

THE SUBMANIFOLD OF SELF-DUAL CODES IN A GRASSMANN MANIFOLD

SHIGERU KOBAYASHI AND ICHIRO TAKADA

(Received May 16, 1994)

1. Introduction

By a $[N, m]$ -linear code over a finite field F , we mean an m -dimensional vector subspace of an N -dimensional vector space V over F . Let C^\perp be the orthogonal complement of a $[N, m]$ -linear code C in V , that is $C^\perp = \{v \in V | \langle v, c \rangle = 0 \text{ for any } c \in C\}$, where $\langle \cdot, \cdot \rangle$ denotes a fixed inner product of V . This is called the dual code of C which is a $[N, N-m]$ -linear code. C is called self-orthogonal (resp. self-dual) if and only if $C \subset C^\perp$ (resp. $C = C^\perp$). For any linear code, it may be known that there exists a self-dual embedding, and so every linear code can be made from a self-dual code. Therefore we are interested in self-dual codes. Since a linear code C is a vector space, C can be thought as an element of the Grassmann manifold $GM(m, V)$. Similarly, C^\perp can be thought as an element of $GM(N-m, V)$. As a set, $GM(m, V)$ and $GM(N-m, V)$ are isomorphic so that C and C^\perp correspond each other as elements of the Grassmann manifolds. In this paper, we shall study the self-orthogonality and the self-duality of linear codes through the Grassmann manifolds. In section 1, we shall give a constructive proof of self-dual embedding of linear codes. In section 2, we shall summarize about the Grassmann manifolds and give an elementary result about the self-duality using a projective embedding. In section 3, we shall give our main theorem on self-orthogonality and self-duality of linear codes. This theorem shows that self-orthogonal codes and self-dual codes are on a quadratic surface in the projective space. Combining our results, we can see that every linear code can be obtained from a self-dual code, and every self-dual code is a special case of a self-orthogonal code.

2. Self-dual embedding of linear codes

In this section, we assume $N = n + m$. Let C be a $[N, m]$ -linear code over a finite field F . We shall construct a self-dual code which contains C as an embedding image. It may be known, but this is a motive for studying self-dual codes and so we shall give the proof. Since C can be thought as a subspace of

F^N , we can write

$$C = \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{pmatrix} \begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix} \\ \leftarrow N \rightarrow$$

where $\xi^{(i)}$ ($i=0, \dots, m-1$) are column vectors of F^N . First assume that $ch(F)=2$ and consider the equation

$$\langle \xi^{(0)}, \xi^{(0)} \rangle + X^2 = 0. \tag{2.1}$$

where \langle , \rangle means the inner product of F^N . Since the Frobenius map $x \rightarrow x^2$ is an automorphism of F , the equation (2.1) has solution, say $X = a_{0,0}$. Further consider the equations

$$\langle \xi^{(i)}, \xi^{(0)} \rangle + a_{0,0} X_i = 0 \quad (i=0, \dots, m-1).$$

Since these equations are linear, they has solutions, say $X_i = a_{0,i}$ ($i=0, \dots, m-1$). Now the following matrix

$$\begin{pmatrix} \xi^{(0)} & a_{0,0} \\ \xi^{(1)} & a_{0,1} \\ \vdots & \vdots \\ \xi^{(m-1)} & a_{0,m-1} \end{pmatrix} = \begin{pmatrix} \xi_1^{(0)} \\ \xi_1^{(1)} \\ \vdots \\ \xi_1^{(m-1)} \end{pmatrix} \begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix} \\ \leftarrow N \rightarrow$$

satisfies $\langle \xi_1^{(0)}, \xi_1^{(j)} \rangle = 0$ ($j=0, \dots, m-1$), where $\xi_1^{(j)} = (\xi^{(j)}, a_{0,j})$ are column vectors in F^N . Next consider the equation

$$\langle \xi^{(1)}, \xi^{(1)} \rangle + X^2 = 0.$$

We can obtain the solution as above, say $X = a_{1,1}$. Further consider equations

$$\langle \xi^{(1)}, \xi^{(i)} \rangle + a_{1,1} X_i = 0 \quad (i=1, \dots, m-1).$$

