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ON RINGS WHOSE CYCLIC FAITHFUL MODULES ARE GENERATORS

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0. Introduction

For each positive integer n, we temporarily say that a ring R is n-PF (n-pseudo-Frobenius) if every faithful right R-module generated by at most n elements is a generator for the category of all right R-modules. As well known, the ring which is n-PF for all positive integers n is called a right FPF (finitely pseudo-Frobenius) ring, and every FPF ring splits in a ring with essential singular ideal and a nonsingular ring. Nonsingular FPF rings were investigated in S. Kobayashi [9] and S. Page [11], [12], [13], etc.; in particular, S. Page [11] characterized (von Neumann) regular right FPF rings as self-injective regular rings having bounded index, and S. Kobayashi [9] gave a characterization of nonsingular right FPF rings. The aim of this paper is to study nonsingular 1-PF rings, which were to some extent investigated in G.F. Birkenmeier [2], [3] and S. Kobayashi [10].

Modifying the proof of [10, Proposition 1] and observing that the converse of the proposition is also true, we see, as will be noted in §3, that for a fixed integer $n \geq 2$, a ring R is right nonsingular and n-PF if and only if R satisfies the condition (C_n) that:

(i) R is right bounded, i.e., every essential right ideal of R contains a two-sided ideal which is essential in R as a right ideal,

(ii) For every right ideal A generated by at most n elements, $R = Tr_R(A) \oplus r_R(A)$, where $Tr_R(A)$ (respectively, $r_R(A)$) is the trace (resp. the right annihilator) ideal of A, and

(iii) Every nonsingular right R-module generated by at most n elements can be embedded in a free right R-module.

However, such the result as above is, in general, false in the case $n=1$. Moreover, for regular or commutative semiprime rings, the FPF condition is, as noted in [10], equivalent to the n-PF condition for each $n \geq 2$, although it is not to the 1-PF condition. Thus it seems that the 1-PF condition is not so much related to the (C_n) and to the FPF as the n-PF $(n\geq 2)$; actually, the class of 1-PF rings is much larger than that of n-PF $(n\geq 2)$ rings.

In this paper we shall refer 1-PF rings to as right GFC rings as in [2], [3], and be concerned with nonsingular right GFC rings. We shall show that the quasi-Baer right GFC rings are precisely the rings satisfying the condition (C_1), so that they may be regarded as a natural generalization of nonsingular n-PF $(n\geq 2)$ rings, and hence, of nonsingular right FPF rings. On the other hand, the structure of right GFC regular rings R, under the assumption that every nonzero two-sided ideal of R contains a nonzero central idempotent, was determined, in [10, Theorem 1], as finite direct products of abelian regular rings and full matrix rings over self-injective abelian regular rings. We shall generally show that even without the assumption, any right GFC regular ring has the same structure and is characterized as a regular ring having bounded index such that every cyclic faithful nonsingular right module is projective.

In Section 1 of this paper is assembled a summary of notation and terminology. Section 2 contains preliminary results on right GFC rings, which will be used afterward. There we shall show that if R is a right nonsingular right GFC ring, then the maximal right quotient ring of R is FPF (Theorem 2.8), which is the key to our study in the following sections. Section 3 is concerned with quasi-Baer or right p.p. right GFC rings. We shall characterize quasi-Baer right GFC rings as rings satisfying the condition (C_1) (Theorem 3.3), and in addition give characterizations of right p.p. (and quasi-Baer) right GFC rings (Theorem 3.7, Theorem 3.8). The last section is devoted to (von Neumann) regular rings. We shall determine the structure of regular right GFC rings, and present other characterizations of those rings (Theorem 4.3). As a consequence, we see that the GFC condition is left-right symmetric for regular rings (Corollary 4.4).

1. Notation and terminology

Throughout this paper all rings are associative with identity, and all modules are unitary.

Let R be a ring, M an R-module, and X a subset of M. We denote by $r_R(X)$ (respectively, $l_R(X)$) the right (resp. left) annihilator of X in R, by $Tr_R(M)$ the trace ideal of M, i.e., $Tr_R(M)=\sum\{\mathrm{Im}\ \varphi|\varphi\in \mathrm{Hom}_R(M, R)\}$, and by $Z(M)$ the (right) singular submodule of M, i.e., $Z(M)=\{x\in M|r_R(x)$ is essential in $R\}$. Given a positive integer n, we denote by $M^{(n)}$ the direct sum of n copies of M. By ideals we usually mean two-sided ideals of R. The notation $N\leq M$ (resp. $N\leq_e M$) means that N is a submodule (resp. an essential submodule) of M. In particular, the notation $A\leq R_R$ signifies that A is a right ideal of R. We use $B(R)$ to denote the set of all central idempotents in R. A *complement* for N in M is any submodule L of M which is maximal with respect to the property $N\cap L=0$.

We call a ring R a *right GFC* (resp. *right FPF*) ring if every cyclic (resp.

finitely generated) faithful right R-module is a generator for Mod-R, the category of all right R-modules. A *left GFC*, or *left FPF* ring is defined similarly.

2. Properties of GFC rings

In this section, we shall provide preliminary results on right GFC rings, which will be used repeatedly throughout the sequel.

Lemma 2.1. (1) *Let I be an ideal of a ring R, and let A be a right ideal of R such that $I+A$ is essential in R. If R/A is a generator for* Mod-R, *then I is essential in R as a right ideal.*

(2) *Let R be a right GFC ring, and let I be an ideal of R. Then R/I is nonsingular as a right R-module if and only if I is a semiprime ideal which has no proper essential extensions in R as a right ideal.*

In particular, R is right nonsingular if and only if it is a semiprime ring.

Proof. See [15, Lemma 2 and Corollaries 5 and 6].

Lemma 2.2. *Let R be a right GFC ring. Let I be an ideal of R such that R/I is nonsingular as a right R-module, let A be a complement for I in R_R, and set $J=r_R(R/A)$. Then*

(1) $I=r_R(J)=l_R(J)$.

(2) *If R is right nonsingular, then $A=J=r_R(I)=l_R(I)$.*

Proof. (1). If B is a complement for J in A_R, then R/B is faithful, whence Lemma 2.1(1) implies that $I\oplus J\leq_e R_R$. Thus $(J\oplus I)/I$ becomes an essential right ideal of the right nonsingular ring R/I, because $(R/I)_R$ is nonsingular. Consequently, we have $I=l_R(J)$. Moreover, noting by Lemma 2.1(2) that I is a semiprime ideal, we obtain $l_R(J)=r_R(J)$.

(2). Since R/A is an essential extension of I_R and since R is right nonsingular, the R-module R/A, and hence $(R/J)_R$, is nonsingular. Thus, it follows from the same argument as in (1) that $J=r_R(I)=l_R(I)=A$ as well.

Lemma 2.3. *Let R be a right GFC ring, and let I be an ideal of R such that R/I is nonsingular as a right R-module. Then R/I is a right nonsingular right GFC ring.*

Proof. Since R/I is a nonsingular right R-module, it is a right nonsingular ring. Let A be a complement for I in R_R, and set $J=r_R(R/A)$. If B is a right ideal of R such that $r_R(R/B)=I$, then $r_R(R/BJ)\leq I\cap J=0$, so that $Tr_R(R/BJ)=l_R(BJ)R=R$. Setting $X=\{r\in R \mid rB\leq I\}$, by Lemma 2.2 we have $l_R(BJ)\leq X$, and hence $XR=R$. This means that R/B generates R/I. Thus R/I is a right GFC ring.

