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THE FUNDAMENTAL GROUP OF THE SMOOTH PART OF A LOG FANO VARIETY

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Introduction

Let X be a normal projective variety over the complex number field $\boldsymbol{C}$. We call X a *Fano variety* if X is $\boldsymbol{Q}$-Gorenstein and the anti-canonical divisor $-K_X$ is ample. A Fano variety X is said to be a *log Fano variety* if X has only log terminal singularities (cf. [6]). A Fano variety X is called a *canonical Fano variety* if X has only canonical singularities (cf. [6]). The *Cartier index* $c(X)$ is the smallest positive integer such that $c(X)K_X$ is a Cartier divisor. The *Fano index*, denoted by $r(X)$, is the largest positive rational number such that $-K_X \sim_{\boldsymbol{Q}} r(X)H$ ($\boldsymbol{Q}$-linear equivalence) for a Cartier divisor H.

This note consists of two sections. In §1, we shall consider canonical Fano 3-folds and prove the following:

Theorem 1. *Let X be a canonical Fano 3-fold. Let $X^o := X - \operatorname{Sing} X$ be the smooth part of X. Assume that X has only isolated singularities. Then we have:*
(1) *Suppose the Fano index $r(X)$ is* 1. *Then $\pi_1(X^o) = \boldsymbol{Z}/c(X)\boldsymbol{Z}$* (cf. Remark 1.1 in §1).
(2) *Suppose that the canonical divisor K_X is a Cartier divisor. Then X^o is simply connected.*

Remark. (1) The assumption that X has only isolated singularities is used to prove Lemma 1.3 in §1.

(2) Using the same proof (see §1) one can show that $\pi_1(X^o) = \boldsymbol{Z}/c(X)\boldsymbol{Z}$ when X is a log Fano variety of Fano index one and with only isolated singularities because even in this case the $\boldsymbol{Z}/c(X)\boldsymbol{Z}$-covering Y constructed in §1 has only isolated canonical singularities.

In [19], we shall give a universal bound for $c(X)$. Under the much stronger condition that X has only terminal and cyclic quotient singularities, T. Sano proved that $c(X) \leq 2$ (cf. [15]).

In §2, we shall consider n-dimensional log Fano varieties and prove the following:

Theorem 2. *Let X be a log Fano variety of Fano index $r(X) > \dim X - 2$. Let $X^o := X - \operatorname{Sing} X$ be the smooth part of X. Then we have*:

(1) *The fundamental group $\pi_1(X^o)$ of X^o is a finite group.*

(2) *Suppose that X has only canonical singularities. Then $\pi_1(X^o)$ is an abelian group of order ≤ 9.* (See [9] for the unique canonical Fano surface X with $|\pi_1(X^o)| = 9$.)

(3) *Suppose that $r(X) \geq \dim X - 1$. Then $\pi_1(X^o)$ is a finite abelian group generated by two elements, and has order ≤ 9.*

(4) Suppose $r(X) > \dim X - 1$. Then the smooth part X^o of X is simply connected.

To prove Theorem 2, we manage to reduce to the dimension two case. In the dimension two case, Theorem 2(1) was proved in our joint works [3, 4] (see [18], and also [2] for a differential geometric proof), and Theorem 2(2) proved in [8, 9]. In this reduction process, in order to apply the crucial theorem due to Alexeev, we need the hypothesis about the Fano index $r(X)$. We hope this hypothesis can be eventually dropped.

Note that in Theorem 2(2) the hypothesis "X has only canonical singularities" is necessary. Indeed, in [17] we have a lot of examples of log Fano surfaces X with exactly one triple point and several double points but with a non-abelian $\pi_1(X^o)$.

We want to remove the condition about the Fano index $r(X)$ in the above theorems and raise the following question:

Let X be a Fano variety. Let $X^o := X - \operatorname{Sing} X$ be the smooth part of X. Suppose that X has only log-terminal (or canonical, or terminal) singularities. Is $\pi_1(X^o)$ a finite group?

In general, it is not true that the smooth part of a rational variety with only log-terminal singularities has finite fundamental group. Just consider the example $(\boldsymbol{P}^1 \times E)/\tau$ in [3, §1.15] where E is an elliptic curve and τ is an involution acting on both $\boldsymbol{P}^1$ and E non-trivially and diagonally. So, the ampleness of $-K_X$ is an essential condition.

