \leq C_{21}(\frac{\rho}{\rho_o})^2\sigma_o+C_{22}\rho^{-2a}\sigma^2+C_{23}\rho^{-2a}(\frac{\rho}{\rho_o})\sigma$,

where $C_9 \sim C_{23}$ are constants independent of ρ, μ and σ.

Proof. (1) Set $\Phi_{ij}^{\alpha}(\psi,s):=h_{ij}^{\alpha}(\bar{g}_j(s)+\psi,s)-h_{ij}^{\alpha}(\bar{g}_j(s),s)-\Sigma_{\beta=1}^{n}\frac{\partial h_{ij}^{\alpha}}{\partial z_j^{\beta}}(z_j,0)\psi^{\beta}$. If we denote $\psi_j^{\beta}(s)=g_j^{(\mu-1)\beta}(s)-\bar{g}_j^{\beta}(s)$, then we have

$$g_i^{(\mu-1)\alpha}(s)-h_{ij}^{\alpha}(g_j^{(\mu-1)}(s),s)=$$

$$(\bar{g}_i^{\alpha}(s)-h_{ij}^{\alpha}(\bar{g}_j^{\alpha}(s),s))+\psi_i^{\alpha}(s)-\sum_{\beta=1}^{n}\frac{\partial h_{ij}^{\alpha}}{\partial z_j^{\beta}}(z_j,0)\psi_j^{\beta}(s)+\Phi_{ij}^{\alpha}(\psi_j(s),s)$$

and $Q(\psi_j^{\beta}(s))$ are polynomials of order μ reduced with respect to $\mathscr{M}$. Hence $(\sigma_{ij(1)|\mu}(s),\cdots,\sigma_{ij(m)|\mu}(s))$ is the homogeneous term of degree μ of $\mathrm{red}_{\mathscr{M}}\circ Q(\bar{\sigma}_{ij}^{\alpha}(s)+\Phi_{ij}^{\alpha}(\psi_j(s),s))$. Therefore (1) follows from (5.8) and $(5.10)_v$ $(1\le v\le\mu-1)$ using Lemmas 5.3 and 3.3.

(2) follows from (1).

(3) Set $\Psi(\eta,s):=\phi(\bar{\tau}(s)+\eta)-\phi(\bar{\tau}(s))-\frac{\partial\phi}{\partial t}(0)\eta$. If we denote $\psi_i(s)=g_i^{(\mu-1)}(s)-\bar{g}_i(s)$ and $\delta(s)=\tau^{(\mu-1)}(s)-\bar{\tau}(s)$, then

$$(\bar{\partial}-\phi(\tau^{(\mu-1)}))(g_i^{(\mu-1)}(s)+g'_{i|\mu}(s))=(\bar{\partial}-\phi(\bar{\tau}))\bar{g}_i(s)+\bar{\partial}(\psi_i(s)+g'_{i|\mu}(s))-\phi_1\delta(s)z_i$$
$$-\phi(\bar{\tau}(s))(\psi_i(s)+g'_{i|\mu}(s))-\phi_1\delta(s)(g_i^{(\mu-1)}(s)+g'_{i|\mu}(s)-z_i)+\Psi(\delta(s),s)(g_i^{(\mu-1)}(s)+g'_{i|\mu}(s))$$

and $Q(\psi_i(s))$ and $Q(\delta(s))$ are polynomials of degree μ reduced with respect to $\mathscr{M}$. Hence $(\xi_{(1)\mu}(s),\cdots,\xi_{(m)\mu}(s))$ is the homogeneous term of degree μ of $\mathrm{red}_{\mathscr{M}}\circ Q(\bar{\xi}_i(s)+\bar{\partial}g'_{i|\mu}(s)-\phi(\bar{\tau}(s))\psi_i(s)-\phi_1\delta(s)(g_i^{(\mu-1)}(s)-z_i)-\Psi(\delta(s),s)g_i^{(\mu-1)}(s))$. Therefore (3) follows from (5.9) and the inductive assumption $(5.10)_v$ and $(5.11)_v$ $(v\le\mu-1)$, using (5.5) and Lemmas 5.4 and 3.3.