Lemma 2.4. *Let R be a right GFC ring, and let I be an ideal of R. If R/I is nonsingular as a right R-module, then it is nonsingular also as a left R-module.*

Proof. Set $\bar{R}=R/I$. If $\bar{R}_R$ is nonsingular, then by Lemma 2.2 there exists an ideal J of R such that $I=r_R(J)$ and $I\cap J=0$. Given any essential left ideal L of R, we see that $(L+I)/I\leq_e {}_R\bar{R}$. Set $A/I=r_{\bar{R}}((L+I)/I)$ where $A\leq R_R$, and let B/I be a complement for A/I in $\bar{R}_{\bar{R}}$ where $B\leq R_R$. Then the nonsingularity of $\bar{R}_R$ implies that $(R/B)_R$ is nonsingular and that $l_{\bar{R}}(A/I)\cap l_{\bar{R}}(B/I)=0$; hence $l_{\bar{R}}(B/I)=0$, because $(L+I)/I\leq_e l_{\bar{R}}(A/I)\leq_e {}_{\bar{R}}\bar{R}$. On the other hand, by Lemma 2.3 the R-module R/B generates $R/r_R(R/B)$; hence $XR=R$, where $X=\{r\in R|rB\leq r_R(R/B)\}$. Since $AXB\leq A\cap r_R(R/B)=I$ and $l_{\bar{R}}(B/I)=0$, it follows that $A=AXR=I$. Thus we obtain $r_{\bar{R}}((L+I)/I)=0$, which shows that ${}_R\bar{R}$ is nonsingular.

As an easy consequence of the lemmas above, we obtain the following results on left and right GFC rings.

Corollary (c.f. [12, Proposition 4]). *If R is a right nonsingular ring which is right and left GFC, then the maximal right quotient ring of R is also the maximal left quotient ring of R.*

Proof. Note by Lemma 2.4 that R is also left nonsingular. Then, by virtue of Utumi's Theorem (c.f. [5, Theorem 2.38]), it suffices to show that if A is a right ideal of R such that R/A is nonsingular, then A is a right annihilator ideal.

We shall show that A is essential in $r_Rl_R(A)_R$, which will obviously imply that $A=r_Rl_R(A)$, as desired. So, let B be a right ideal of R such that $B\leq r_Rl_R(A)$ and $A\cap B=0$, and let C be a complement for B in R_R such that $A\leq C$. It then follows from Lemma 2.3 that $XR=R$, where $X=\{r\in R|rC\leq r_R(R/C)\}$. Since $BXC=0$ and $l_R(C)\leq l_R(C)\cap l_R(B)=0$, we have $BX=0$; hence $B=0$. Thus A is essential in $r_Rl_R(A)_R$.

We give a proof for the following easy fact, which will be well known.

Lemma 2.5. *Let R be a right nonsingular and semiprime ring, and let Q be the maximal right quotient ring of R. For every ideal I of R, there exists $e\in B(Q)$ such that $I\leq_e eQ_R$.*

Proof. An idempotent e in Q can be taken to satisfy $I\leq_e eQ_R$. We claim that e is central in Q. To see this, let A be an essential right ideal of R such that $eA\leq I$. Then $(1-e)ReA\leq(1-e)I=0$. Since Q_R is nonsingular, it follows that $(1-e)Re=0$. Also, setting $B=eR(1-e)R\cap R$, we see by the semiprimeness of R

that $B=0$, and hence $eR(1-e)=0$. Thus e commutes with all elements of R. Now, given any $x\in Q$, we take $C\leq_e R_R$ such that $xC\leq R$. It follows that $(ex-xe)C=0$, so that $ex=xe$. Therefore e is indeed central in Q.

If every essential right ideal of R contains an ideal which is essential in R as a right ideal, then R is said to be *right essentially bounded*. Note that in [9] such rings are referred to as *right bounded* rings.

Lemma 2.6 ([9, Lemma 2]). *Let R be a right nonsingular ring which is right essentially bounded. If M is a finitely generated faithful right R-module, then $M/Z(M)$ is also faithful.*

Lemma 2.7. *Let R be a right GFC ring. Then*

(1) *R is right essentially bounded.*

(2) *If N is an essential submodule of a finitely generated nonsingular right R-module M, then $r_R(N)=r_R(M)$.*

Proof. (1). See [15, Proposition 4].

(2). Let $x_1, \cdots, x_n$ generate M_R, and for each $i=1, \cdots, n$, set $A_i=\{a\in R|x_ia\in N\}$. Then, by the essentiality of N, each A_i is essential in R ; hence, according to (1) we see, by noting $\bigcap_{i=1}^{n} r_R(R/A_i)\leq r_R(M/N)$, that $r_R(M/N)\leq_e R_R$. As a result, $r_R(M/N)/r_R(M)$ is an essential right ideal of the ring $R/r_R(M)$, because $(R/r_R(M))_R$ is nonsingular. Moreover, since $r_R(M)$ is a semiprime ideal by Lemma 2.1(2) and since $r_R(M/N)r_R(N)\leq r_R(M)$, it follows that $r_R(N)r_R(M/N)\leq r_R(M)$. Thus the nonsingularity of $R/r_R(M)$ implies that $r_R(N)=r_R(M)$.

S. Page proved in [12, Theorem 2] that if R is a right nonsingular right FPF ring, then the maximal right quotient ring of R is also FPF, while G.F. Birkenmeier [3, Corollary 3.6] obtained the same result for right nonsingular quasi-Baer right GFC rings. The next theorem more generally shows that for any right nonsingular ring, the right GFC condition has the same effect on the maximal right quotient ring. This is useful to our study in the following sections.

Theorem 2.8. *Let R be a right nonsingular right GFC ring. Then the maximal right quotient ring of R is a left and right FPF ring.*

Proof. Let Q denote the maximal right quotient ring of R. By [2, Theorem 2] and [11, Corollary 9.2] it suffices to prove that Q is right GFC. Thus, given any right ideal X of Q such that Q/X is faithful, we must show that Q/X is a generator for Mod-Q. To this end, set $Y/X=Z((Q/X)_Q)$, the singular submodule of Q/X as a Q-module, and note that Y/X is the singular submodule of Q/X

also as an R-module; hence Q/Y is nonsingular both as a Q-module and as an R-module.

First we claim that Q/Y is faithful. Since $(Q/r_Q(Q/Y))_Q$ is a nonsingular Q-module, there exists $e\in B(Q)$ such that $r_Q(Q/Y)=eQ$. Observing that the cyclic R-module $(eR+X)/X$ is singular because it is contained in the singular module Y/X, we see by Lemma 2.7(1) that $r_R((eR+X)/X)$ is essential in R_R. Now, let a be an arbitrary element of $eR\cap r_R((eR+X)/X)$. Set $I=l_Rr_R(aR)$, and let x be an idempotent in Q such that $aQ=xQ$, and so $x=ay$ for some $y\in Q$. Also, by Lemma 2.1(2) and Lemma 2.5, let $f\in B(Q)$ such that $I\leq_e fR_R$. Then, $ef=0$. To see this, note that aR is essential in xR_R, and then by Lemma 2.7(2) that $r_R(xR)=r_R(aR)=r_R(I)=r_R(fR)$. Thus, according to Lemma 2.3, the R-module xR generates fR, whence there exist $\varphi_1, \cdots, \varphi_n\in \mathrm{Hom}_R(xR, fR)$ such that $f\in \sum_{i=1}^{n}\varphi_i(xR)$. Each R-homomorphism φ_i may be extended to a Q-homomorphism from xQ to fQ, so that $ef\in\sum_{i=1}^{n}e\varphi_i(xR)=\sum_{i=1}^{n}e\varphi_i(x)xR\leq efRayR\leq eRafyR\leq X$ (the last inclusion of which is obtained from $a\in r_R((eR+X)/X)$, i.e., $ef\in X$. Since ef is central and Q/X is faithful, we obtain $ef=0$, as desired. This implies that $a=0$, because $a\in eR\cap fR$. Thus $eR\cap r_R((eR+X)/X)=0$, whence the essentiality of $r_R((eR+X)/X)$ shows that $e=0$. Therefore, $r_Q(Q/Y)$ must be zero, as claimed.