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1. Proof of Theorem 1

We shall first show that Theorem 1(1) follows from Theorem 1(2). Let X be as in Theorem 1(1). We have a linear equivalence $-cK_X \sim cH$ where $c=c(X)$ is the Cartier index and H is a primitive Cartier divisor. Here we say that a Cartier divisor H on X is primitive if $\mathcal{O}(H)$ is not divisible in the Picard group of X. Let

$$Y := \operatorname{Spec} \bigoplus_{i=0}^{c-1} \mathcal{O}_X(i(K_X+H)).$$

Then the natural $\boldsymbol{Z}/c\boldsymbol{Z}$-covering morphism $f: Y \to X$ is etale over the smooth part $X^o := X - \operatorname{Sing} X$ of X. Moreover, $-K_Y = -f^*K_X \sim f^*H$. So, Y is a (Gorenstein) Fano variety with only canonical singularities. Thus, the hypotheses in Theorem 1(2) are satisfied by Y. Note that $\pi_1(Y^o) = \pi_1(f^{-1}(X^o))$ because $Y^o - f^{-1}(X^o)$ has codimension ≥ 2 in the smooth 3-fold $Y^o := Y - \operatorname{Sing} Y$. So, Theorem 1(1) will follow form Theorem 1(2). Moreover, we have

REMARK 1.1. $f^{-1}(X^o)$ is the universal covering of X^o.

Now we shall prove Theorem 1(2). Let X be as in Theorem 1(2). We can write $-K_X = rH$ where $r = r(X) \in \boldsymbol{Z}_{>0}$ is the Fano index and H an ample Cartier divisor. Let $S \in |-K_X|$ be a general member. By [12], we know that S is a K3-surface possibly with rational double singularities. Let $\sigma: T \to S$ be a minimal resolution of singularities. Then T is a K3-surface. The first assertion of the following Lemma 1.2 is from [13] or [10, Theorem 5] and the second is a consequence of the first.

Lemma 1.2. *Let T be a K3-surface defined over $\boldsymbol{C}$. Let L be a nonzero numerically effective divisor on T. Then we have:*
(1) $|L|$ has base points if and only if there exist irreducible curves E, Γ, and an integer $k \geq 2$ such that $L \sim kE + \Gamma$, $(E^2)=0$, $(\Gamma^2)=-2$, $E.\Gamma=1$. In this case, every member of $|\Gamma|$ is of the form $E'+\Gamma$, where E' is a sum of k effective divisors $E_1, \cdots, E_k$ such that $E_i \sim E$ for all i; in particular, there is an elliptic fibration $\varphi: T \to \boldsymbol{P}^1$ such that E is a fiber and Γ is a cross-section.
(2) $|sL|$ is base point free for all $s \geq 2$.

We need the following lemma which is proved in [16, Theorem 0.5].

Lemma 1.3. (1) *The singular locus* $\operatorname{Sing} S$ *of a general member $S \in |-K_X|$ contains $S \cap \operatorname{Sing} X$.*
(2) *If $r(X)>1$ then $|-K_X|$ is base point free. Hence a general member $S \in |-K_X|$ is disjoint from* $\operatorname{Sing} X$.
(3) *Let* $\operatorname{Bs}|-K_X|$ *be the base locus. If* $\dim \operatorname{Bs}|-K_X|=1$ *then* $\operatorname{Bs}|-K_X|$ *is*

disjoint from Sing X. *Hence a general member* $S \in |-K_X|$ *is disjoint from* Sing X.
(4) *If* $\dim \mathrm{Bs}|-K_X|=0$ *then* $P:=\mathrm{Bs}|-K_X|$ *is a single point and* P *is a rational double point of* S *of Dynkin type* A_1. *So,* $S \cap \mathrm{Sing}\, X = \phi$ *or* $\{P\}$. (Indeed, $S \cap \mathrm{Sing}\, X = \{P\}$ (cf. [16])).