(4) and (5) follow from (3) taking account of (5.6). Q.E.D.

Let $V_{\beta,C}$ be the open set of poly-radious in Theorem 3.1 (4) with respect to a and $V':=\{\rho=(\eta^{1+\delta_1},\cdots,\eta^{1+\delta_q})\in V_{\beta,C}\,|\,0<\delta_j<\varepsilon_j\ \ (j=1,\cdots,q)\}$, where $(\varepsilon_1,\cdots,\varepsilon_q)$ has the property in Lemma 3.4 and satisfies $\max_j\varepsilon_j<1$. Choose a $\rho\in V'$ such that $C_6(\frac{\rho}{\rho_o})<1/3$, $C_7\rho^{-2a}(\frac{\rho}{\rho_o})\sigma_o<1/3$, and $C_8\rho^{-2a}(\frac{\rho}{\rho_o})<1/3$ hold. This is possible because we have $\rho^{-2a}(\frac{\rho}{\rho_o})<\eta^{\frac{1}{2}}/\inf_i(\rho_o)_i$. In fact, $\rho^{2a}=\eta^{2\Sigma_{j=1}^{q}a_j+2<\delta,a>}\le\eta^{\frac{1}{4}}$ holds for $0<\delta_j<\varepsilon_j$ $(1\le j\le q)$ since $a_1+\cdots a_q<1/8$ and $\max_j\varepsilon_j<1$. Let $\sigma=(\frac{\rho}{\rho_o})\sigma_o$. Then, by Proposition 5.5, $(5.10)_\mu$ and $(5.11)_\mu$ hold for $\mu=1,2,\cdots$, since $C_6(\frac{\rho}{\rho_o})^2\sigma_o<\sigma/3$, $C_7\rho^{-2a}\sigma^2=C_7\rho^{-2a}(\frac{\rho}{\rho_o})\sigma_o\sigma<\sigma/3$ and $C_8\rho^{-2a}(\frac{\rho}{\rho_o})^2\sigma<\sigma/3$ hold. Thus we proved the convergence of $g_{(\kappa)i}(s)=\lim_{\mu\to+\infty}g_{(\kappa)i}^{(\mu)}(s)$ $(\kappa=1,\cdots,m)$ and $\tau_{(\kappa)}(s)=\lim_{\mu\to+\infty}\tau_{(\kappa)}^{(\mu)}(s)$ $(\kappa=1,\cdots,m)$. This completes the proof of Proposition 5.2.

Corollary 5.7. *Let $\alpha<\varepsilon'<\varepsilon_*$. Then there exist an isomorphism $\tau:T_\varepsilon\to T_{\varepsilon'}$ and an embedding $g:\bar{\Omega}_\varepsilon\times T_\varepsilon\to\mathscr{X}_\varepsilon$ which is holomorphic with respect to $\phi_{\varepsilon'}(\tau(t))$ $(t\in T_\varepsilon)$. In particular, $(\mathscr{X}_{\varepsilon'})_{\mathrm{red}}\to(T_{\varepsilon'})_{\mathrm{red}}$ is embedded in $(\mathscr{X}_\varepsilon)_{\mathrm{red}}\to(T_\varepsilon)_{\mathrm{red}}$ as an open part.*