Now, set $N=(R+Y)/Y$. According to Lemma 2.5, there exists $g\in B(Q)$ such that $r_R(N)\leq_e gR$, and then $NgB=0$ for some $B\leq_e R_R$. Noting that Q/Y is nonsingular, we have $Ng=0$, and hence $g\in Y$. Since the Q-module Q/Y, as seen above, is faithful, the central idempotent g is zero, that is, N_R is faithful. It then follows from the hypothesis of R that the cyclic faithful R-module N is a generator for Mod-R; hence $l_R(R\cap Y)R=R$. On the other hand, the essentiality of $R\cap Y$ in Y_R implies that $l_R(R\cap Y)\leq l_Q(Y)$. Therefore we obtain $l_Q(Y)Q=Q$, which means that Q/Y is a generator for Mod-Q. Obviously, Q/X generates Q/Y, so that Q/X is indeed a generator for Mod-Q, which completes the proof of the theorem.

A ring R has *bounded index* if there exists a positive integer n such that $x^n=0$ for all nilpotent elements x of R. The least such positive integer is called the (*bounded*) *index* of R.

The theorem above implies the following.

Corollary 2.9. *Let R be a right nonsingular right GFC ring. Then there exists a positive integer k, and R is a subdirect product of prime rings, each of which is contained in a simple artinian ring of length at most k.*

Proof. Let Q denote the maximal right quotient ring of R. It then follows from [11, Theorem 9] and [6, Theorem 6.2 and Corollary 7.10] that Q has index k for some positive integer k, and that for every prime ideal P of Q, the ring Q/P is a simple artinian ring of length at most k. Thus it suffices to show that for every prime ideal P of Q, the ring $R/(R\cap P)$ is a prime ring. So, let $a\in R$, and let I be an ideal of R such that $a\notin P$ and $aI\leq R\cap P$. Also, set $J=l_R r_R(aR)$. By Lemma 2.1(2) and Lemma 2.5, there exists $e\in B(Q)$ such that $J\leq_e eR_R$ and then by Lemma 2.7(2), $r_R(aR)=r_R(J)=r_R(eR)$. It then follows from Lemma 2.3 that aR generates eR, whence there exists $\varphi_1, \cdots, \varphi_n\in \mathrm{Hom}_R(aR, eR)$ such that $e\in \sum_{i=1}^{n}\varphi_i(aR)$. Extending each φ_i to a Q-homomorphism $Q\rightarrow Q$, we have $eI\leq \sum_{i=1}^{n}\varphi_i(aR)I\leq \sum_{i=1}^{n}\varphi_i(1)aI\leq P$. On the other hand, $e\notin P$, because $a\notin P$ and $a\in eR$. Thus the primeness of P implies that $I\leq P\cap R$. Therefore, $R/(R\cap P)$ is a prime ring, as desired.

3. quasi-Baer or p.p. GFC rings

In this section we shall study quasi-Baer or right p.p. right GFC rings, and give characterizations of those rings.

Following [4], we call a ring R a *quasi-Baer* ring if the right annihilator of every ideal in R is generated by an idempotent in R. A *right p.p.* ring is one in which every principal right ideal is projective.

First we note the following on right nonsingular and semiprime rings, which will be well known.

Lemma 3.1. *Let R be a right nonsingular and semiprime ring, and let Q be the maximal right quotient ring of R. Then the following conditions are equivalent :*

(a) *R is quasi-Baer.*

(b) *For every ideal I of R such that R/I is nonsingular as a right R-module, I is generated by an idempotent in R.*

(c) $B(R)=B(Q)$.

Proof. (a)⇒(b). It is only to note that if I is an ideal of a semiprime ring R such that $(R/I)_R$ is nonsingular, then we have $I=r_R l_R(I)$.

(b)⇒(c). Since any element of Q commuting with all elements of R is central in Q, we have only to show that $B(Q)\subseteq B(R)$. Thus, given any $e\in B(Q)$, we see by (b) that there exists $f\in B(R)$ such that $eQ\cap R=fR$. It follows that $eQ=fQ$, whence $e=f\in B(R)$.

(c)⇒(a). This follows immediately from Lemma 2.5.

As a result of the lemma above, we note by [13, Proposition 1] that right nonsingular right FPF rings are quasi-Baer right GFC rings.

Recall that a regular ring R is *abelian* if all idempotents in R are central, or equivalently, R has bounded index 1. Also, an idempotent e in a regular ring R is said to be *abelian* whenever the ring eRe is abelian.

We need the following lemma as in [11, Lemma A].

Lemma 3.2. *Let R be a right nonsiugular right GFC ring, and let Q be the maximal right quotient ring of R. If M is a cyclic nonsingular right R-module, then there exist finitely many $e_1, \cdots, e_n \in B(Q)$ such that M can be embedded in $e_1R \oplus \cdots \oplus e_nR$.*

Proof. Let M be a cyclic nonsingular right R-module. Then there exists an idempotent f in Q such that $M \cong fR$. Thus we may assume that $M = fR$. Note from [6, Theorem 7.20] that any idempotent in a right self-injective regular ring S having bounded index is a finite sum of orthogonal abelian idempotents in S. Since Q has bounded index by Theorem 2.8 and [11, Theorem 9], the idempotent f can be actually expressed as $f = \sum_{j=1}^{k} f_j$, where $f_1, \cdots, f_k$ are orthogonal abelian idempotents in Q. Thus M is contained in $f_1R \oplus \cdots \oplus f_kR$, so that it suffices to embed each f_jR in $(e_jR)^{(n_j)}$ for some integer n_j and for some $e_j \in B(Q)$. Therefore we may furthermore assume that f itself is abelian in Q.

Let A be a complement for $r_R(M)$ in R_R. It then follows from Lemma 2.2 that $A = l_R r_R(M)$ and $r_R(A) = r_R(M)$, and from Lemma 2.5 that there exists $e \in B(Q)$ for which $A \leq_e eR_R$, and then by Lemma 2.7(2), $r_R(M) = r_R(eR)$. Consequently, Lemma 2.3 shows that M generates eR, whence there exist $\varphi_1, \cdots, \varphi_n \in \mathrm{Hom}_R(M, eR)$ such that $e \in \sum_{i=1}^{n} \varphi_i(M)$. Extending each φ_i, we may assume that $\varphi_i \in \mathrm{Hom}_Q(fQ, eQ)$. Now, we consider a homomorphism $\varphi : fQ \to (eQ)^{(n)}$ defined by $\varphi(x) = (\varphi_i(x))_{i=1}^{n}$ for $x \in fQ$, and claim that φ is monic. To this end, set $K = \mathrm{Ker}\ \varphi$. Then, $fQ_Q = K \oplus N$ for some $N \leq fQ_Q$. Since f is an abelian idempotent in Q, it follows from [6, Theorem 3.4] that $NK = 0$. Thus, $K \leq r_Q(fQ/K)$, that is, $\varphi_i(fQK) = 0$ for all $i = 1, \cdots, n$, from which we obtain $e(K \cap R) \leq \sum_{i=1}^{n} \varphi_i(fR)(K \cap R) \leq \sum_{i=1}^{n} \varphi_i(fR(K \cap R)) = 0$. On the other hand, $K \cap R \leq M \cap R \leq l_R r_R(M) \leq eR$, so that $K \cap R$, and hence K, must be zero. Therefore, φ is indeed monic, whence the restriction of φ to M obviously embeds M into $(eR)^{(n)}$, which completes the proof of the lemma.

REMARK. The proof of the lemma above shows that if f is an abelian idempotent in Q, then fR_R can be embedded in $(eR)^{(n)}$ for some integer n and for some $e \in B(Q)$.

As a corollary, we obtain the following.