Proof. (1) follows from the condition that the divisor S is a Cartier divisor on X.
(2) and (3) are proved in [16]. Moreover, we have $\mathrm{Bs}|-K_X|=\mathrm{Bs}|(-K_X)_{|S}|$.
(4) By Lemma 1.2 and by $\dim \mathrm{Bs}|(-K_X)_{|S}|$ $(=\dim \mathrm{Bs}|-K_X|)=0$, we see that $|\sigma^*((-K_X)_{|S})|=|kE|+\Gamma$. Here E is a fiber of an elliptic fibration $\varphi: T \to \boldsymbol{P}^1$, Γ is a cross-section of φ and $P:=\sigma(\Gamma)$ is a singularity on S with $\sigma^{-1}(P)=\Gamma$. So, P is a rational double point of S of Dynkin type A_1. We have also $\mathrm{Bs}|-K_X|=\{P\}$. (4) is proved.

Since the Kodaira D-dimension $\kappa(X, S) \geq 2$, the following natural homomorphism is surjective (see Theorem 2.4 below and the arguments after Theorem 2.4)

$$\pi_1(S - S \cap \mathrm{Sing}\, X) \to \pi_1(X^o).$$

To finish the proof of Theorem 1(2), we need to prove that $\pi_1(S - S \cap \mathrm{Sing}\, X)=(1)$. If $S \cap \mathrm{Sing}\, X=\phi$, then $\pi_1(S)=(1)$ and we are done because S is a K3-surface possibly with rational double singularities. Suppose $P:=S \cap \mathrm{Sing}\, X \neq \phi$, then by Lemma 1.3, P is a single point and a rational double point of S of Dynkin type A_1. Using the notations in Lemmas 1.2 and 1.3, we have $S - S \cap \mathrm{Sing}\, X = S - \{P\} = T' - \Gamma$. Here T' is the resolution of the singularity P on S. Without loss of generality, we may assume that $T'=T$ and it suffices to prove the following lemma in order to finish the proof of Theorem 1(2).

Lemma 1.4. *Let* T *be a K3-surface. Let* $\varphi: T \to \boldsymbol{P}^1$ *be an elliptic fibration. Let* Γ *be a* (-2)*-curve which is a cross-section of* φ. *Then* $T - \Gamma$ *is simply connected.*

Proof. Consider the long cohomology exact sequence:

$$H^2(T, \boldsymbol{Z}) \to H^2(\Gamma, \boldsymbol{Z}) \to H^3(T, \Gamma; \boldsymbol{Z}) \to H^3(T, \boldsymbol{Z}).$$

Since T is simply connected we have $H^3(T, \boldsymbol{Z})=0$. Let E be a fiber of φ. Then $E.\Gamma=1$. So the map $H^2(T, \boldsymbol{Z}) \to H^2(\Gamma, \boldsymbol{Z})$ takes E to the generator. Thus, by the duality we have $H_1(T-\Gamma, \boldsymbol{Z})=(0)$.

To prove Lemma 1.4, we have only to show that $\pi_1(T-\Gamma)$ is abelian. Let E_1 be a singular fiber of φ. Since the blowing-up of smooth points in the open surface $T-\Gamma$ does not affect the fundamental group of this open surface, one may assume that E_1 is simple normal crossing and hence E_1 consists of $\mathbf{P}^1$'s. Clearly, $\kappa(T,$

$E_1+\Gamma)=2$. So, we can apply [11, Cor. 2.3, see Theorem 2.4 below]. Let U_1, U_2 be (open) tubular neighborhoods of sufficiently small radii in T of the curves E_1, Γ, respectively. Then $U_1\cup U_2$ is an open tubular neighborhood of $E_1+\Gamma$. Hence the following natural homomorphism is surjective:

$$\pi_1((U_1-\Gamma\cap U_1)\cup(U_2-\Gamma))\rightarrow\pi_1(T-\Gamma).$$

Thus, to finish the proof of Lemma 1.4, we have only to prove that $\pi_1((U_1-\Gamma\cap U_1)\cup(U_2-\Gamma))$ is abelian. Applying Van-Kampen Theorem, we see easily that $\pi_1((U_1-\Gamma\cap U_1)\cup(U_2-\Gamma))=\pi_1(U_1-\Gamma\cap U_1)$. Indeed, $\pi_1(U_2-\Gamma)$ is generated by a loop σ around Γ, and we may assume that σ is taken from $(U_1-\Gamma\cap U_1)\cap(U_2-\Gamma)$. We may even assume that σ is a loop in $E_1-\Gamma\cap E_1$ around the point $\Gamma\cap E_1$. So, σ is contractible in $U_1-\Gamma\cap U_1$ because $E_1-\Gamma\cap E_1$ is a union of one $\boldsymbol{A}^1$ and several $\boldsymbol{P}^1$'s.

