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Osaka J. Math.
32 (1995), 547-563

ON STANDARD L-FUNCTIONS ATTACHED TO $\mathrm{ALT}^{n-1}(C^n)$-VALUED SIEGEL MODULAR FORMS

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(Received February 8, 1993)

Introduction

In [23], we studied some properties of standard L-functions attached to $\mathrm{sym}^l(V)$-valued Siegel modular forms of weight $\det^k \otimes \mathrm{sym}^l$. More precisely, let $\det^k \otimes \mathrm{sym}^l$ be an irreducible rational representation of $GL(n, \boldsymbol{C})$ with representation space $\mathrm{sym}^l(V)$, where V is isomorphic to $\boldsymbol{C}^n$ and $\mathrm{sym}^l(V)$ is the l-th symmetric tensor product of V. Let f be a $\mathrm{sym}^l(V)$-valued holomorphic cusp form of weight $\det^k \otimes \mathrm{sym}^l$ for $Sp(n, \boldsymbol{Z})$ (size $2n$). Suppose f is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra. Then we define the standard L-function attached to f by

$$(0.1)\qquad L(s, f, \underline{\mathrm{St}}):=\prod_p\left\{(1-p^{-s})\prod_{j=1}^n(1-\alpha_j(p)^{-1}p^{-s})(1-\alpha_j(p)p^{-s})\right\}^{-1},$$

where p runs over all prime numbers and $\alpha_j(p)$ $(1\leq j\leq n)$ are the Satake p-parameters of f. The right-hand side of (0.1) converges absolutely and locally uniformly for $\mathrm{Re}(s)>n+1$. We put

$$\Lambda(s, f, \underline{\mathrm{St}}):=\Gamma_{\boldsymbol{R}}(s+\varepsilon)\Gamma_{\boldsymbol{C}}(s+k+l-1)\left\{\prod_{j=2}^n \Gamma_{\boldsymbol{C}}(s+k-j)\right\}L(s, f, \underline{\mathrm{St}}),$$

with

$$\Gamma_{\boldsymbol{R}}(s):=\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right),\ \Gamma_{\boldsymbol{C}}(s):=2(2\pi)^{-s}\Gamma(s)$$

and

$$\varepsilon:=\begin{cases}0 \text{ for } n \text{ even,}\\ 1 \text{ for } n \text{ odd.}\end{cases}$$

Then we have the following (cf. Andrianov and Kalinin [Z], Böcherer [5] and Mizumoto [19] for $l=0$).

Theorem. ([23, Theorems 2 and 3]) *For k, $l \in 2\boldsymbol{Z}$, k, $l>0$, $\Lambda(s, f, \underline{\mathrm{St}})$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation*

$$\Lambda(s, f, \underline{\mathrm{St}})=\Lambda(1-s, f, \underline{\mathrm{St}}).$$

Suppose $k>n$. Then $\Lambda(s, f, \underline{\mathrm{St}})$ is holomorphic except for possible simple poles at $s=0$ and $s=1$; it has a pole at $s=1$ (or equivalently, $s=0$) if and only if f belongs to the $\boldsymbol{C}$-vector space spanned by certain theta series in [24] which is invariant under the action of the Hecke algebra.

If we note that the signature of $\det^k \otimes \mathrm{sym}^l$ is $(k+l, k, \cdots, k) \in \boldsymbol{Z}^n$, we expect the following [23, §3.1 Remark]:

($\boldsymbol{C}$). *Let ρ be an irreducible rational representation of $GL(n, \boldsymbol{C})$ with representation space $\boldsymbol{V}$ whose signature is $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \boldsymbol{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let f be a $\boldsymbol{V}$-valued holomorphic cusp form of weight ρ for $Sp(n, \boldsymbol{Z})$. Suppose that f is an eigenform. Then, it is expected that the completed Dirichlet series*

$$\Lambda(s, f, \underline{\mathrm{St}}):=\Gamma_{\boldsymbol{R}}(s+\varepsilon)\prod_{j=1}^{n} \Gamma_{\boldsymbol{C}}(s+\lambda_j-j) L(s, f, \underline{\mathrm{St}})$$

should satisfy a functional equation.

Unfortunately, within our knowledge it is not verified so far whether ($\boldsymbol{C}$) holds or not except $\det^k$ and $\det^k \otimes \mathrm{sym}^l$ cases. We will give another example satisfying ($\boldsymbol{C}$).

For $l \in \boldsymbol{Z}$, $0 \leq l \leq n$, let $\det^k \otimes \mathrm{alt}^l$ be an irreducible rational representation of $GL(n, \boldsymbol{C})$ with representation space $\mathrm{alt}^l(V)$, where V is isomorphic to $\boldsymbol{C}^n$ and $\mathrm{alt}^l(V)$ is the l-th alternating tensor product of V. Let $M_k^n(\mathrm{alt}^l(V))$ (resp. $S_k^n(\mathrm{alt}^l(V))$) be the $\boldsymbol{C}$-vector space consisting of $\mathrm{alt}^l(V)$-valued holomorphic modular (resp. cusp) forms of weight $\det^k \otimes \mathrm{alt}^l$ for $Sp(n, \boldsymbol{Z})$.