Clearly we have solutions, say $X_i = a_{1,i}$ ($i=1, \dots, m-1$). Hence the following matrix

$$\begin{pmatrix} \xi_1^{(0)} & 0 \\ \xi_1^{(1)} & a_{1,1} \\ \vdots & \vdots \\ \xi_1^{(m-1)} & a_{1,m-1} \end{pmatrix} = \begin{pmatrix} \xi_2^{(0)} \\ \xi_2^{(1)} \\ \vdots \\ \xi_2^{(m-1)} \end{pmatrix} \begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix}$$

← N →

satisfies

$$\begin{aligned} \langle \xi_2^{(0)}, \xi_2^{(j)} \rangle &= 0 \quad (j=0, 1, \dots, m-1) \\ \langle \xi_2^{(1)}, \xi_2^{(k)} \rangle &= 0 \quad (k=1, 2, \dots, m-1) \end{aligned}$$

where $\xi_2^{(0)} = (\xi_1^{(0)}, 0)$ and $\xi_2^{(i)} = (\xi_1^{(i)}, a_{1,i})$ ($i=1, \dots, m-1$). We continue this process, so that we have the following matrix

$$C \begin{pmatrix} a_{0,0} & \cdots & \cdots & \mathbf{0} \\ a_{0,1} & a_{1,1} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{0,m-1} & a_{1,m-1} & \cdots & a_{m-1,m-1} \end{pmatrix} = \begin{pmatrix} \xi_{m-1}^{(0)} \\ \xi_{m-1}^{(1)} \\ \vdots \\ \xi_{m-1}^{(m-1)} \end{pmatrix}$$

← N+m →

We can express this matrix in the form

$$\begin{pmatrix} C & A \end{pmatrix} = \begin{pmatrix} \xi_{m-1}^{(0)} \\ \xi_{m-1}^{(1)} \\ \vdots \\ \xi_{m-1}^{(m-1)} \end{pmatrix}$$

← N+m →

where A is the following $m \times m$ matrix

$$\begin{pmatrix} a_{0,0} & \cdots & \cdots & \mathbf{0} \\ a_{0,1} & a_{1,1} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{0,m-1} & a_{1,m-1} & \cdots & a_{m-1,m-1} \end{pmatrix}. \tag{2.2}$$

Clearly the matrix (2.2) satisfies

$$\langle \xi_{m-1}^{(i)}, \xi_{m-1}^{(j)} \rangle = 0 \quad (i, j=0, 1, \dots, m-1).$$

Thus this matrix gives a self-orthogonal code. On the other hand, consider

the dual code C^\perp . Then the same argument can be applied to the dual code C^\perp . Since $N=m+n$, we can express C^\perp in the form

$$C^\perp = \begin{pmatrix} \eta^{(0)} \\ \eta^{(1)} \\ \vdots \\ \eta^{(n-1)} \end{pmatrix} \begin{matrix} \uparrow \\ n \\ \downarrow \end{matrix} \cdot \\ \leftarrow N \rightarrow$$

We can also obtain a self-orthogonal code from C^\perp and express in the form

$$(C^\perp \ B)$$

where B is an $n \times n$ matrix obtained from C^\perp as well as A . To make a self-dual code, we take the following matrix

$$\hat{C} = \begin{pmatrix} C & A & 0 \\ C^\perp & 0 & B \end{pmatrix} \begin{matrix} \uparrow \\ m+n \\ \downarrow \end{matrix} \cdot \\ \leftarrow N+m+n \rightarrow$$

This is a self-dual $[2N, N]$ -code because C^\perp is a dual vector space of F^N / C . Next we assume $ch(F)=p>2$ and consider an equation

$$X_1^2 + X_2^2 + X_3^2 + \langle \xi^{(0)}, \xi^{(0)} \rangle = 0.$$

Then a theorem of Chevalley-Waring (cf.[3]) shows that this equation have a solution, say $(a_{0,0}^{(1)}, a_{0,0}^{(2)}, a_{0,0}^{(3)})$. Further we consider following equations

$$\langle \xi^{(0)}, \xi^{(i)} \rangle + a_{0,0}^{(1)} X_i = 0 \quad (i=1, \dots, m-1).$$