Corollary (c.f. [15, Corollary 9]). *Let R be a right GFC ring. Let M be a cyclic nonsingular right R-module which has finite Goldie dimension. Then the endomorphism ring* $\mathrm{End}_R(M)$ *of M_R is a right order in a semisimple artinian ring.*

Furthermore, if R is also left GFC, then $\mathrm{End}_R(M)$ *is a two-sided order in a semisimple artinian ring.*

Proof. According to Lemma 2.3, we may assume, by passing from R to $R/r_R(M)$, that R is right nonsingular and M is faithful. Thus M generates R, whence R can be embedded in a finite direct sum of copies of M. As a result, R_R has finite Goldie dimension, because so does M_R. It then follows from Lemma 2.1(2) and [5, Corollary 3.32] that R is a semiprime right Goldie ring, i.e., R is a right order in a semisimple artinian ring Q. On the other hand, Lemma 3.2 shows that M can be embedded in $e_1R\oplus\cdots\oplus e_nR$ for some central idempotents $e_1, \cdots, e_n$ in Q. Since each e_iR ($\cong R/r_R(e_iQ\cap R)$) can be embedded in a direct product of copies of R_R, so is M. Therefore, [8, Theorem 2.2.14 and Theorem 2.2.17] implies that $\mathrm{End}_R(M)$ is a right order in a semisimple artinian ring.

If R is also left GFC, then by Lemma 2.4 we may also assume, as in the proof above, that R is right and left nonsingular and M is faithful. Thus it follows from Corollary following Lemma 2.4 and [5, Theorems 2.38 and 3.14] that the semiprime ring R is right and left Goldie, so that [8, Theorem 2.2.17] again implies that $\mathrm{End}_R(M)$ is a two-sided order in a semisimple artinian ring.

Recall that a ring R is n-PF for some positive integer n if every faithful right R-module generated by at most n elements is a generator for Mod-R, and also that a ring R satisfies the condition (C_n) for some positive integer n if R satisfies the following three conditions:

(i) R is right essentially bounded,

(ii) For every right ideal A generated by at most n elements, $R=Tr_R(A)\oplus r_R(A)$,

(iii) Every nonsingular right R-module generated by at most n elements can be embedded in a free right R-module.

It is obvious that for each positive integer n, the following implications hold:

$$\text{FPF}\Rightarrow(n+1)\text{-PF}\Rightarrow n\text{-PF}\Rightarrow 1\text{-PF}\Leftrightarrow\text{GFC};$$

$$(C_{n+1})\Rightarrow(C_n).$$

Here we note, as mentioned in Section 0, that for each $n\geq 2$, the conditions n-PF and (C_n) on right nonsingular rings are equivalent.

Proposition A (c.f. [10, Proposition 1]). *Let n (≥ 2) be an integer. Then a ring R is right nonsingular and n-PF if and only if R satisfies* (C_n).

Proof. The "only if" part is obtained by modifying the proof of [10, Proposition 1] and noting [5, Theorem 5.17].

Conversely, assume that R satisfies (C_n), and then note by the conditions (i) (ii) of (C_n) that R is right nonsingular. Let M be a faithful right R-module generated by at most n elements.

To prove that M is a generator, according to Lemma 2.6 we may assume, by replacing M with $M/Z(M)$, that M is nonsingular; hence by (iii) there exists a positive integer k, and a monomorphism $\varphi: M \rightarrow R^{(k)}$. For each $i=1, \cdots, k$, letting $p_i: R^{(k)} \rightarrow R$ be the i-th projection, we see by (ii) that $R = Tr_R(p_i\varphi(M)) \oplus r_R(p_i\varphi(M))$ for all i. Since $\bigcap_{i=1}^{k} r_R(p_i\varphi(M)) = r_R(M) = 0$, it follows that $R = \sum_{i=1}^{k} Tr_R(p_i\varphi(M)) = Tr_R(M)$, as desired.

In the case $n=1$, the proposition above can not hold in general. However, in the next theorem, we shall show that the right GFC (i.e., 1-PF) rings with quasi-Baer condition are precisely the rings satisfying the condition (C_1). Thus it seems that the right GFC rings, under the quasi-Baer condition, fairly behave as well as nonsingular n-PF ($n \geq 2$) rings, and hence, as FPF rings.

Theorem 3.3. *For a ring R, the following conditions are equivalent:*

(a) *R is a quasi-Baer right GFC ring.*

(b) (i) *R is right essentially bounded,*

(ii) *For every $a \in R$,*

$$R = Tr_R(aR) \oplus r_R(aR),$$

(iii) *Every cyclic nonsingular right R-module can be embedded in a free right R-module.*

(c) (i) *R is right nonsingular and right essentially bounded,*

(ii) *For every cyclic nonsingular right R-module M,*

$$R = Tr_R(M) \oplus r_R(M).$$

Proof. (a)$\Rightarrow$(b). The condition (b)(i) follows from Lemma 2.7(1). Given any $x \in R$ such that $r_R(x) \leq_e R$, we see by the right essentially boundedness of R that $r_R(xR) \leq_e R_R$. Since R is quasi-Baer, the ideal $r_R(xR)$ is generated by an idempotent; hence $x=0$. Thus R is right nonsingular, whence (b)(iii) is obtained from Lemma 2.1(2), Lemma 3.1 and Lemma 3.2. To prove (b)(ii), let $a \in R$. Since R is quasi-Baer, there exists an ideal I of R such that $R = r_R(aR) \oplus I$. Obviously, $Tr_R(aR) = l_R r_R(a)R \leq I$, while conversely, it follows from Lemma 2.3 that $I \leq$

$Tr_R(aR)$. Therefore we obtain $R = Tr_R(aR) \oplus r_R(aR)$.

(b)$\Rightarrow$(c). The conditions (b)(i)(ii) immediately shows that R is right nonsingular. For (c)(ii), let M be a cyclic nonsingular right R-module generated by x. According to (b)(iii), there exists a monomorphism $\varphi : M \to R^{(n)}$ for some positive integer n. Setting $\varphi(x) = (a_1, \cdots, a_n) \in R^{(n)}$, we have $r_R(M) = \bigcap_{i=1}^{n} r_R(a_iR)$. Also, according to (b)(ii), for each $i = 1, \cdots, n$, there exists $e_i \in B(R)$ such that $Tr_R(a_iR) = e_iR$ and $r_R(a_iR) = (1-e_i)R$. The element $\prod_{i=1}^{n}(1-e_i)$ belongs to $r_R(M)$, so that we have $r_R(M) + \sum_{i=1}^{n} e_iR = R$. It is easy to see that $Tr_R(a_iR) \leq Tr_R(M)$ for all i; hence $R = Tr_R(M) + r_R(M)$. Moreover, since $(Tr_R(M) \cap r_R(M))^2 = 0$, the condition (b)(ii) implies that $Tr_R(M) \cap r_R(M) = 0$. Therefore we obtain $R = Tr_R(M) \oplus r_R(M)$.

(c)$\Rightarrow$(a). Let M be a cyclic faithful right R-module. It then follows from (c)(i) and Lemma 2.6 that $M/Z(M)$ is also faithful, and then from (c)(ii) that $R = Tr_R(M/Z(M))$. Thus $M/Z(M)$, and hence M, is a generator for Mod-R. Therefore R is right GFC. Next, to prove that R is quasi-Baer, let I be an ideal of R, and set $J/I = Z((R/I)_R)$, where $J \leq R_R$. Noting that J becomes an ideal such that $(R/J)_R$ is nonsingular, by (c)(ii) we have $R = J \oplus K$, where $K = Tr_R((R/J)_R)$. Since R is right nonsingular, I is essential in J_R, whence Lemma 2.7(2) shows that $r_R(I) = r_R(J) = K$. Thus R is quasi-Baer, which completes the proof of the theorem.

REMARK. As can be seen above, the following implications on rings hold:
$(C_1)\Leftrightarrow$quasi-Baer and right GFC (i.e., 1-PF);
$(C_n)\Leftrightarrow$right nonsingular and n-PF for each $n \geq 2$.

Concerning rings satisfying the condition (C_n) for $n \geq 1$, we may improve Corollary 2.9 on nonsingular GFC rings as follows.