Proof. Let $S' := T_{\varepsilon'}$, $S := \{o\}$, $\tau^{(0)} := 0$ and $g^{(0)} := i : \bar{\Omega}_\varepsilon \to \pi_{\varepsilon'}^{-1}(o)$. Then, by Proposition 5.2, we have a holomorphic map $\tau : T_{\varepsilon'} \to T_\varepsilon$ and a C^{k-n}-map $g : \bar{\Omega}_\varepsilon \times T_{\varepsilon'} \to \mathscr{X}_{\varepsilon'}$ satisfying $(5.1)''_\infty \sim (5.4)_\infty$ of Proposition 5.2. On the contrary, since $w_{\varepsilon'}$ is formally versal, we have a formal map $\hat{\sigma} : \hat{T}_\varepsilon \to \hat{T}_{\varepsilon'}$. $d(\hat{\sigma} \circ \hat{\tau}) = id$ and $d(\hat{\tau} \circ \hat{\sigma}) = id$ follow from the effectiveness of $w_{\varepsilon'}$ and ϕ_ε combined with the Dolbeault isomorphism. Hence, according to [4, Annex, Note1], we have that $\hat{\tau}$ is an isomorphism. Therefore τ is an isomorphism. The latter half part follows by [19, Satz 4.2]. Q.E.D.

Proof of Theorem 5.1. Let $\varpi : \mathscr{Y} \to S' = S'_{\mathrm{red}}$ be a smooth map such that $\varpi^{-1}(o)$ is a neighbourhood of K. Suppose that there exist a holomorphic map $\tau_o : S' \supset S \to T_\varepsilon$ and an isomorphism $G_o : \mathscr{Y}'_{|S} \to \mathscr{X}_\varepsilon \times_{T_\varepsilon} S$. Choose an $\alpha < \varepsilon' < \varepsilon_*$ and denote the isomorphism $\tau : T_\varepsilon \to T_{\varepsilon'}$, the embedding $g : \bar{\Omega}_\varepsilon \times T_\varepsilon \to \mathscr{X}_\varepsilon$ and the isomorphism $G : (\mathscr{X}'_\varepsilon)_{\mathrm{red}} \to (\mathscr{X}_{\varepsilon'})_{\mathrm{red}}$, obtained in Corollary 5.7 respectively, where $\mathscr{X}'_\varepsilon$ denotes a neighbourhood of K. Let $\tau'_o := \tau \circ \tau_o : S \to T_{\varepsilon'}$ and $g'_o := G_o^{-1} \circ g_o : \bar{\Omega}_{\varepsilon'} \times S \to \mathscr{Y}_{|S}$ where g_o denotes the embedding $\bar{\Omega}_{\varepsilon'} \times S \to \mathscr{X}_\varepsilon \times_{T_\varepsilon} S$ induced from g and τ_o. By Proposition 5.2, we have liftings $\tilde{\tau}' : S' \to T_{\varepsilon'}$ and $\tilde{g}' : \bar{\Omega}_{\varepsilon'} \times S' \to \mathscr{Y}$, of τ'_o and g'_o respectively, satisfying $(5.1)''_\infty \sim (5.4)_\infty$ of Proposition 5.2. Since S' is reduced, by $(5.3)_\infty$ and [19, Sats 4.2], we have that $\tilde{g}'$ induces an isomorphism $\tilde{G}' : \mathscr{X}_{\varepsilon'} \times_{T_{\varepsilon'}} S' \to \mathscr{Y}$ in F_K.

Now we remark that $Im\, \tilde{\tau}' \subset (T_{\varepsilon'})_{\mathrm{red}}$ since S' is reduced. Then $Im\, \tau'_o \subset (T_{\varepsilon'})_{\mathrm{red}}$ and $Im\, \tau_o \subset (T_\varepsilon)_{\mathrm{red}}$. Hence $g'_o = G_o^{-1} \circ G_o$ where G_o denotes the isomorphism $(\mathscr{X}'_\varepsilon)_{\mathrm{red}} \times_{(T_\varepsilon)_{\mathrm{red}}} S \to (\mathscr{X}_{\varepsilon'})_{\mathrm{red}} \times_{(T_{\varepsilon'})_{\mathrm{red}}} S$ induced from G, τ_o and τ'_o.