Suppose that $f \in S_k^n(\mathrm{alt}^{n-1}(V))$ is an eigenform. We note that the signature of $\det^k \otimes \mathrm{alt}^{n-1}$ is $(k+1, \cdots, k+1, k)$. We put

$$\Lambda(s, f, \underline{\mathrm{St}}):=\Gamma_{\boldsymbol{R}}(s+1)\left\{\prod_{j=1}^{n-1} \Gamma_{\boldsymbol{C}}(s+k+1-j)\right\}\Gamma_{\boldsymbol{C}}(s+k-n) L(s, f, \underline{\mathrm{St}}).$$

Then the main result of this paper is the following (cf. Piatetski-Shapiro and Rallis [21], Weissauer [24]).

Theorem 1. *Let k be an even integer, n an odd integer and $2k \geq n>2$. Then $\Lambda(s, f, \underline{\mathrm{St}})$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation*

$$\Lambda(s, f, \underline{\mathrm{St}})=\Lambda(1-s, f, \underline{\mathrm{St}}).$$

Moreover, suppose $k>n$. Then, $\Lambda(s, f, \underline{\mathrm{St}})$ is entire.

NOTATION.

1°. As usual, $\boldsymbol{Z}$ is the ring of rational integers, $\boldsymbol{Q}$ the field of rational numbers, $\boldsymbol{R}$ the field of real numbers, $\boldsymbol{C}$ the field of complex numbers.

2°. Let $m, n\in\boldsymbol{Z}$, $m, n>0$. If A is an $m\times n$-matrix, then we write it also as $A^{(m,n)}$, and as $A^{(m)}$ if $m=n$. The identity matrix of size n is denoted by 1_n.

3°. For $m, n\in\boldsymbol{Z}$, $m, n>0$, and a commutative ring R containing 1, let $R^{(m,n)}$ (resp. $R^{(n)}$) be the R-module of all $m\times n$ (resp. $n\times n$) matrices with entries in R.

4°. For a real symmetric positive definite matrix S, $S^{1/2}$ is the unique real symmetric positive definite matrix such that $(S^{1/2})^2=S$.

5°. For matrix $A^{(m)}$, $B^{(m,n)}$, we define $A[B]:={}^t\overline{B}AB$, where ${}^tB$ is the transpose of B and $\overline{B}$ is the complex conjugate of B.

6°. For a matrix $A^{(m)}=(a_{jh})_{1\le j,h\le m}$, $\widetilde{a_{jh}}$ is the cofactor of a_{jh} and $\widetilde{A}=(\widetilde{a_{jh}})$.

7°. For $n\in\boldsymbol{Z}$, $n>0$, we put

$$\boldsymbol{T}^{(n)}:=\left\{T=\begin{pmatrix} t_1 & & & 0\\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n\end{pmatrix}\in\boldsymbol{Z}^{(n)}\;\middle|\; t_j>0(1\le j\le n),\ t_1|\cdots|t_n\right\}.$$

8°. For $n\in\boldsymbol{Z}$, $n>0$, let $\Gamma^n:=Sp(n, \boldsymbol{Z})$ be the Siegel modular group of degree n and let $\mathfrak{H}_n$ be the Siegel upper half space of degree n, that is,

$$\mathfrak{H}_n:=\{Z=X+iY\in\boldsymbol{C}^{(n)}|{}^tZ=Z,\ Y>0\}.$$

For each $r\in\boldsymbol{Z}$ with $0\le r\le n$, we put

$$P_{n,r}:=\left\{\begin{pmatrix} * & * \\ C^{(n)} & D^{(n)}\end{pmatrix}\in\Gamma^n\;\middle|\; C=\begin{pmatrix}0 & 0\\ 0 & C_4^{(r)}\end{pmatrix},\ D=\begin{pmatrix} * & 0\\ * & D_4^{(r)}\end{pmatrix}\right\}.$$

All these are subgroups of Γ^n.

9°. For $n\in\boldsymbol{Z}$, $n\ge 0$, we put

$$\Gamma_n(s):=\prod_{j=1}^{n}\Gamma\left(s-\frac{j-1}{2}\right),$$

and

$$\gamma(s):=\begin{cases}\dfrac{\Gamma_n\left(\dfrac{s+n}{2}\right)}{\Gamma_n\left(\dfrac{s}{2}\right)} & \text{for } n \text{ even},\\[2ex] \dfrac{\Gamma_{n-1}\left(\dfrac{s+n}{2}\right)}{\Gamma_{n-1}\left(\dfrac{s-1}{2}\right)} & \text{for } n \text{ odd},\end{cases}$$

where $\Gamma(s)$ is the gamma function. We note that

$$\gamma(s)=\gamma(1-s)$$

Moreover, we put

$$\xi(s):=\Gamma_{R}(s)\zeta(s)=\xi(1-s),$$

where $\zeta(s)$ is the Riemann zeta function.

Throughout the paper we understand that a product (resp. a sum) over an empty set is equal to 1 (resp. 0).