These equations have a solution since the equations are linear. We set a solution as

$$(x_1, \dots, x_{m-1}) = (a_{0,1}, a_{0,2}, \dots, a_{0,m-1}).$$

Then the following matrix

$$\begin{pmatrix} C & \begin{matrix} a_{0,0}^{(1)} & a_{0,0}^{(2)} & a_{0,0}^{(3)} \\ a_{0,1} & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ a_{0,m-1} & \cdots & \mathbf{0} \end{matrix} \end{pmatrix} = \begin{pmatrix} \xi_1^{(0)} \\ \xi_1^{(1)} \\ \vdots \\ \xi_1^{(m-1)} \end{pmatrix} \cdot$$

satisfies

$$\langle \xi_1^{(0)}, \xi_1^{(i)} \rangle = 0 \quad (i=0, 1, \dots, m-1).$$

Next consider

$$X_1^2 + X_2^2 + X_3^2 + \langle \xi_5^{(1)}, \xi_5^{(1)} \rangle = 0.$$

Let $(a_{1,1}^{(1)}, a_{1,1}^{(2)}, a_{1,1}^{(3)})$ and $a_{1,j}$ be a solution of

$$\langle \xi_1^{(1)}, \xi_1^{(j)} \rangle + a_{1,1}^{(1)} x_j = 0 \quad (j=1, 2, \dots, m-1).$$

Then the following matrix

$$\begin{pmatrix} \xi_1^{(1)} \\ \xi_1^{(2)} \\ \vdots \\ \xi_1^{(m-1)} \end{pmatrix} \begin{pmatrix} a_{1,1}^{(1)} & a_{1,1}^{(2)} & a_{1,1}^{(3)} \\ a_{1,2} & 0 & 0 \\ a_{1,2} & \ddots & \vdots \\ a_{1,m-1} & \cdots & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \xi_2^{(1)} \\ \xi_2^{(2)} \\ \vdots \\ \xi_2^{(m-1)} \end{pmatrix},$$

satisfies

$$\langle \xi_2^{(1)}, \xi_2^{(j)} \rangle = 0 \quad (j=1, 2, \dots, m-1).$$

We continue this process, so that we have the following matrix

$$(C \ A) = \begin{pmatrix} \xi_{m-1}^{(0)} \\ \xi_{m-1}^{(1)} \\ \vdots \\ \xi_{m-1}^{(m-1)} \end{pmatrix},$$

where A is the following $m \times 3m$ matrix

$$A = \begin{pmatrix} a_{0,0}^{(1)} & a_{0,0}^{(2)} & a_{0,0}^{(3)} & \cdots & \cdots & \mathbf{0} \\ a_{0,1} & 0 & 0 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ a_{0,m-1} & \cdots & \mathbf{0} & a_{m-1,m-1}^{(1)} & a_{m-1,m-1}^{(2)} & a_{m-1,m-1}^{(3)} \end{pmatrix},$$

which satisfies:

$$\langle \xi_{m-1}^{(i)}, \xi_{m-1}^{(j)} \rangle = 0 \quad (i, j=0, 1, \dots, m-1).$$

Thus this matrix gives a self-orthogonal code. Further we can apply the same method to the dual code C^\perp . By using the same notation as above, we have a self-orthogonal code for C^\perp

$$\left(\begin{pmatrix} \eta^{(0)} \\ \eta^{(1)} \\ \vdots \\ \eta^{(n-1)} \end{pmatrix} B \right)$$

where B is an $n \times 3n$ matrix obtained from C^\perp as well as A . For $k \geq 5$, consider the following equations

$$f_1(X_1, \dots, X_k) = \sum_{i=1}^k X_i^2 = 0$$

$$f_2(X_1, \dots, X_k) = \sum_{i=1}^{k-1} X_i X_{i+1} = 0$$

Since $\sum_{i=1}^2 \deg f_i = 4 < k$, we can use a theorem of Chevalley-Waring again, so that there exists a non-trivial solution