Proposition B. *Let R be a ring satisfying the condition (C_n) for some positive integer n. Then there exists a positive integer k, and R is a subdirect product of prime ring R_i's, where each R_i is contained in a simple artinian ring of length at most k such that every nonzero right ideal of R_i generated by at most n elements is a generator for* Mod-R_i.

In particular, if R is a right nonsingular right FPF ring, then each the ring R_i above may be taken to satisfy the condition that every nonzero finitely generated right ideal of R_i is a generator for Mod-R_i.

Proof. Let Q denote the maximal right quotient ring of R. Since the intersection of all minimal prime ideals of Q is zero, it suffices, as in the proof of

Corollary 2.9, to show that for every minimal prime ideal P of Q, every nonzero right ideal of the ring $R/(R\cap P)$ generated by at most n elements is a generator for Mod-$(R/(R\cap P))$. So, let $a_1, \cdots, a_n \in R$ such that $a_1 \notin P$, and set $A=\sum_{i=1}^{n} a_i R$. Then we must show that $(A+(R\cap P))/(R\cap P)\cong A/(A\cap P)$ generates $R/(R\cap P)$. By hypothesis, there exists $e\in B(R)$ $(=B(Q)$ by Lemma 3.1) such that $r_R(A)=eR$ and $Tr_R(A)=(1-e)R$. Since $(1-e)a_1=a_1\notin P$, it follows that $1-e\notin P$; hence $e\in P$. Thus we have $r_R(A)\leq R\cap P$, whence A generates $R/(R\cap P)$, that is, there exists an epimorphism $\varphi: A^{(m)}\rightarrow R/(R\cap P)$ for some positive integer m. Now, observing by [6, Theorem 8.26 and Corollary 9.15] that $P=\{ex \mid e\in P\cap B(R);\ x\in Q\}$ and hence $R\cap P=\{er \mid e\in P\cap B(R);\ r\in R\}$, we see that φ induces an epimorphism from $(A/(A\cap P))^{(m)}$ onto $R/(R\cap P)$. Therefore, $A/(A\cap P)$ generates $R/(R\cap P)$, as desired.

The second assertion is now obvious.

Here we shall present the following examples to illustrate the conditions of Theorem 3.3.

EXAMPLE 1. (1) There exists a ring R which satisfies the conditions (b)(i) (ii) and (c)(i) of Theorem 3.3, but R is not quasi-Baer right GFC.

Choose a commutative domain D, set $D_n=D$ for all $n=1, 2, \cdots$, and set $T=\prod_{n=1}^{\infty} D_n$ and $R=D\cdot 1_T+\bigoplus_{n=1}^{\infty} D_n\subset T$. Then, we see that R satisfies the conditions (b) (i) (ii) and (c)(i) of Theorem 3.3.

Set $x=(x_n)\in T$ such that $x_n=0$ if n is odd; $x_n=1$ if n is even. Then, it is easy to see that xR can not be embedded in a free R-module. Therefore R is not a quasi-Baer right GFC ring.

(2) There exists a ring R which satisfies the conditions (b)(ii) (iii) and (c)(ii) of Theorem 3.3, but R is not quasi-Baer right GFC.

Let R be a simple noetherian ring which is not artinian (e.g. the Weyl algebra over a field of characteristic 0). Obviously, R satisfies the condition (b)(ii) of Theorem 3.3, while by [8, Theorem 2.2.15] it does also the condition (b)(iii) and (c)(ii) of Theorem 3.3.

But, R is not right essentially bounded, because R is a simple non-artinian ring. Therefore R is not a quasi-Baer right GFC ring.

(3) There exists a ring R which satisfies the conditions (b)(i) (iii) of Theorem 3.3, but R is not quasi-Baer right GFC.

Let R be a right artinian ring such that $Z(R_R)\leq_e R_R$ (e.g. let p be a prime number and k (≥ 2), n positive integers, and let R be the ring of all lower triangular $n\times n$ matrices over $\boldsymbol{Z}/p^k\boldsymbol{Z}$, where $\boldsymbol{Z}$ is the ring of integers). Then R

obviously satisfies the conditions (b)(i) (iii) of Theorem 3.3.

But, choosing $0 \neq a \in Z(R_R)$, we see that $r_R(aR) \cap Tr_R(aR) \neq 0$. Therefore R is not a quasi-Baer right GFC ring.

From now on, we shall be concerned with right p.p. right GFC rings.

We need the following lemma on right p.p ring.

Lemma 3.4 (c.f. [14, Proposition I, 6.9]). *Let R be a right p.p. ring. Then every cyclic submodule of a free right R-module is isomorphic to a direct sum of principal right ideals of R; in particular, it is projective.*

Proof. Let M be a cyclic submodule of a free right R-module $F = \bigoplus_{i=1}^{n} R_i$, where each $R_i = R$. The proof is by induction on n.

The case $n=1$ is clear. Now let $n>1$, and let p be the n-th projection $F \to R_n$. Since R is right p.p., the epimorphism $p: M \to p(M)$ splits, so that $M \cong p(M) \oplus (\operatorname{Ker} p \cap M)$. Noting that $\operatorname{Ker} p \cap M$ is a cyclic submodule of $\bigoplus_{i=1}^{n-1} R_i$, we see by the induction hypothesis that $\operatorname{Ker} p \cap M$, and hence M, is isomorphic to a direct sum of principal right ideals of R.

We call an idempotent e in R a *faithful* idempotent if the R-module eR is faithful.

The following is a categorical result on right p.p. right GFC rings (c.f. [13, Corollary 1B]).

Proposition 3.5. *Let R be a right p.p. right GFC ring, and let Q be the maximal right quotient ring of R. Then there exists a faithful idempotent e in R such that eQe is a self-injective abelian regular ring which is the maximal right quotient ring of eRe.*

In particular, R is Morita equivalent to a right nonsingular ring whose the maximal right quotient ring is a self-injective abelian regular ring.

Proof. By virtue of Theorem 2.8 and [11, Theorem 9], there exists an idempotent f in Q such that f is faithful and abelian, which means that the Q-module fQ is faithful and the regular ring fQf is abelian. According to Lemma 2.5, there exists $g \in B(Q)$ for which $r_R(fR) \leq_e gR$, and then $gA \leq r_R(fR)$ for some $A \leq_e R_R$. Noting that R is right nonsingular and that $fgA=0$, we have $fg=0$. Since fQ is faithful, it follows that $g=0$, that is, the R-module fR is faithful. Now, by Remark following Lemma 3.2, and Lemma 3.4, the faithful module $fR \cong R/((1-f)Q \cap R)$ is projective, whence there exists an idempotent e in R such that $(1-f)Q \cap R = (1-e)R$, and then $eR \cong fR$ is faithful. Moreover, we have $eQe \cong$

$\mathrm{End}_Q(eQ)\cong\mathrm{End}_Q(fQ)\cong fQf$, whence eQe is a self-injective abelian regular ring. Thus, to show the first assertion, it suffices to prove that eRe is essential in $(eQe)_{eRe}$. To this end, given any nonzero element exe in eQe, where $x\in Q$, we take an essential right ideal B of R such that $0\neq exeB\leq R$. Since R is semiprime by Lemma 2.1(2), it follows that $(exeB)^2$, and hence $exe(eRe)\cap eRe$, is nonzero. Consequently, eRe is essential in $(eQe)_{eRe}$ as desired.

The assumption of R also implies that eR is a generator for Mod-R. Therefore, R is Morita equivalent to the right nonsingular ring eRe whose the maximal right quotient ring is a self-injective abelian regular ring, thereby completing the proof of the proposition.

Lemma 3.6. *Let R be a right nonsingular right GFC ring, and let f be an idempotent in R. Then $R=(RfR)\oplus r_R(fR)$.*

Proof. According to Lemma 2.3, the R-module fR generates $R/r_R(fR)$, whence there exist $a_1, \cdots, a_n, b_1, \cdots, b_n\in R$ such that $\sum_{i=1}^{n} a_ib_i=1$ and $a_i(1-f)\in r_R(fR)$ for $i=1, \cdots, n$. Thus we have $1=\sum_{i=1}^{n} a_i(1-f)b_i+\sum_{i=1}^{n} a_ifb_i$, which implies that $R=r_R(fR)+RfR$. Moreover, since R is a semiprime ring by Lemma 2.1(2), it follows that $r_R(fR)\cap RfR=0$.