1. Preliminaries

Let ρ be a finite-dimensional representation of $GL(n, \boldsymbol{C})$ with representation space V. By definition, $\boldsymbol{V}$-valued C^{∞}-Siegel modular forms of weight ρ are C^{∞}-functions from $\mathfrak{H}_n$ to $\boldsymbol{V}$ satisfying

$$(1.1) \qquad (f|_{\rho}M)(Z)=f(Z)$$

for all $Z\in\mathfrak{H}_n$ and $M=\begin{pmatrix}A^{(n)} & B^{(n)}\\ C^{(n)} & D^{(n)}\end{pmatrix}\in\Gamma^n$, where

$$(f|_{\rho}M)(Z):=\rho((CZ+D)^{-1})f(M\langle Z\rangle) \text{ and } M\langle Z\rangle:=(AZ+B)(CZ+D)^{-1}.$$

The space of all such functions is denoted by $M_{\rho}^{n}(\boldsymbol{V})^{\infty}$.

We write $|_k$ for $\rho=\det^k$ and we omit subscripts ρ, k when there is no fear of confusion.

A holomorphic function f from $\mathfrak{H}_n$ to $\boldsymbol{V}$ is called a $\boldsymbol{V}$-valued Siegel modular form of weight ρ if it satisfies (1.1) and if it is holomorphic at the cusps when $n=1$. The space of $\boldsymbol{V}$-valued Siegel modular forms of weight ρ is denoted by $M_{\rho}^{n}(\boldsymbol{V})$.

We define the Siegel operator on $M_{\rho}^{n}(\boldsymbol{V})$ by

$$(\Phi f)(Z):=\lim_{t\to\infty} f\left(\begin{pmatrix}Z & 0\\ 0 & it\end{pmatrix}\right)$$

for $Z\in\mathfrak{H}_{n-1}$. Let $\boldsymbol{W}$ be the subspace of $\boldsymbol{V}$ generated by the values of Φf for all $f\in M_{\rho}^{n}(\boldsymbol{V})$. Then $\boldsymbol{W}$ is invariant under the transformations

$$\rho\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right),\ g \in GL(n-1,\ \boldsymbol{C}).$$

If we assume $\boldsymbol{W} \neq \{0\}$, we get the representation σ of $\mathrm{GL}(n-1,\ \boldsymbol{C})$ with representation space $\boldsymbol{W}$. Thus the operator Φ defines the map

$$\Phi:\ M_\rho^n(\boldsymbol{V}) \to M_\sigma^{n-1}(\boldsymbol{W}).$$

Suppose $f \in M_\rho^n(\boldsymbol{V})$. Then it is called a cusp form if it satisfies $\Phi f = 0$, and we put

$$S_\rho^n(\boldsymbol{V}):\ = \{f \in M_\rho^n(\boldsymbol{V}) | f \text{ is a cusp form}\}.$$

If ρ is an irreducible rational representation, ρ is equivalent to an irreducible rational representation $\tilde{\rho}$ satisfying the following condition: Let $\tilde{\boldsymbol{V}}$ be the representation space of $\tilde{\rho}$. Then, there exists a unique one-dimensional vector subspace $\boldsymbol{C}\tilde{v}$ of $\tilde{\boldsymbol{V}}$ such that for any upper triangular matrix of $GL(n,\ \boldsymbol{C})$,

$$\tilde{\rho}\left(\begin{pmatrix} g_{11} & & * \\ & \ddots & \\ 0 & & g_{nn} \end{pmatrix}\right)\tilde{\boldsymbol{v}} = \left(\prod_{j=1}^{n} g_{jj}^{\lambda_j}\right)\tilde{\boldsymbol{v}},$$

where $(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n) \in \boldsymbol{Z}^n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Then we call $(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n)$ the signature of ρ.

REMARK. Suppose the signature of ρ is $(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n)$. We note that $M_\rho^n(\boldsymbol{V}) = \{0\}$ if $\lambda_n < 0$ and that $M_\rho^n(\boldsymbol{V})^\infty = \{0\}$ if $\lambda_1 + \cdots + \lambda_n \not\equiv 0 \bmod 2$.

Now, we put

$$G^+Sp(n,\ \boldsymbol{Q}):\ = \left\{M \in GL(2n,\ \boldsymbol{Q}) \middle| {}^tM\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}M = \mu(M)\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},\ \mu(M) > 0\right\}.$$

For $g \in G^+Sp(n,\ \boldsymbol{Q})$, let $\Gamma^n g \Gamma^n = \cup_{j=1}^r \Gamma^n g_j$ be a decomposition of the double coset $\Gamma^n g \Gamma^n$ into left cosets. For $f \in M_\rho^n(\boldsymbol{V})$ (resp. $S_\rho^n(\boldsymbol{V}),\ M_\rho^n(\boldsymbol{V})^\infty$), we define the Hecke operator $(\Gamma^n g \Gamma^n)$ by

$$f|(\Gamma^n g \Gamma^n):\ = \sum_{j=1}^{r} f|g_j.$$

Let $f \in S_\rho^n(\boldsymbol{V})$ be an eigenform. We define the standard L-function attached to f by (0.1). We also define the following series:

$$D(s,\ f):\ = \sum_{T \in \boldsymbol{T}^{(n)}} \lambda(f,\ T)\det(T)^{-s}, \tag{1.2}$$

where $\lambda(f,\ T)$ is the eigenvalue on f of the Hecke operator $\left(\Gamma^n\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}\Gamma^n\right)$, $T \in \boldsymbol{T}^{(n)}$. By Böcherer [6], we have:

$$(1.3) \qquad \zeta(s)\prod_{j=1}^{n}\zeta(2s-2j)D(s, f)=L(s-n, f, \underline{\mathrm{St}}).$$