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k).$$

Since α is non-trivial, we may assume that $\alpha_1 \neq 0$. We set

$$M = \begin{pmatrix} \alpha_1 & \cdots & \alpha_k & & \mathbf{0} \\ 0 & \alpha_1 & \cdots & & \\ & & & \ddots & \vdots \\ & & & & \alpha_1 & \cdots & \alpha_k \end{pmatrix} \begin{matrix} \uparrow \\ (k+1)N \\ \downarrow \end{matrix}$$

$\leftarrow \quad 2kN \quad \rightarrow$

Then the following matrix

$$\begin{pmatrix} C & A & 0 & \mathbf{0} \\ C^\perp & 0 & B & \\ \mathbf{0} & & & M \end{pmatrix}$$

gives a self-dual $[(2k+4)N, (k+2)N]$ -linear code.

Therefore we obtain the following theorem.

Theorem 1. *Let C be a $[N, m]$ -linear code over a finite field F . Then there exist a self-dual code \hat{C} such that C is embedded in \hat{C} . More precisely, we can take \hat{C} as follows:*

- (1) *if $ch(F) = 2$, \hat{C} is self-dual $[2N, N]$ -linear code.*
- (2) *if $ch(F) = p > 2$, then for any integer $k \geq 5$, \hat{C} is a self-dual $[(2k+4)N, (k+2)N]$ -linear code.*

3. Grassmann Manifold

In this section, we summarize about Grassmanian manifolds. Let $N=n+m$ and $V=V(N)$ be an N -dimensional vector space over a field F . Put $GM(m,V)=\{m\text{-dimensional subspace of } V\}$. Take a basis $\{e_0, e_1, \dots, e_{N-1}\}$ of V . Then $V=Fe_0 \oplus Fe_1 \oplus Fe_2 \dots \oplus Fe_{N-1}$. Let V^* be the dual space of V and $\{f_0, f_1, \dots, f_{N-1}\}$ be a dual basis with $\langle e_i, f_j \rangle = \delta_{i,j}$, where δ_{ij} denotes Kronecker'delta. Let $V^*=Ff_0 \oplus Ff_1 \oplus \dots \oplus Ff_{N-1}$. For a subspace $V_0 \subseteq V$, define $V_0^\perp = \{f \in V^* \mid f(V_0) = 0\}$. Then there is a one to one correspondence between V_0 and V_0^\perp , so that $GM(m,V)$ is isomorphic to $GM(n,V^*)$ as a set. Let $\wedge^m V$ be the space of m -th exterior products of V . $\wedge^m V$ is the $\binom{N}{m}$ -dimensional vector space over F with basis $\{e_{i_0} \wedge e_{i_1} \wedge \dots \wedge e_{i_{m-1}}; 0 \leq i_0 \leq i_1 \leq \dots \leq i_{m-1} \leq N\}$. We define the projective embedding of $GM(m,V)$ as follows:

$$GM(m,V) \rightarrow P(\wedge^m V)$$

$$\xi = \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{pmatrix} \mapsto \xi^{(0)} \wedge \dots \wedge \xi^{(m-1)}.$$

For $\xi \in GM(m,V)$, we can write $\xi^{(j)} = \sum_{0 \leq i \leq N} \xi_{ji}^{(j)} e_i$. Then

$$\xi^{(0)} \wedge \dots \wedge \xi^{(m-1)} = \sum_{0 \leq l_0 < \dots < l_{m-1} \leq N} \xi_{l_0, \dots, l_{m-1}} e_{l_0} \wedge \dots \wedge e_{l_{m-1}}$$

where $\xi_{l_0, \dots, l_{m-1}}$ is the determinant of the matrix obtained by picking out the l_0, \dots, l_{m-1} columns of ξ .

The above projective embedding can be translated as follows:

$$GM(m,V) \rightarrow P^{\binom{N}{m}-1}(F)$$

$$\xi = \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{pmatrix} \mapsto (\xi_{l_0, \dots, l_{m-1}})_{0 \leq l_0 < \dots < l_{m-1} \leq N}. \tag{3.1}$$

Further, this projective embedding satisfies the *Plücker* relation

$$\sum_{0 \leq i \leq N} (-1)^i \xi_{k_0, \dots, k_{m-2}, i} \xi_{l_0, \dots, \check{l}_i, \dots, l_m} = 0$$

for

$$0 \leq k_0 < \dots < k_{m-2} < N, 0 \leq l_0 < \dots < l_m \leq N$$

where \bar{I}_i means removing I_i .