Now the proof of Lemma 1.4 is reduced to the proof that $\pi_1(U_1-\Gamma\cap U_1)$ is abelian. Note that $\pi_1(U_1-\Gamma\cap U_1)=\pi_1(E_1-$the smooth point $\Gamma\cap E_1$ of $E_1)$ because E_1 is a strong deformation retract of U_1, and $\pi_1(E_1-$the smooth point $\Gamma\cap E_1$ of $E_1)=\pi_1(E_1)$ because E_1 is a divisor with simple normal crossing whose irreducible components are all isomorphic to $\boldsymbol{P}^1$ and E_1 meets Γ transversally in a single point. Note that $\pi_1(E_1)$ is equal to (1) when E_1 is a tree of $\boldsymbol{P}^1$'s, and equal to $\boldsymbol{Z}$ when E_1 is a simple loop of $\boldsymbol{P}^1$'s. So $\pi_1(U_1-\Gamma\cap U_1)$ is abelian.

The proof of Lemma 1.4 is completed. This proof does not work when Γ is a multiple section.

2. Proof of Theorem 2

Let X be a log Fano variety of dimension $d (d\geq 2)$ satisfying the hypothesis in Theorem 2. Write $-K_X\sim_{\boldsymbol{Q}} rH$ where r is a positive rational number such that $r>d-2$ and H is an ample Cartier divisor. This is possible by the hypothesis in Theorem 2. We need the following:

Theorem 2.1 (cf. Theorem 0.5 in [1]). *Let X be a log Fano variety. With the above notations and assumptions, then $|H|$ is non-empty and base component free, and a general member X_{d-1} in $|H|$ is a normal projective variety with only log terminal singularities.*

By the adjunction formula, we have $-K_{X_{d-1}}\sim_{\boldsymbol{Q}}(r-1)H_{|H}$. Hence when $d\geq 3$, X_{d-1} is a log Fano variety by the definition and its Fano index $r(X_{d-1})\geq r-1>\dim X_{d-1}-2$.

Applying Alexeev's Theorem $(d-1)$-times, we get a *ladder* below such that the assertions in the following lemma hold true:

$$(X_d=X,\ H_d=H),\ (X_{d-1},\ H_{d-1}),\ \cdots,\ (X_1,\ H_1).$$

Lemma 2.2. (1) *We have* $X_i \in |H_{i+1}|$, $H_i = H_{i+1|X_i}$ *and* $-K_{X_i} \sim_{\mathbf{Q}} (r+i-d)H_i$ *for all* $i \geq 1$. *Moreover,* X_i*'s* $(i \geq 2)$ *are log Fano varieties of Fano index* $r(X_i) > \dim X_i - 2$.
(2) *If* $r = d-1$, *then* X_1 *is a nonsingular elliptic curve.*
(3) *If* $r > d-1$, *then* X_1 *is a nonsingular rational curve.*

The following lemma is, though easy, crucial in order to reduce to the dimension two case.

Lemma 2.3. *Let* X *be a log Fano variety of dimension* d $(d \geq 2)$. *With the notations and assumptions at the beginning of the section, we have*:
(1) $|H|$ *has at most isolated base points and the base locus* $\mathrm{Bs}|H|$ *is contained in the smooth part* X^o *of* X.
(2) *The singular locus* $\mathrm{Sing}\, X_{d-1}$ *of* X_{d-1} *contains* $X_{d-1} \cap \mathrm{Sing}\, X$.
(3) *If* X *has only canonical singularity then so does* X_{d-1}.

Proof. Note that (2) follows from the condition that X_{d-1} is a Cartier divisor on X. The assertion (3) follows from (1) and the definiton of canonical singularity.