For $k\in 2\boldsymbol{Z}$, $k>0$, $s\in \boldsymbol{C}$ and $Z=(z_{jh})\in\mathfrak{H}_n$ with $z_{jh}:=x_{jh}+iy_{jh}$, we define the Eisenstein series by

$$E_k^n(Z, s):=\sum_{M=\begin{pmatrix} * & * \\ C^{(n)} & D^{(n)}\end{pmatrix}\in P_{n,0}\backslash\Gamma^n}\det(CZ+D)^{-k}\det(\mathrm{Im}(M\langle Z\rangle))^s.$$

Then $E_k^n(Z, s)\in M_k^{n\infty}$, where $M_k^{n\infty}$ is the space of C^∞-Siegel modular forms of weight k. The function $E_k^n(Z, s)\det(\mathrm{Im}(Z))^{-s}$ converges absolutely and locally uniformly for $k+2\mathrm{Re}(s)>n+1$. Moreover, we have the following:

Theorem 2. (Langlands [18], Kalinin [13] and Mizumoto [19, 20]) *Let $n\in\boldsymbol{Z}$, $k\in 2\boldsymbol{Z}$ and $n, k>0$. Then for $Z\in\mathfrak{H}_n$,*

$$\boldsymbol{E}_k^n(Z, s):=\frac{\Gamma_n\left(s+\frac{k}{2}\right)}{\Gamma_n(s)}\xi(2s)\prod_{j=1}^{\left[\frac{n}{2}\right]}\xi(4s-2j)E_k^n\left(Z, s-\frac{k}{2}\right)$$

is invariant under $s\to\frac{n+1}{2}-s$ and it is an entire function in s.

It is also known that every partial derivative (in z_{jh}'s) of the Eisenstein series $E_k^n(Z, s)$ is slowly increasing (locally uniformly in s).

Theorem 3. (Mizumoto [20]) *Let $n\in\boldsymbol{Z}$, $k\in 2\boldsymbol{Z}$ and $n, k>0$.*
(i) *For each $s_0\in\boldsymbol{C}$, there exist constants $\delta>0$ and $d\in\boldsymbol{Z}(d\geq 0)$, depending only on n, k and s_0, such that*

$$(s-s_0)^d E_k^n(X+iY, s)$$

is holomorphic in s for $|s-s_0|<\delta$, and is C^∞ in (X, Y).
(ii) *Furthermore, for given $\varepsilon>0$ and $N\in\boldsymbol{Z}$ ($N\geq 0$), there exist constants $\alpha>0$ and $\beta>0$ depending only on n, k, d, s_0, ε, δ and N such that*

$$|(s-s_0)^d D_{X,Y}E_k^n(X+iY, s)|\leq\alpha\det(\mathrm{Im}(Z))^\beta$$

for $Y\geq\varepsilon 1_n$ and $|s-s_0|<\delta$, where $D_{X,Y}$ is an arbitary monomial of degree N in $\frac{\partial}{\partial x_{jh}}$ and $\frac{\partial}{\partial y_{jh}}$ $(1\leq j, h\leq n)$.

The assertion above for the case $N=0$ has been proved by Langlands [18] and Kalinin [13].

2. Differential operators

In what follows, we put

$$V_1 = \boldsymbol{C}e_1 \oplus \cdots \oplus \boldsymbol{C}e_n, \ \boldsymbol{e}_1 = (e_1, \cdots, e_n),$$
$$V_2 = \boldsymbol{C}e_{n+1} \oplus \cdots \oplus \boldsymbol{C}e_{2n}, \ \boldsymbol{e}_2 = (e_{n+1}, \cdots, e_{2n}).$$

Let $\mathrm{alt}^{n-1}(V_1)$ (resp. $\mathrm{alt}^{n-1}(V_2)$) be the $(n-1)$-th alternating tensor product of V_1 (resp. V_2). If we put

$$t_j :=(-1)^{j-1} e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n,$$
$$t_{n+j} :=(-1)^{j-1} e_{n+1} \wedge \cdots \wedge e_{n+j-1} \wedge e_{n+j+1} \wedge \cdots \wedge e_{2n} \ (1 \leq j \leq n),$$

we can write

$$\mathrm{alt}^{n-1}(V_1) = \boldsymbol{C}t_1 \oplus \cdots \oplus \boldsymbol{C}t_n \text{ and } \mathrm{alt}^{n-1}(V_2) = \boldsymbol{C}t_{n+1} \oplus \cdots \oplus \boldsymbol{C}t_{2n}.$$

Moreover, we put

$$\boldsymbol{t}_1 :=(t_1, \cdots, t_n) \text{ and } \boldsymbol{t}_2 :=(t_{n+1}, \cdots, t_{2n}).$$