Concerning right p.p. (and quasi-Baer) rings, we obtain the following results.

Theorem 3.7. *For a right p.p. ring R, the following conditions are equivalent*:

(a) *R is a right GFC ring.*

(b) (i) *R is right essentially bounded,*

(ii) *For every idempotent f in R, the ideal RfR is generated by a central idempotent in R,*

(iii) *Every cyclic faithful nonsingular right R-module has a direct summand which is faithful and projective.*

Proof. (a)⟹(b). The conditions (b)(i) (ii) follow from Lemma 2.7(1) and Lemma 3.6.

For (b)(iii), let C be a cyclic faithful nonsingular right R-module. Then, $C\cong fR_R$ for some idempotent f in the maximal right quotient ring Q of R. As in the proof of Lemma 3.2, the idempotent f can be expressed as $f=\sum_{j=1}^{k} f_j$, where $f_1, \cdots, f_k$ are orthogonal abelian idempotents in Q. Since fQ_Q is faithful, i.e., f is a faithful idempotent in Q, we may take f_1 to be a faithful idempotent in Q, so that by the first half of the proof of Proposition 3.5, the R-module f_1R_R is faithful and projective. Thus we have a split epimorphism $C\cong(f_1+\cdots+f_k)R\rightarrow f_1R$, which implies (b)(iii).

(b)⇒(a). To prove that R is right GFC, it suffices by (b)(i) (iii) and Lemma 2.6 to show that every cyclic faithful projective right R-module M is a generator for Mod-R. Since M is cyclic projective, $M\cong fR$ for some idempotent f in R. By (b)(ii), there exists $e\in B(R)$ such that $RfR=eR$, and then $1-e\in r_R(fR)=r_R(M)=0$. Thus we obtain $Tr_R(M)=RfR=R$, as desired.

Theorem 3.8. *For a ring R, the following conditions are equivalent:*

(a) *R is a quasi-Baer right p.p. right GFC ring.*

(b) (i) *R is right nonsingular and right essentially bounded,*

(ii) *For every idempotent f in R, the ideal RfR is generated by a central idempotent in R,*

(iii) *Every cyclic nonsingular right R-module is projective.*

Proof. (a)⇒(b). This follows immediately from Lemma 3.1, Lemma 3.2, Lemma 3.4 and Theorem 3.7.

(b)⇒(a). It follows from (b)(i) (iii) that R is a quasi-Baer right p.p. ring, while Theorem 3.7 implies that R is right GFC.

Let S be a ring, and let n be a positive integer. We denote by $M_n(S)$ the ring of all $n\times n$ matrices over S, and by e_{ij} $(1\leq i, j\leq n)$ the matrix units in $M_n(S)$, i.e., e_{ij} has a 1_S in the (i, j) position as its only nonzero entry.

We shall illustrate the conditions of Theorem 3.7 by the following examples, in which all rings considered are regular rings.

EXAMPLE 2. (1) There exists a regular ring R which satisfies the conditions (b) (i) (ii) of Theorem 3.7, but R is not right GFC.

Choose an abelian regular ring S which is not self-injective (e.g., let $S=D\cdot 1+\bigoplus_{n=1}^{\infty}D_n$ be as in Example 1(1), where each $D_n=D$ is a division ring). Let n be an integer ≥ 2, and set $R=M_n(S)$ and $Q=M_n(Q(S))$, where $Q(S)$ is the maximal quotient ring of S. It then follows from [6, Lemma 6.20] that R satisfies the condition (b)(i) of Theorem 3.7, while it is easy to see that R satisfies the condition (b)(ii) of Theorem 3.7.

But, there exists a cyclic faithful nonsingular right R-module which has no faithful and projective direct summands. Indeed, choose $x\in Q(S)-S$ and set $e=e_{11}+xe_{1n}$, and $C=eR_R=\begin{pmatrix} S+xS & \cdots & S+xS \\ 0 & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & 0 \end{pmatrix}$. Then we see that e is a faithful abelian idempotent in Q, and C_R is a cyclic faithful nonsingular R-module. Now, suppose that $C_R=C_1\oplus C_2$, where C_1 is faithful and projective. Then, $eQ_Q=E(C_1)\oplus E(C_2)$, where $E(C_i)$ is the R-injective hull of C_i. Since $E(C_1)$ is a faithful principal right ideal of Q, and hence, a generator for Mod-Q, and since by [6, Theorem 3.4], $\mathrm{Hom}_Q(E(C_1), E(C_2))=0$, it follows that $E(C_2)=0$, so that $C_R=C_1$ is projective. Consequently, the S-module $S+xS$ is projective, whence $x\in S$, a contradiction. Thus, C_R must have no faithful and projective direct summands. Therefore R is not a right GFC ring.

(2) There exists a regular ring R which satisfies the conditions (b)(ii) (iii) of Theorem 3.7, but R is not right GFC.

Choose an infinite dimensional vector space V over a field F, and set $S=\mathrm{End}_F(V)$ and $K=\{x\in S|\dim_F(xV)<\dim_F(V)\}$. Let R be the maximal right quotient ring of S/K (See [6, Example 10.11]). Since R is a simple right self-injective regular ring, it obviously satisfies the conditions (b)(ii) (iii) of Theorem 3.7.

But, R is a simple non-artinian ring, whence it is not right essentially bounded. Therefore R is not a right GFC ring.

(3) There exists a regular ring R which satisfies the conditions (b)(i) (iii) of Theorem 3.7, but R is not right GFC.

For each $n=1, 2, \cdots$, choose a regular ring R_n having bounded index i_n such that the supremum of all i_n's is infinite (e.g. as a simple such R_n, we may take a simple artinian ring of length n), and set $R=\prod_{n=1}^{\infty}R_n$. First we shall show that R does not satisfy the condition (b)(ii) of Theorem 3.7. By [6, Theorem 7.2], for each n, there exist nonzero orthogonal idempotents $f_{n,1}, f_{n,2}, \cdots, f_{n,i_n}$ in R_n such that $f_{n,1}R_n\cong f_{n,j}R_n$ for all $j\in\{1, 2, \cdots, i_n\}$. Now, set $f=(f_{1,1}, f_{2,1}, \cdots)\in R$, and claim that the ideal RfR can not be generated by any central idempotents in R. Suppose, to the contrary, that $RfR=gR$ for some $g\in B(R)$. Obviously, $g=(g_1, g_2, \cdots)$, where $g_n\in B(R_n)$ for all n. For each n, we have $\bigoplus_{j=1}^{i_n}f_{n,j}R_n\leq Tr_{R_n}(f_{n,1}R_n)=R_nf_{n,1}R_n=g_nR_n$; hence $(\bigoplus_{j=1}^{i_n}f_{n,j}R_n)\oplus X_n=g_nR_n$ for some R_n-submodule X_n of g_nR_n. On the other hand, since $g\in RfR$, there exists a positive integer s, and for each n there exist $x_{n,1}, x_{n,2}, \cdots, x_{n,s}, y_{n,1}, y_{n,2}, \cdots, y_{n,s}\in R_n$ such that $g_n=\sum_{k=1}^{s}x_{n,k}f_{n,1}y_{n,k}$. By the assumption of R_n, an integer m can be taken to satisfy $i_m\geq s+1$. Defining a map $\varphi: (f_{m,1}R_m)^{(s)}\to R_m$ by $(f_{m,1}z_k)_{k=1}^{s}\mapsto\sum_{k=1}^{s}x_{m,k}f_{m,1}z_k$ for $z_k\in R_m$, and noting that $g_mR_m\leq\mathrm{Im}\ \varphi$, we obtain $(f_{m,1}R_m)^{(s)}\cong g_mR_m\oplus Y$ for some right

R_m-module Y. Thus, it follows that $g_mR_m=(\bigoplus_{j=1}^{im} f_{m,j}R_m)\oplus X_m\cong(f_{m,1}R_m)^{(im)}\oplus X_m\cong g_mR_m\oplus Y\oplus(f_{m,1}R_m)^{(im-s)}\oplus X_m$. But, this contradicts the fact that every finitely generated projective R_m-module is directly finite (see [6, Corollary 7.11 and Proposition 5.2]). Therefore R does not satisfy the condition (b)(ii) of Theorem 3.7.