Let C be a $[N, m]$ -linear code which is an element of $GM(m, V)$ and write

$$C = \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{pmatrix} .$$

Likewise, let

$$C^\perp = \begin{pmatrix} \eta^{(0)} \\ \vdots \\ \eta^{(n-1)} \end{pmatrix} ,$$

which is an element of $GM(n, V)$. According to (3.1), $GM(m, V)$ has a projective embedding into $\mathbf{P}^{\binom{N}{m}-1}(F)$ and similarly $GM(n, V)$ has a projective embedding into $\mathbf{P}^{\binom{N}{n}-1}(F)$. Since $\mathbf{P}^{\binom{N}{m}-1}(F) = \mathbf{P}^{\binom{N}{n}-1}(F)$, we have an easy criterion of self-duality of C as follows:

Theorem 2. *Let C be a $[N, m]$ -linear code over a finite field F and let C^\perp be the dual code of C . Assume that C and C^\perp are as above. Then C is self-dual if and only if $(\xi_{I_0, \dots, I_{m-1}})_{0 \leq I_0 < \dots < I_{m-1} \leq N} = (\eta_{s_0, \dots, s_{n-1}})_{0 \leq s_0 < \dots < s_{n-1} \leq N}$ in $\mathbf{P}^{\binom{N}{m}-1}$ and $N = 2m$.*

Proof. First assume that C is a self-dual code. Then since $C = C^\perp$, the theorem is clear. Conversely, assume that $(\xi_{I_0, \dots, I_{m-1}})_{0 \leq I_0 < \dots < I_{m-1} \leq N} = (\eta_{s_0, \dots, s_{n-1}})_{0 \leq s_0 < \dots < s_{n-1} \leq N}$ in $\mathbf{P}^{\binom{N}{m}-1}$ and $N = 2m$. Then clearly $(\xi^{(0)} \wedge \dots \wedge \xi^{(m-1)}) = (\eta^{(0)} \wedge \dots \wedge \eta^{(n-1)})$ and $\xi^{(0)} \wedge \dots \wedge \xi^{(m-1)} = a\eta^{(0)} \wedge \dots \wedge \eta^{(n-1)}$ for some non zero element a of F . Hence $\xi^{(0)} \wedge \dots \wedge \xi^{(m-1)} \wedge \eta^{(i)} = a\eta^{(0)} \wedge \dots \wedge \eta^{(i)} \wedge \eta^{(n-1)} \wedge \eta^{(i)} = 0$ ($i = 0, \dots, m-1$), that is $\eta^{(i)} \in F\xi^{(0)} \oplus \dots \oplus F\xi^{(m-1)}$. Similarly, we have $\xi^{(i)} \in F\eta^{(0)} \oplus \dots \oplus F\eta^{(n-1)}$. This implies that $F\xi^{(0)} \oplus \dots \oplus F\xi^{(m-1)} = F\eta^{(0)} \oplus \dots \oplus \eta^{(n-1)}$ and we have $C = C^\perp$.

4. Self-duality of linear codes

In this section, we shall study self-orthogonal (resp. self-dual) codes in the Grassmann manifolds.

Theorem 3. *Let $C = F\xi^{(0)} \oplus \dots \oplus F\xi^{(m-1)}$ be a $[N, m]$ -linear code over a finite field F . Then C is a self-orthogonal (resp. self-dual) code if and only if C is a point of the Grassmann manifolds which satisfies the Plücker's relations and is on the quadratic surface defined by*

$$\sum_{0 \leq I_0 < \dots < I_{m-1} \leq N} \xi_{I_0, \dots, I_{m-1}}^2 = 0 \quad (\text{resp. further } N = 2m),$$

where $\xi_{l_0, \dots, l_{m-1}}$ is the determinant of the matrix obtained by picking out the m columns l_0, \dots, l_{m-1} of C .