Since $\tilde{g}'_{|\bar{\Omega}_{\varepsilon'} \times S} = g'_o$, we have that $\tilde{G}'_{|\mathscr{X}_{\varepsilon'} \times_{T_{\varepsilon'}} S} = G_o^{-1} \circ G_o$ ideal theoretically, by [20, Korollar 2.4 (for $A = S'$, $B = \phi$)].

Therefore if we set $\tilde{\tau} := \tau^{-1} \circ \tilde{\tau}' : S' \to (T_\varepsilon)_{\mathrm{red}}$ and $\tilde{G} := \tilde{G}' \circ G_{S'} : (\mathscr{X}'_\varepsilon)_{\mathrm{red}} \times_{(T_\varepsilon)_{\mathrm{red}}} S' \to \mathscr{Y}$, where $G_{S'}$ denotes the isomorphism $(\mathscr{X}'_\varepsilon)_{\mathrm{red}} \times_{(T_\varepsilon)_{\mathrm{red}}} S' \to (\mathscr{X}_{\varepsilon'})_{\mathrm{red}} \times_{(T_{\varepsilon'})_{\mathrm{red}}} S'$ induced from G, $\tilde{\tau}$ and $\tilde{\tau}'$, $(\tilde{G}, \tilde{\tau})$ is a lifting of (G_o, τ_o) in F_K. Q.E.D.

6. Deformations of s.p.c. domains-proof of Theorem 2

Let X be a s.p.c. manifold with $\dim_{\mathbf{C}} X \geq 4$ and with the exceptional set E. Let $p : X \to \bar{X}$ be the Remmert quotient and $p(E) = \{q_1, \cdots, q_k\}$. Let $r : X \to [-\infty, +\infty)$ be an exhaustion function having the follwing properties:

(6.1) r is a strictly plurisubharmonic C^∞-function on $X \backslash E$.

(6.2) $dr \neq 0$ on $\{x \in X \mid -\infty < r(x) < c_1\} \cup \{x \in X \mid c_2 < r(x) < +\infty\}$ for some $-\infty < c_1 < c_2 < +\infty$.

(6.3) $E = \cap_{-\infty < \varepsilon} \Omega_\varepsilon$.

Proposition 6.1. *There exists a formally semi-universal convergent family $\mathscr{X} \to T$ of deformations of X which is semi-universal in $F_{\bar{\Omega}_\alpha}$ for all $-\infty \leq \alpha < c_1$ or*

$c_2 \leq \alpha < +\infty$, *where we denote* $F_{\bar{\Omega}_{-\infty}} := F_E$ *and* $F_{\bar{\Omega}_{c_2}} := \cap_{\varepsilon > c_2} F_{\Omega_\varepsilon}$.

Proof. Choose a strictly increasing sequence $c_2 < \varepsilon_2 < \varepsilon_3 < \cdots$ such that $\lim_{i\to\infty} \varepsilon_i = +\infty$. By Theorem 1, there exists a $\mathscr{X}_i \to T_i$ $(i=2,3,\cdots)$ which is a semi-universal in $F_{\bar{\Omega}_\varepsilon}$ for all $c_2 \leq \varepsilon < \varepsilon_i$ respectively. Since semi-universal elements are isomorphic to each other, we may assume that $\mathscr{X}_i \to T_i$ is embedded in $\mathscr{X}_{i+1} \to T_{i+1}$ as an open part. Hence if we set $T := T_i$ and $\mathscr{X} := \cup_{i=2}^{\infty} \mathscr{X}_i$, $\mathscr{X} \to T$ is a family of deformations of whole X which is semi-universal in $F_{\bar{\Omega}_\varepsilon}$ for all $c_2 \leq \varepsilon < \infty$. Next, fix an $-\infty < \varepsilon_1 < c_1$. Then, by Theorem 1, we have a family $\mathscr{X}_1 \to T_1$ of deformations of Ω_{ε_1} which is semi-universal in $F_{\bar{\Omega}_\varepsilon}$ for all $-\infty \leq \varepsilon < \varepsilon_1$. Hence, we have a morphism $(\mathscr{X} \to T) \to (\mathscr{X}_1 \to T_1)$ in $F_{\bar{\Omega}_{\varepsilon_1 - 1}}$. Since $(\mathscr{X}_2 \to T_2)$ is formally semi-universal in $F_{\bar{\Omega}_\varepsilon}$ for all $-\infty \leq \varepsilon < c_1$ (cf. [16, §2]), we have a formal map $\hat{T}_1 \to \hat{T}$ and a formal isomorphism $\hat{\mathscr{X}}_1 \to \hat{\mathscr{X}} \times_{\hat{T}} \hat{T}_1$ in $F_{\bar{\Omega}_{\varepsilon_1 - 1}}$. By the effectiveness, the above holomorphic maps are all isomorphisms (cf. [4, Annex, Notel]). Hence $\mathscr{X}_1 \to T_1$ is embedded in $\mathscr{X} \to T$ as an open part. Therefore we conclude that $\mathscr{X} \to T$ is also a semi-universal element in $F_{\bar{\Omega}_\varepsilon}$ for all $-\infty \leq \varepsilon < c_1$.