If for each $g \in GL(n, \boldsymbol{C})$, g acts on $\boldsymbol{e}_j$ $(j=1, 2)$ by $\boldsymbol{e}_j g$, then $\det^k \otimes \mathrm{alt}^{n-1}(g)$ acts on $\boldsymbol{t}_j$ $(j=1, 2)$ by

$$\det{}^k \otimes \mathrm{alt}^{n-1}(g) \boldsymbol{t}_j :=\det(g)^k \boldsymbol{t}_j \tilde{g} = \det(g)^{k+1} \boldsymbol{t}_j {}^t g^{-1}.$$

If we put $a=(a_1, \cdots, a_n) \in \boldsymbol{C}^n$, $\det^k \otimes \mathrm{alt}^{n-1}(g)$ acts on $\sum_{j=1}^n a_j t_j = \boldsymbol{t}_1 {}^t a \in \mathrm{alt}^{n-1}(V_1)$ and $\boldsymbol{t}_2 {}^t a \in \mathrm{alt}^{n-1}(V_2)$ by

$$\det{}^k \otimes \mathrm{alt}^{n-1}(g)(\boldsymbol{t}_j {}^t a) :=\det(g)^k \boldsymbol{t}_j \tilde{g} {}^t a = \det(g)^{k+1} \boldsymbol{t}_j {}^t g^{-1} {}^t a \quad (j=1, 2).$$

Thus we get the action of $\det^k \otimes \mathrm{alt}^{n-1}$ on $\mathrm{alt}^{n-1}(V_j)$ $(j=1, 2)$.

Let ι be the isomorphism from V_1 to V_2 defined by $\iota(e_j) = e_{n+j}$ $(1 \leq j \leq n)$. It induces the isomorphism (also denoted by ι) from $\mathrm{alt}^{n-1}(V_1)$ to $\mathrm{alt}^{n-1}(V_2)$. For a $\mathrm{alt}^{n-1}(V_1)$-valued function f on $\mathfrak{H}_n$ and for $Z \in \mathfrak{H}_n$, we define $\iota(f)$ by

$$(\iota(f))(Z) :=\iota(f(Z)).$$

For a function F on $\mathfrak{H}_{2n}$, $\begin{pmatrix} Z^{(n)} & U^{(n)} \\ {}^t U^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we define the pullback d^* by

$$(d^* F)\left(\begin{pmatrix} Z & U \\ {}^t U & W \end{pmatrix}\right) :=F\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right).$$

We consider $\Gamma^n \times \Gamma^n$ imbedded in Γ^{2n} by

$$\begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \times \begin{pmatrix} A'^{(n)} & B'^{(n)} \\ C'^{(n)} & D'^{(n)} \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix},$$

and when convenient will identify $\Gamma^n \times \Gamma^n$ with its image in Γ^{2n}.

We summarize some facts on differential operators obtained from invariant pluri-harmonic polynomials in Ibukiyama [12]. Let ρ_0 (resp. ρ_0') be an irreducible rational representation of $GL(n, \boldsymbol{C})$ with representation space $\boldsymbol{V}$ (resp. $\boldsymbol{V}'$), where ρ_0 is equivalent to ρ_0'. For $n, k \in \boldsymbol{Z}$, $n, k>0$, let $X=(x_{jv})$be a variable on $\boldsymbol{C}^{(n,2k)}$. We put

$$\Delta_{jh}:=\sum_{v=1}^{2k}\frac{\partial^2}{\partial x_{jv}\partial x_{hv}}.$$

A polynomial $P(X)$ on $\boldsymbol{C}^{(n,2k)}$ is called pluri-harmonic if $\Delta_{jh}P=0$ for each j, h with $1\leq j\leq h\leq n$.

From now on, we assume that $2k\geq n$. Suppose that a polynomial map

$$P:\ \boldsymbol{C}^{(n,2k)}\times \boldsymbol{C}^{(n,2k)}\rightarrow \boldsymbol{V}\otimes \boldsymbol{V}'$$

satisfies the following three conditions:

(2.1) $P(X_1, X_2)$ is pluri-harmonic for each X_j $(j=1, 2)$,

(2.2) $P(X_1g, X_2g)=P(X_1, X_2)$ for each $g\in O(2k)$,

(2.3) $P(a_1X_1, a_2X_2)=(\rho_0(a_1)\otimes \rho'_0(a_2))P(X_1, X_2)$ for each $a_j\in GL(n, \boldsymbol{C})$ $(j=1, 2)$.

Then there exists a unigue polynomial map Q on $\boldsymbol{C}^{(2n)}$ such that

$$P(X_1, X_2)=Q\begin{pmatrix} X_1{}^tX_1 & X_1{}^tX_2 \\ X_2{}^tX_1 & X_2{}^tX_2\end{pmatrix}.$$

Let $\mathfrak{Z}=(z_{jh})$ be a variable on $\mathfrak{H}_{2n}$. We put

$$\frac{\partial}{\partial \mathfrak{Z}}:=\left(\frac{1+\delta_{jh}}{2}\frac{\partial}{\partial z_{jh}}\right)_{1\leq j,h\leq 2n},$$

where, for $z_{jh}=x_{jh}+iy_{jh}$,

$$\frac{\partial}{\partial z_{jh}}=\frac{1}{2}\left(\frac{\partial}{\partial x_{jh}}-i\frac{\partial}{\partial y_{jh}}\right),\ \frac{\partial}{\partial \bar{z}_{jh}}=\frac{1}{2}\left(\frac{\partial}{\partial x_{jh}}+i\frac{\partial}{\partial y_{jh}}\right).$$