Now, assume, in addition, that each R_n is self-injective. Since $R=\prod_{n=1}^{\infty} R_n$ is self-injective, it obviously satisfies the condition (b)(iii) of Theorem 3.7. Furthermore, it is easy to see that any direct product of right essentially bounded rings is also right essentially bounded, whence [6, Lemma 6.20 and Corollary 7.10] implies that R satisfies the condition (b)(i) of Theorem 3.7.

4. Regular GFC rings

In this section we shall characterize regular right GFC rings and determine the structure of those rings.

We use the following easy fact on matrix rings.

Lemma 4.1. *Let S be a ring, and let $n\geq 2$ be an integer, and set $T=M_n(S)$. For each $a\in S$ and for i, $j\in\{1, \cdots, n\}$ with $i\neq j$, set $a^{(i,j)}=e_{ii}+ae_{ij}\in T$. Then*

(1) *Both $a^{(i,j)}$ and $1-a^{(i,j)}$ are faithful idempotents in T.*

(2) *The ring T is generated by all the idempotents $a^{(i,j)}$ ($a\in S$; $1\leq i$, $j\leq n$ with $i\neq j$) as a ring.*

Proof. (1). Since
$$a^{(i,j)}T=\begin{pmatrix} 0 & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & 0 \\ S & \cdots & S \\ 0 & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & 0 \end{pmatrix}\leftarrow i\text{-th}$$
and
$$(1-a^{(i,j)})T\cong T/a^{(i,j)}T\cong\begin{pmatrix} S & \cdots & S \\ & \cdots & \\ S & \cdots & S \\ 0 & \cdots & 0 \\ S & \cdots & S \\ & \cdots & \\ S & \cdots & S \end{pmatrix}\leftarrow i\text{-th},$$
it follows that both the idempotents $a^{(i,j)}$ and $1-a^{(i,j)}$ are faithful.

(2). It suffices to show that for any $x\in S$ and for i, $j\in\{1, \cdots, n\}$, the element xe_{ij} is expressed as a product of such the elements $a^{(i,j)}$. If $i\neq j$, then $xe_{ij}=x^{(i,j)}\cdot 0^{(j,i)}$. On the other hand, in case $i=j$, choose $k\neq i$ so that we obtain $xe_{ii}=$

$x^{(i,k)}\cdot 0^{(k,i)}\cdot 1^{(k,i)}\cdot 0^{(i,k)}$, thereby completing the proof of the lemma.

Lemma 4.2. *Let R be a semiprime ring, and let Q be the maximal right quotient ring of R. Then the following conditions are equivalent*:

(a) (i) *R contains all the faithful idempotents in Q,*

(ii) *R has bounded index.*

(b) (i) *For right ideals A_1 and A_2 of R such that both R/A_1 and R/A_2 are faithful and nonsingular and $A_1\cap A_2=0$, the sum $A_1\oplus A_2$ is a direct summand of R_R,*

(ii) *R has bounded index.*

(c) *R is isomorphic to a finite direct product of a ring whose the maximal right quotient ring is a self-injective abelian regular ring, and full matrix rings over self-injective abelian regular rings.*

Proof. Note from [7, Proposition 4] that any semiprime ring having bounded index is right (and left) nonsingular, and hence its maximal right quotient ring is a right self-injective regular ring.

(a)⇒(b). Let A_1, A_2 be right ideals of R such that both R/A_1 and R/A_2 are faithful and nonsingular and $A_1\cap A_2=0$. Taking idempotents $e_1, e_2\in Q$ to satisfy $A_i\leq_e e_iQ$ for $i=1, 2$, we have $e_1Q\cap e_2Q=0$. Since Q is a regular ring, we may assume that e_1, e_2 are orthogonal. Also, $A_i=e_iQ\cap R$, because R/A_i is nonsingular. Thus $R/A_i\cong(1-e_i)R_R$ is faithful, whence so is $(1-e_i)Q_Q$. The condition (a)(i) now implies that each e_i belongs to R, from which we obtain $A_i=e_iR$ for $i=1, 2$. Therefore $A_1\oplus A_2=(e_1+e_2)R$ is a direct summand of R.

(b)⇒(c). By (b)(ii), R has index $n\geq 1$. First we claim that Q has index at most n. Suppose not. Then, according to [6, Theorem 7.2], Q contains a direct sum of $n+1$ nonzero pairwise isomorphic right ideals; hence there exist nonzero orthogonal idempotents $e_1, e_2, \cdots, e_{n+1}$ in Q such that $e_iQ\cong e_jQ$ for all i, j, because Q is a regular ring. Observe that for each $i=1, \cdots, n$, both $(1-(e_1+\cdots+e_i))Q_Q$ and $(1-e_{i+1})Q_Q$ are faithful. It then follows, as in the first half of the proof of Proposition 3.5, that for each i, both $(1-(e_1+\cdots+e_i))R_R\cong R/((e_1+\cdots+e_i)Q\cap R)$ and $(1-e_{i+1})R_R\cong R/(e_{i+1}Q\cap R)$ are faithful. Noting that if $(e_1Q\cap R)\oplus\cdots\oplus(e_iQ\cap R)$ is a direct summand of R_R, then $(e_1Q\cap R)\oplus\cdots\oplus(e_iQ\cap R)=(e_1+\cdots+e_i)Q\cap R$, and using (b)(i) n times in succession, we conclude that $(e_1Q\cap R)\oplus(e_2Q\cap R)\oplus\cdots\oplus(e_{n+1}Q\cap R)$ is a direct summand of R_R, so that there exist orthogonal idempotents $e_1', \cdots, e_{n+1}'$ in R such that $e_iQ\cap R=e_i'R$ for all i. Since $e_iQ=e_i'Q$, we may take each e_i to be in R. For each $i=1, \cdots, n$, let φ_i: $e_iQ\rightarrow e_{i+1}Q$ be an isomorphism. Then, as in the proof of [6, Corollary 7.4], it is easy to see by the essentiality of e_iR in e_iQ_R that there exists a nonzero submodule A_1 of e_1R_R such that $\varphi_i\varphi_{i-1}\cdots\varphi_1(A_1)$ is a nonzero submodule of $e_{i+1}R_R$ for all i. For each $i=2, 3, \cdots, n+1$, set $A_i=\varphi_{i-1}\varphi_{i-2}\cdots\varphi_1(A_1)$. Then, $A_1A_2\cdots A_{n+1}$ must be zero. Indeed, for each $i=1, \cdots, n$, let a_i be an arbitrary element of A_i, and as in the proof of (a)⟹(c) in [6, Theorem 7.2], set $a=a_1e_2+a_2e_3+\cdots+a_ne_{n+1}$. Then, each a_i belongs to e_iR, so that $a^n=a_1a_2\cdots a_ne_{n+1}$ and then $a^{n+1}=0$; hence $a^n=0$, because R has index n. This shows that $A_1A_2\cdots A_{n+1}\leq A_1A_2\cdots A_ne_{n+1}R=0$. Thus there exists $k\in\{2, 3, \cdots, n+1\}$ such that $A_{k-1}A_k\cdots A_{n+1}=0$ and $A_kA_{k+1}\cdots A_{n+1}\neq 0$. But, $A_kA_kA_{k+1}\cdots A_{n+1}=\varphi_{k-1}(A_{k-1})A_kA_{k+1}\cdots A_{n+1}=\varphi_{k-1}(A_{k-1}A_k\cdots A_{n+1})=0$, whence $(A_kA_{k+1}\cdots A_{n+1})^2=0$. Since R is a semiprime ring, it follows that $A_kA_{k+1}\cdots A_{n+1}=0$, which is a contradiction. Therefore Q has index at most n, as claimed.