Formally semi-universality of $\mathscr{X} \to T$ follows by comparing $\mathscr{X} \to T$ with the formally semi-universal formal family (cf. [17, Theorem 2.11]). Q.E.D.

Let $E_j := p^{-1}(q_j)$ $(j=1,2,\cdots,k)$ and $\mathscr{U}_j \to T_j$ be a semi-universal family in F_{E_j} $(j=1,2,\cdots,k)$. It is clear that $\cup_{\lambda=1}^{k'} \mathscr{U}_{j_\lambda} \times_{T_{j_\lambda}} (\Pi_{\nu=1}^{k'} T_{j_\nu}) \to \Pi_{\lambda=1}^{k'} T_{j_\lambda}$ is a semi-universal family in $F_{\cup_{\lambda=1}^{k'} E_{j_\lambda}}$. Then we have

Lemma 6.2. $\cup_{j=1}^{k} \mathscr{U}_j \times_{T_j} (\Pi_{l=1}^{k} T_l) \to \Pi_{j=1}^{k} T_j$ *is isomorphic to* $\mathscr{X} \to T$ *in* F_E.

Corollary 6.3. *Any* $\cup_{\lambda=1}^{k'} \mathscr{U}_{j_\lambda} \times_{T_{j_\lambda}} (\Pi_{\nu=1}^{k'} T_{j_\nu}) \to \Pi_{\lambda=1}^{k'} T_{j_\lambda}$ *is embedded in* $\mathscr{X} \to T$ *as an open part of* $\mathscr{X}_{|\Pi_{\lambda=1}^{k'} T_{j_\lambda}} \to \Pi_{\lambda=1}^{k'} T_{j_\lambda}$.