If we put

$$\boldsymbol{D}:=d^*Q\left(\frac{\partial}{\partial \mathfrak{Z}}\right),$$

we have the following:

Theorem 4. (Ibukiyama [12]) *Let n, $k\in \boldsymbol{Z}$ and $2k\geq n>0$.*
(i) *Let F be any* **C**-*valued C^∞-function on $\mathfrak{H}_{2n}$. If we put $\rho=\det^k\otimes \rho_0$ and*

$\rho'=\det^k\otimes\rho'_0$, *then for each* $(g, g')\in\Gamma^n\times\Gamma^n$ *and* $\mathfrak{Z}=\begin{pmatrix} Z^{(n)} & U^{(n)} \\ {}^tU^{(n)} & W^{(n)}\end{pmatrix}\in\mathfrak{H}_{2n}$, *we get the following commutation relation*:

$$((\boldsymbol{D}F)|_\rho(g)_Z|_{\rho'}(g')_W)(\mathfrak{Z})=(\boldsymbol{D}(F|_k(g, g')))(\mathfrak{Z}),$$

where $(\quad)_Z$ (*resp.* $(\quad)_W$) *denotes the action on* Z (*resp.* W).

(ii) *The operator* $\boldsymbol{D}$ *sends modular forms to modular forms*:

$$\boldsymbol{D}:\ M_k^{2n\infty}\to M_\rho^n(V)^\infty\otimes M_{\rho'}^n(V')^\infty.$$

Moreover, $\boldsymbol{D}$ *is a holomorphic operator and it satisfies*

$$\boldsymbol{D}:\ M_k^{2n}\to M_\rho^n(V)\otimes M_{\rho'}^n(V').$$

Now we apply it to $\det^k\otimes\mathrm{alt}^{n-1}$ cases. Let $\rho_0=\mathrm{alt}^{n-1}$ (resp. $\rho'_0=\mathrm{alt}^{n-1}$) be the representation of $GL(n, \boldsymbol{C})$ with representation space $\mathrm{alt}^{n-1}(V_1)$ (resp. $\mathrm{alt}^{n-1}(V_2)$). For a variable $\mathfrak{Z}=(z_{jh})$ on $\mathfrak{H}_{2n}$, we put

$$u_{jh}:=z_{j\,n+h}(1\leq j,\ h\leq n),\ U^{(n)}:=(u_{jh}) \text{ and } \frac{\partial}{\partial U}:=\left(\frac{\partial}{\partial u_{jh}}\right)_{1\leq j,h\leq n}.$$

For functions on $\mathfrak{H}_{2n}$, we define the differential operator $\mathscr{D}$ by

$$\mathscr{D}:=d^*\left(\boldsymbol{t}_1\widetilde{\frac{\partial}{\partial U}}{}^t\boldsymbol{t}_2\right).$$

Then we have:

Proposition 1. *Let* $n,\ k\in\boldsymbol{Z}$ *and* $2k\geq n>2$.

(i) *Let* F *be any* $\boldsymbol{C}$*-valued* C^∞*-function on* $\mathfrak{H}_{2n}$. *Then for each* $(g, g')\in\Gamma^n\times\Gamma^n$ *and* $\mathfrak{Z}=\begin{pmatrix} Z & U \\ {}^tU & W\end{pmatrix}\in\mathfrak{H}_{2n}$, *we get the following commutation relation*:

$$((\mathscr{D}F)|_\rho(g)_Z|_{\rho'}(g')_W)(\mathfrak{Z})=(\mathscr{D}(F|_k(g, g')))(\mathfrak{Z}).$$

(ii) *The operator* $\mathscr{D}$ *sends modular forms to modular forms*:

$$\mathscr{D}:\ M_k^{2n\infty}\to M_k^n(\mathrm{alt}^{n-1}(V_1))^\infty\otimes M_k^n(\mathrm{alt}^{n-1}(V_2))^\infty.$$

Moreover, $\mathscr{D}$ *is a holomorphic operator and it satisfies*

$$\mathscr{D}:\ M_k^{2n}\to M_k^n(\mathrm{alt}^{n-1}(V_1))\otimes M_k^n(\mathrm{alt}^{n-1}(V_2)).$$

Proof. Let X_j $(j=1, 2)$ be variables on $\boldsymbol{C}^{(n,2k)}$. If we put

$$\frac{\partial}{\partial U}=X_1{}^tX_2,$$

the polynomial $\boldsymbol{t}_1\widetilde{X_1{}^tX_2}{}^t\boldsymbol{t}_2$ satisfies the three conditions (2. 1), (2. 2), (2. 3).

Therefore we get Proposition 1 by Theorem 4. $\square$

3. Proof of Theorem 1

We prove Theorem 1 according to Böcherer's method in [5]. We first apply the differential operator $\mathcal{D}$ to the Eisenstein series $E_k^{2n}(\mathfrak{Z}, s)$. For this, we use the coset decomposition by Garrett:

Lemma 1. (Garrett [9] and Mizumoto [19, Appendix B])

(i) *The double coset* $P_{2n,0}\backslash\Gamma^{2n}/\Gamma^n\times\Gamma^n$ *has an irredundant set of coset representatives*

$$g_{\hat{T}}=\begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & \hat{T}^{(n)} & 1_n & 0 \\ \hat{T}^{(n)} & 0 & 0 & 1_n \end{pmatrix},$$

where $\hat{T}=\begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}$, $T\in\boldsymbol{T}^{(r)}$ $(0\leq r\leq n)$.