Proof. As explained in the previous section, C can be thought as a point of the Grassmann manifolds which satisfies the *Plücker's* relations. So we must prove that C is self-orthogonal if and only if C is on the quadratic surface defined as above. First assume that C is a self-orthogonal code. Let

$$C = \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{pmatrix} \begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix} \\ \leftarrow N \rightarrow$$

Since C is contained in C^\perp , we have

$$\begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix} \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{pmatrix} \begin{pmatrix} {}^t\xi^{(0)} & \dots & {}^t\xi^{(m-1)} \end{pmatrix} \begin{matrix} \uparrow \\ N \\ \downarrow \end{matrix} = 0, \\ \leftarrow N \rightarrow \qquad \leftarrow m \rightarrow$$

where ${}^t\xi^{(i)}$ is the transpose of $\xi^{(i)}$. Then we have

$$\det \left\{ \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{pmatrix} \begin{pmatrix} {}^t\xi^{(0)} & \dots & {}^t\xi^{(m-1)} \end{pmatrix} \right\} = 0.$$

In this case, Binet-Cauchy formula (cf.[1]) implies

$$\begin{aligned} & \det \left\{ \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{pmatrix} \begin{pmatrix} {}^t\xi^{(0)} & \dots & {}^t\xi^{(m-1)} \end{pmatrix} \right\} \\ &= \sum_{\square} \det(\square) \det(\square) \\ &= \sum_{\square} \det \square \square \\ &= \sum \xi_{l_0, \dots, l_{m-1}}^2 = 0 \end{aligned}$$

where \square is an $m \times m$ matrix obtained by picking out m columns of C and summation is taken over all $m \times m$ matrices.

Conversely, we assume that

$$\sum \xi_{i_0 \dots i_{m-1}}^2 = 0.$$

Then Binet-Cauchy formula implies

$$\det \left\{ \left(\begin{array}{c} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{array} \right) \left(\begin{array}{ccc} {}_t\xi^{(0)} & \dots & {}_t\xi^{(m-1)} \end{array} \right) \right\} = 0$$

since

$$\begin{aligned} & \det \left\{ \left(\begin{array}{c} \xi^{(0)} \\ \vdots \\ \xi^{(m-1)} \end{array} \right) \left(\begin{array}{ccc} {}_t\xi^{(0)} & \dots & {}_t\xi^{(m-1)} \end{array} \right) \right\} \\ &= \det \left(\begin{array}{ccc} \langle \xi^{(0)}, \xi^{(0)} \rangle, & \dots, & \langle \xi^{(0)}, \xi^{(m-1)} \rangle \\ \vdots & \dots & \vdots \\ \langle \xi^{(m-1)}, \xi^{(0)} \rangle, & \dots, & \langle \xi^{(m-1)}, \xi^{(m-1)} \rangle \end{array} \right) = 0 \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ means canonical inner product in F^N .

This shows that for any i ($i=0, \dots, m-1$),

$$X_0 \langle \xi^{(0)}, \xi^{(i)} \rangle + \dots + X_{m-1} \langle \xi^{(m-1)}, \xi^{(i)} \rangle = 0$$

has a non-trivial solution $(\lambda_0, \dots, \lambda_{m-1})$. In particular,

$$\langle \lambda_0 \xi^{(0)} + \dots + \lambda_{m-1} \xi^{(m-1)}, \xi^{(i)} \rangle = 0$$

so that

$$\lambda_0 \xi^{(0)} + \dots + \lambda_{m-1} \xi^{(m-1)}$$

is contained in $C \cap C^\perp$. We set

$$\eta^{(m-1)} = \lambda_0 \xi^{(0)} + \dots + \lambda_{m-1} \xi^{(m-1)}$$

which satisfies

$$\langle \xi^{(i)}, \eta^{(m-1)} \rangle = 0 \quad (i=0, \dots, m-1).$$

By renumbering $\lambda_0, \dots, \lambda_{m-1}$, we may assume that $\lambda_{m-1} \neq 0$. We claim that

$$\xi^{(0)}, \dots, \xi^{(m-2)}, \eta^{(m-1)}$$

are linearly independent over F . Assume that

$$a_0 \xi^{(0)} + \dots + a_{m-2} \xi^{(m-2)} + a_{m-1} \eta^{(m-1)} = 0$$

for $a_0, \dots, a_{m-1} \in F$. Then

$$a_0 \xi^{(0)} + \dots + a_{m-2} \xi^{(m-2)} + a_{m-1} (\lambda_0 \xi^{(0)} + \dots + \lambda_{m-1} \xi^{(m-1)}) = 0.$$