According to [6, Theorem 7.20], $Q\cong\prod_{h=1}^{m}M_{n(h)}(D_h)$, where $n(1)=1$ and $n(h)\neq 1$ for $h=2, 3, \cdots, m$, and where $D_1, D_2, \cdots, D_m$ are self-injective abelian regular rings. If Q is abelian, i.e., $Q=D_1$, then (c) obviously holds. Thus assume that Q is not abelian. Let $f_1, f_2, \cdots, f_m$ denote the complete set of orthogonal central idempotents in Q such that $f_1Q=D_1$ and $f_hQ=M_{n(h)}(D_h)$ for $h=2, 3, \cdots, m$. Then, to obtain (c), it suffices to prove that $f_hQ\leq R$ for $h=2, 3, \cdots, m$. Let $h\in\{2, 3,\cdots, m\}$ be fixed. In view of Lemma 4.1(2), it furthermore suffices to prove that $a^{(i,j)}\in R$ for all $a\in D_h$ and for all $i, j\in\{1, 2, \cdots, n(h)\}$ with $i\neq j$, where $a^{(i,j)}=e_{ii}+ae_{ij}\in M_{n(h)}(D_h)$, and where e_{ij} $(1\leq i, j\leq n(h))$ are the (i, j) matrix units in $M_{n(h)}(D_h)$. To this end, set $g_1=a^{(i,j)}$ and $g_2=f_h-g_1$, and set $g_1^*=1-g_2$ and $g_2^*=1-g_1$. Since by Lemma 4.1(1) both g_1 and g_2 are faithful idempotents in the ring f_hQ, it follows that both $g_1^*Q_Q$ and $g_2^*Q_Q$ are faithful, whence $g_i^*R_R\cong R/(g_jQ\cap R)$ is faithful for $i\neq j$. Thus, the condition (b)(i) implies that $(g_1Q\cap R)\oplus(g_2Q\cap R)$ is a direct summand of R_R. Consequently, there exist orthogonal idempotents g_1', g_2' in R such that $g_iQ\cap R=g_i'R$ for $i=1, 2$, from which we have $g_1=g_1'g_1=g_1'(g_1+g_2)=g_1'f_h=g_1'$. Therefore we conclude that $a^{(i,j)}=g_1=g_1'\in R$, as desired.

(c)⟹(a). If $R\cong R_1\times R_2$, where the maximal right quotient ring of R_1 is a self-injective abelian regular ring, and where R_2 is a finite direct product of full matrix rings over self-injective abelian regular rings, then R_1 has bounded index at most 1 and R_2 also has bounded index by [6, Theorem 7.12]; hence R itself has bounded index. On the other hand, since both the rings R_1 and R_2 obviously satisfy the condition (a)(i), so does the ring R, which completes the proof of the lemma.

Let R be a regular ring having bounded index with Q the maximal right quotient ring. Then by [6, Corollary 7.4 and Theorem 7.20], $Q\cong\prod_{h=1}^{m}M_{n(h)}(D_h)=\prod_{h=1}^{m}f_hQ$, as in the proof of (b)⟹(c) in Lemma 4.2. For each h, let $e_{ij}^{(h)}$ denote the (i, j) matrix unit in $M_{n(h)}(D_h)$. Now, let g be an arbitrary faithful abelian idempotent in $M_{n(h)}(D_h)$. Then it is easy to see that $(f_1+\cdots+f_{h-1}+g+f_{h+1}+\cdots+f_m)R_R$ can be embedded in $\oplus\{(e_{i_1i_1}^{(1)}+\cdots+e_{i_{h-1}i_{h-1}}^{(h-1)}+g+e_{i_{h+1}i_{h+1}}^{(h+1)}+\cdots+e_{i_mi_m}^{(m)})R\mid 1\leq i_j\leq n(j)$ for $j\in\{1, \cdots, h-1, h+1, \cdots, m\}\}$, and each $e_{i_1i_1}^{(1)}+\cdots+e_{i_{h-1}i_{h-1}}^{(h-1)}+g+e_{i_{h+1}i_{h+1}}^{(h+1)}$

$+\cdots+e^{(m)}_{i_mi_m}$ is a faithful abelian idempotent in Q. On the other hand, we see, by observing the proof of (b)⇒(c) in Lemma 4.2, that if all such the R-modules $(e^{(1)}_{i_1i_1}+\cdots+e^{(h-1)}_{i_{h-1}i_{h-1}}+g+e^{(h+1)}_{i_{h+1}i_{h+1}}+\cdots+e^{(m)}_{i_mi_m})R_R$, and hence $(f_1+\cdots+f_{h-1}+g+f_{h+1}+\cdots+f_m)R_R$, are projective, then the R-modules $g_i^*R_R\cong R/(g_jQ\cap R)$ (in the proof) are projective, that is, $(g_1Q\cap R)\oplus(g_2Q\cap R)$ is a direct summand of R_R, whence R has the same structure as in (c) of Lemma 4.2.

Thus we remark the following for the proof of the next theorem.

REMARK. For a regular ring R with Q the maximal right quotient ring, the equivalent conditions (a),(b),(c) of Lemma 4.2 are also equivalent to the following condition:

(b') (i) *For every faithful abelian idempotent f in Q, the R-module fR_R is projective,*

(ii) *R has bounded index.*

At this point, applying the previous results to regular rings, we obtain the following theorem, in which the equivalence (a)⇔(d), under the assumption that every nonzero ideal of R contains a nonzero central idempotent, is given in S. Kobayashi [10, Theorem 1].

Theorem 4.3. *Let R be a regular ring, and let Q be the maximal right quotient ring of R. Then the following conditions are equivalent:*

(a) *R is a right GFC ring.*

(b) (i) *R contains all the faithful idempotents in Q,*

(ii) *R has bounded index.*

(c) (i) *Every cyclic faithful nonsingular right R-module is projective,*

(ii) *R has bounded index.*

(d) *R is isomorphic to a finite direct product of an abelian regular ring and full matrix rings over self-injective abelian regular rings.*

Proof. (a)⇒(b). It follows from Theorem 2.8 and [11, Theorem 9] that Q, and hence R, has bounded index, while the first half of the proof of Proposition 3.5 shows that for every faithful abelian idempotent f in Q, the R-module fR is projective. Thus, the implication (a)⇒(b) is obtained by the remark above.

(b)⇔(c)⇔(d). This follows immediately from Lemma 4.2 and [6, Theorem 3.8].

(d)⇒(a). Since every one-sided ideal of abelian regular rings is two-sided, those rings are obviously right GFC. Also, according to [11, Theorem 9] and [6, Theorem 7.12], full matrix rings over self-injective abelian regular rings are FPF rings. Thus it follows that (d) implies (a).

Using [11, Corollary 9.2], we obtain the following corollary.

Corollary 4.4. *A regular ring R is right GFC if and only if R is left GFC.*

The following corollaries are immediate.

Corollary 4.5. *Let R be a regular ring which contains no nonzero abelian central idempotents. Then, R is right GFC if and only if R is right FPF.*

Corollary 4.6. *Let R be an indecomposable regular ring. If R is right GFC, then it is a simple artinian ring.*

REMARK. Since the matrix ring $M_n(R)$ $(n\geq 2)$ over any ring R contains no nonzero abelian central idempotents, it follows immediately from Corollary 4.5 that a regular ring R is right FPF if and only if the matrix ring $M_n(R)$ is right GFC for some $n\geq 2$. This fact is also noted in [10].

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