(ii) *The left coset* $P_{2n,0}\backslash P_{2n,0}\ g_{\hat{T}}(\Gamma^n\times\Gamma^n)$ *has an irredundant set of coset representatives* $g_{\hat{T}}\hat{g}_1 g_2 \hat{g}'_1 g'_2$,

$$\hat{g}_1\in G_{n,r},\ g_2\in P_{n,r}\backslash\Gamma^n,\ \hat{g}'_1\in\Gamma^r(T)\backslash G_{n,r},\ g'_2\in P_{n,r}\backslash\Gamma^n,$$

where

$$G_{n,r}:=\left\{\begin{pmatrix} \hat{A}^{(n)} & \hat{B}^{(n)} \\ \hat{C}^{(n)} & \hat{D}^{(n)} \end{pmatrix}=\begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & A^{(r)} & 0 & B^{(r)} \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & C^{(r)} & 0 & D^{(r)} \end{pmatrix}\in\Gamma^n \,\middle|\, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma^r\right\}$$

and for $T\in\boldsymbol{T}^{(r)}$,

$$\Gamma^r(T):=\left\{g\in\Gamma^r \,\middle|\, \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} g \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix}\in\Gamma^r\right\}.$$

Now we prove the following (cf. Böcherer [4, Satz 9], [5, Satz 3]):

Proposition 2. *Let k be an even integer, n an odd integer and s a complex number such that $k+2\mathrm{Re}(s)>2n+1$. Suppose that $2k\geq n>2$. For* $\mathfrak{Z}=\begin{pmatrix} Z^{(n)} & U^{(n)} \\ {}^tU^{(n)} & W^{(n)} \end{pmatrix}\in\mathfrak{H}_{2n}$, $\mathfrak{Z}_0=\begin{pmatrix} Z^{(n)} & 0 \\ 0 & W^{(n)} \end{pmatrix}\in\mathfrak{H}_{2n}$, *we get*

$$(\mathcal{D}E_k^{2n})(\mathfrak{Z}, s)$$
$$=\frac{\Gamma(2k+2s+1)}{\Gamma(2k+2s-n+2)}\sum_{T\in\boldsymbol{T}^{(n)}}\left(\mathcal{P}(Z, W, s)\middle|\left(\Gamma^n\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}\Gamma^n\right)_W\right)\det(T)^{-k-2s}$$

$$+\frac{\Gamma(2k+2s+1)}{\Gamma(2k+2s-n+2)}\mathcal{R}(Z,\ W,\ s),$$

where

$$\mathcal{P}(Z,\ W,\ s)$$
$$:=\sum_{g\in\Gamma^n}\{\det(\operatorname{Im}(Z))^s\det(\operatorname{Im}(W))^s|\det(W+Z)|^{-2s}\rho((W+Z)^{-1})(\boldsymbol{t}_1\,{}^t\boldsymbol{t}_2)\}|(g)_Z,$$

and

$$\mathcal{R}(Z,\ W,\ s):=\sum_{T\in\boldsymbol{T}^{(n-1)}}\ \sum_{g_2\in P_{n,n-1}\backslash\Gamma^n}\ \sum_{g'_2\in P_{n,n-1}\backslash\Gamma^n}\ \sum_{\tilde{g}_1\in G_{n,n-1}}\ \sum_{\tilde{g}'_1\in\Gamma^{n-1}(T)\backslash G_{n,n-1}}$$
$$\cdot\{\det(\operatorname{Im}(Z))^s\det(\operatorname{Im}(\tilde{W}))^s|\det(1_n-\hat{T}W\hat{T}Z)|^{-2s}$$
$$\cdot\rho((1_n-\hat{T}W\hat{T}Z)^{-1})(\boldsymbol{t}_1\hat{T}\,{}^t\boldsymbol{t}_2)\}|(\tilde{g}'_1)_W|(\tilde{g}_1)_Z|(g'_2)_W|(g_2)_Z.$$

Proof. It follows from Proposition 1 and Lemma 1 that

$$(\mathcal{D}E_k^{2n})(\mathfrak{Z},\ s)=\sum_{r=0}^{n}\ \sum_{T\in\boldsymbol{T}^{(r)}}\ \sum_{g_2\in P_{n,r}\backslash\Gamma^n}\ \sum_{g'_2\in P_{n,r}\backslash\Gamma^n}\ \sum_{\tilde{g}_1\in G_{n,r}}\ \sum_{\tilde{g}'_1\in\Gamma^r(T)\backslash G_{n,r}}$$
$$\{\mathcal{D}(\det(\operatorname{Im}\mathfrak{Z})^s|_k g_{\hat{T}})\}|(\tilde{g}'_1)_W|(\tilde{g}_1)_Z|(g'_2)_W|(g_2)_Z.$$