Now we shall prove (1). Consider the following exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_{X_{d-1}}(H) \to 0.$$

By Kawamata's vanishing theorem (cf. [6, Theorem 1-2-5]), we have $H^1(X, \mathcal{O}_X) = 0$ and hence we have a surjection:

$$(*) \qquad H^0(X, \mathcal{O}_X(H)) \to H^0(X_{d-1}, \mathcal{O}_{X_{d-1}}(H)).$$

By the result (*), we have $\mathrm{Bs}|H| = \mathrm{Bs}|H_{d-1}|$ where $H_{d-1} := H_{|X_{d-1}}$. So if $|H_{d-1}|$ has at most isolated base points then the same is true for $|H|$. If a point P in $\mathrm{Bs}|H_{d-1}|$ is a smooth point on X_{d-1} then P is also a smooth point on X because X_{d-1} is a Cartier divisor on X. Thus, we are reduced to prove a statement similar to (1) for X_{d-1} (cf. Lemma 2.2). By the same argument, we can reduce to prove (1) for X_2. So to prove (1), we may assume that $\dim X = 2$.

By Alexeev's Theorem, we may assume that H is normal. Hence H is nonsingular because $\dim H = 1$. This, together with the condition that H is a Cartier divisor on X, implies that H is contained in the smooth part of X. Hence follows the second part of (1). Thus (1) follows because $|H|$ is base component free by Alexeev's Theorem.

Now we shall apply the following:

Theorem 2.4 (cf. [11, Cor. 2.3]). *Let* $\tilde{X}$ *be a nonsingular projective variety. Let* $\tilde{H}$ *be a divisor on* $\tilde{X}$ *such that the Kodaira D-diminsion* $\kappa(\tilde{X}, \tilde{H}) \geq 2$. *Let* Δ *be a Zariski-closed proper subset. Let* U *be any open tubular neighborhood of*

$\tilde{H}$ in $\tilde{X}$. Then the following natural homomorphism is a surjection:

$$\pi_1(U-\Delta\cap U)\to\pi_1(\tilde{X}-\Delta).$$

Let $f:\ \tilde{X}\to X$ be a resolution of singularities such that $f^*X_{d-1}+\Delta$ is a normal crossing divisor. Here Δ is the exceptional divisor of f. By Lemma 2.3(2), X_{d-1}^o $:=X_{d-1}-\mathrm{Sing}\,X_{d-1}$ is a Zariski-open subset of $X_{d-1}-\mathrm{Sing}\,X=f^*X_{d-1}-\Delta$. So one has a surjective homomorphism

$$\pi_1(X_{d-1}^o)\to\pi_1(f^*X_{d-1}-\Delta).$$

Applying Nori's Theorem to $\tilde{H}:\ =f^*X_{d-1}$, one obtains a surjective homomorphism:

$$\pi_1(f^*X_{d-1}-\Delta)\to\pi_1(X^o).$$

Combining these two surjections, we have proved the following Lemma 2.5 when $i=d-1$. Applying the same arguments several times, one can prove Lemma 2.5 for all $i\geq 1$.

Lemma 2.5. *The natural homomorphism $\pi_1(X_i^o)\to\pi_1(X_{i+1}^o)$ is surjective for all $i\geq 1$.*

Now Theorem 2(1) follows from Lemma 2.5 because it is true in the dimension two case by [3, 4]. Theorem 2(2) follows from Lemma 2.5 and Lemma 2.3(3) since it is true in the dimesion two case by [8, 9].

Theorem 2(4) follows from Lemma 2.5 and Lemma 2.2(3) because now $X_1^o=X_1\cong\boldsymbol{P}^1$ and $\pi_1(X_1^o)=(1)$.

In view of Lemma 2.5 and Theorem 2(4), it suffices to prove Theorem 2(3) in the case where $d=\dim X=2$ and $r(X)=d-1=1$. It is easy to see that $H_1(X^o, \boldsymbol{Z})$ is finite (cf. e.g. [18, Lemma 1.3]). By Lemma 2.5, $\pi_1(X^o)$ is a surjective image of $\pi_1(X_1^o)=\boldsymbol{Z}\times\boldsymbol{Z}$ because $X_1^o=X_1$ is a nonsingular elliptic curve by Lemma 2.2(2). Theorem 2(3) is proved via [9] for X_2 is Gorenstein now by Th. 2.1.

We have proved Theorem 2 stated in the Introduction. Actually, the proofs for (3) and (4) of Theorem 2 are self-contained. In other words, we obtained a simpler proof for the result in [3, 4] when $r(X)\geq\dim X-1=1$.

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