Proof of Theorem 2. Let $\mathscr{X} \to T$ be the formally seimi-universal family in F_X obtained in Proposition 6.1. Let $K \subset X$ be a strongly pseudo-convex compact subset. First we suppose that $K \cap E = E' := \cup_{\lambda=1}^{k'} E_{j_\lambda}$ and set $K = \cap_{i=1}^{\infty} U^{(i)}$ where all $U^{(i)}$'s are inverse images of Stein spaces in the Remmert quotient. We may assume that $p^{-1}(U^{(i)}) \cap E = E'$ $(i=1,2,\cdots)$. Let $r^{(i)}$ be an exhaustion function on $U^{(i)}$ having the properties (6.1)$\sim$(6.3) for some $-\infty < c_1^{(i)} < c_2^{(i)} < +\infty$ respectively. By Proposition 6.1, we have a family $\mathscr{X}^{(i)} \to T^{(i)}$ which is formally semi-universal in $F_{U^{(i)}}$ and is semi-universal in $F_{\bar{\Omega}_\varepsilon^{(i)}}$ for all $\varepsilon \in [-\infty, c_1^{(i)}) \cup [c_2^{(i)}, +\infty)$. Denote $(\mathscr{X}' \to T') := (\mathscr{X} \times_T (\Pi_{\nu=1}^{k'} T_{j_\nu}) \to \Pi_{\lambda=1}^{k'} T_{j_\lambda})$. We will compare $\mathscr{X}^{(i)} \to T^{(i)}$ with $\mathscr{X}' \to T'$. For each $i=1,2,\cdots$, choose $c_2^{(i)} < \varepsilon^{(i)} < +\infty$ such that $K \subset \Omega_{\varepsilon^{(i)}}^{(i)}$. We may assume that $\Omega_{\varepsilon^{(i+1)}}^{(i+1)} \subset\subset \Omega_{\varepsilon^{(i)}}^{(i)}$. By the semi-universality of $\mathscr{X}^{(i)} \to T^{(i)}$ in $F_{\bar{\Omega}_{\varepsilon^{(i)}}^{(i)}}$, there exists a holomorphic map $\tau^{(i)} : T' \to T^{(i)}$ and an isomorphism between $\mathscr{X}^{(i)} \times_{T^{(i)}} T' \to T'$ and $\mathscr{X}' \to T'$ in $F_{\bar{\Omega}_{\varepsilon^{(i)}}^{(i)}}$. Remark that this is also an isomorphism in $F_{E'}$. By the semi-universality of $\mathscr{X}' \to T'$ in $F_{E'}$, there exists a holomorphic map $\tau : T^{(i)} \to T'$

and an isomorphism between $\mathscr{X}^{(i)} \to T^{(i)}$ and $\mathscr{X}' \times_{T'} T^{(i)} \to T^{(i)}$ in $F_{E'}$. From the effectiveness, τ and $\tau^{(i)}$ are all isomorphic to each other. Hence we may assume that all $\mathscr{X}^{(i)} \to T^{(i)}$ are embedded in $\mathscr{X}' \to T'$ as open parts.

Let $\varpi: \mathscr{Y} \to S'$ be a family with $K \subset \varpi^{-1}(o)$ $(o \in S')$ and a morphism $\begin{array}{ccc} \mathscr{Y}_{|S} & \to & \mathscr{X}' \\ \downarrow & & \downarrow \\ S & \to & T' \end{array}$ be given. Choose i such that $K \subset \Omega^{(i)}_{\varepsilon(i)} \subset\subset \varpi^{-1}(o)$. Then we have a morphism $\begin{array}{ccc} \mathscr{Y}_{|S} & \to & \mathscr{X}^{(i)} \\ \downarrow & & \downarrow \\ S & \to & T^{(i)} \end{array}$ in $F_{\bar{\Omega}^{(i)}_{\varepsilon(i)}}$. By the semi-universality of $\mathscr{X}^{(i)} \to T^{(i)}$, we have a lifting $\begin{array}{ccc} \mathscr{Y} & \to & \mathscr{X}^{(i)} \\ \downarrow & & \downarrow \\ S' & \to & T^{(i)} \end{array}$ in $F_{\bar{\Omega}^{(i)}_{\varepsilon(i)}}$. Therefore we have a lifting $\begin{array}{ccc} \mathscr{Y} & \to & \mathscr{X}' \\ \downarrow & & \downarrow \\ S' & \to & T' \end{array}$ in F_K. This implies that $\mathscr{X} \to T$ is a versal element in F_K.

Next, we suppose that $K \cap E = \phi$. Then we may assume that $p^{-1}(U^{(i)}) \cap E = \phi$ $(i = 1, 2, \cdots)$ in the above argument. Choose an arbitrary point $p \in K$. Then by the same argument as above with $E' = \{p\}$, we have that $\mathscr{X} \to T$ is a versal element in F_K. Q.E.D.

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