Since $\xi^{(0)}, \dots, \xi^{(m-1)}$ are linearly independent, we have

$$a_i + a_{m-1} \lambda_i = 0 \quad (i=0, \dots, m-1), \quad a_{m-1} \lambda_{m-1} = 0.$$

Since $\lambda_{m-1} \neq 0$, we have that $a_{m-1} = 0$. Thus we obtain

$$a_0 \xi^{(0)} + \dots + a_{m-2} \xi^{(m-2)} = 0.$$

Since $\xi^{(0)}, \dots, \xi^{(m-2)}$ are linearly independent, we have

$$a_0 = \dots = a_{m-2} = 0.$$

This shows that $\xi^{(0)}, \dots, \xi^{(m-2)}, \eta^{(m-1)}$ are linearly independent.

Now $\{\xi^{(0)}, \dots, \xi^{(m-2)}, \eta^{(m-1)}\}$ becomes a basis of C . Since

$$\langle \eta^{(m-1)}, \xi^{(i)} \rangle = 0 \quad (i=0, \dots, m-1),$$

we know

$$\det \left\{ \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-2)} \\ \eta^{(m-1)} \end{pmatrix} \left({}^t \xi^{(0)}, \dots, {}^t \xi^{(m-2)}, {}^t \eta^{(m-1)} \right) \right\} = 0$$

which implies

$$\begin{aligned} & \det \left\{ \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-2)} \\ \eta^{(m-1)} \end{pmatrix} \left({}^t \xi^{(0)}, \dots, {}^t \xi^{(m-2)}, {}^t \eta^{(m-1)} \right) \right\} \\ &= \det \left\{ \begin{pmatrix} \xi^{(0)} \\ \vdots \\ \xi^{(m-2)} \end{pmatrix} \left({}^t \xi^{(0)} \dots {}^t \xi^{(m-1)} \right) \right\} = 0. \end{aligned}$$

By the same argument, we see that there exists a non-trivial solution

$$(\mu_0, \dots, \mu_{m-2}) \in F^{m-1}$$

such that

$$\langle \mu_0 \xi^{(0)} + \dots + \mu_{m-2} \xi^{(m-2)}, \xi^{(i)} \rangle = 0 \quad (i=0, \dots, m-2).$$

We set

$$\eta^{(m-2)} = \mu_0 \xi^{(0)} + \cdots + \mu_{m-2} \xi^{(m-2)}$$

which satisfies

$$\langle \xi^{(i)}, \eta^{(m-2)} \rangle = 0 \quad (i=0, \dots, m-2), \quad \langle \eta^{(m-1)}, \eta^{(m-2)} \rangle = 0.$$

Similarly,

$$\{\xi^{(0)}, \dots, \xi^{(m-3)}, \eta^{(m-2)}, \eta^{(m-1)}\}$$

becomes a basis of C . We proceed this process. Then we obtain a basis

$$\{\eta^{(0)}, \dots, \eta^{(m-1)}\}$$

which satisfies

$$\langle \eta^{(i)}, \eta^{(j)} \rangle = 0 \quad (i, j=0, \dots, m-1).$$

Now C becomes a self-orthogonal code. Since the case of a self-dual code is clear, the proof is complete.

References

- [1] S. Lang: Linear Algebra, Springer, New York, 1987.
- [2] J.H. van Lint, G. van der Geer: Introduction to Coding Theory and Algebraic Geometry, Birkhäuser, Basel, 1988.
- [3] J.-P. Serre: Cours d'Arithmétique, P. U. France, 1970.

Department of Mathematics
 Naruto University of Education
 Takashima, Naruto, 772, Japan
 e-mail address: skoba@naruto-u.ac.jp
 and
 Department of Mathematics
 Tokushima University
 Tokushima, 770, JAPAN
 e-mail address: g01536@sinet.ad.jp