If for each $\hat{T}$ we put $g_{\hat{T}}=\begin{pmatrix} * & * \\ \mathfrak{C}^{(2n)} & \mathfrak{D}^{(2n)}\end{pmatrix}$, we get

$$\mathcal{D}(\det(\operatorname{Im}\mathfrak{Z})^s|_k g_{\hat{T}})=\det(\mathfrak{C}\overline{\mathfrak{Z}_0}+\mathfrak{D})^{-s}\mathcal{D}(\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-k-s}\det(\operatorname{Im}(\mathfrak{Z}))^s),$$

by the form of $\mathcal{D}$ and that of $\det(\operatorname{Im}(\mathfrak{Z}))$,

$$=\det(\mathfrak{C}\overline{\mathfrak{Z}_0}+\mathfrak{D})^{-s}\det(\operatorname{Im}(\mathfrak{Z}_0))^s\mathcal{D}(\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-k-s}).$$

As an example, we compute

$$d^*\widetilde{\frac{\partial}{\partial u_{nn}}}(\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-k-s}).$$

Let $\mathfrak{S}_m$ be the symmetric group of degree m. We put

$$\delta:=\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D}),\ \delta_0:=\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D}),\ \partial_{jh}:=\frac{\partial}{\partial u_{jh}}\quad(1\leq j,\ h\leq n)$$

and, for $m,\ q\in\boldsymbol{Z}$, $0<m$ and $0\leq q<m$,

$$L_m^q:=\left\{(l_1,\cdots,\ l_m)\in\boldsymbol{Z}^m\,|\,l_\nu\geq 0\ (1\leq\nu\leq m),\ \sum_{\nu=1}^{m}l_\nu=m-q,\ \sum_{\nu=1}^{m}\nu l_\nu=m\right\}.$$

For $(l_1,\cdots,\ l_m)\in L_m^q$, let $\Lambda(l_1,\cdots,\ l_m)$ be the set consisting of $J\in\mathfrak{S}_m$ such that, if $l_\gamma\neq 0$ $(1\leq\gamma\leq m)$,

$$1\leq J\left(\sum_{\nu=0}^{\gamma-1}\nu l_\nu+\gamma\lambda+1\right)<\cdots<J\left(\sum_{\nu=0}^{\gamma-1}\nu l_\nu+\gamma\lambda+\gamma\right)\leq m\quad(0\leq\lambda<l_\gamma)$$

and

$$1 \leq J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + 1\right) < J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma + 1\right) < \cdots < J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma(l_\gamma - 1) + 1\right) \leq m.$$

Then we get

$$\begin{aligned} d^* \widetilde{\partial_{nn}}(\delta^{-k-s}) &= d^*\Big(\sum_{\tau \in \mathfrak{S}_{n-1}} \mathrm{sgn}(\tau) \partial_{1\tau(1)} \cdots \partial_{n-1\tau(n-1)}\Big)(\delta^{-k-s}) \\ &= \sum_{q=0}^{n-2} \Big\{ \Big(\prod_{\mu=0}^{n-2-q} (-k-s-\mu) \Big) \delta_0^{-k-s-(n-1-q)} \\ &\quad \times d^* \sum_{\tau \in \mathfrak{S}_{n-1}} \sum_{(l_1, \cdots, l_{n-1}) \in L_{n-1}^q} \sum_{J \in \Lambda} \mathrm{sgn}(\tau) \partial_\tau^J(q\,;\,(l_1, \cdots, l_{n-1}))(\delta) \Big\}, \end{aligned}$$

where $\Lambda = \Lambda(l_1, \cdots, l_{n-1})$ and

$$\begin{aligned} \partial_\tau^J(q\,;\,(l_1, \cdots, l_{n-1}))(\delta) = \prod_{\gamma=1}^{n-1} \Big\{ &(\partial_{\tau(J(\alpha^\gamma+1))} \cdots \partial_{\tau(J(\alpha^\gamma+\gamma))})(\delta) \\ &\times \cdots \\ &\times (\partial_{\tau(J(\alpha^\gamma+\gamma(l_\gamma-1)+1))} \cdots \partial_{\tau(J(\alpha^{\gamma+1}))})(\delta) \Big\} \end{aligned}$$

with $\alpha^\gamma := \sum_{\nu=0}^{\gamma-1} \nu l_\nu$, $\partial_{\tau(J(\cdot))} := \partial_{J(\cdot)\,\tau(J(\cdot))}$.

For each q $(0 \leq q \leq n-2)$, $(l_1, \cdots, l_{n-1}) \in L_{n-1}^q$, $\tau \in \mathfrak{S}_{n-1}$ and $J \in \Lambda$, we define

$$(\partial_{J(j)\,\tau(J(h))}) := \begin{pmatrix} ((A_\tau^J)_{\xi\eta})^{(n-1-q)} & * \\ * & \partial_{nn} \end{pmatrix},$$

where, for $\sum_{\nu=1}^{\gamma-1} l_\nu + 1 \leq \xi \leq \sum_{\nu=1}^{\gamma} l_\nu$ and $\sum_{\nu=1}^{\gamma'-1} l_\nu + 1 \leq \eta \leq \sum_{\nu=1}^{\gamma'} l_\nu$, $(A_\tau^J)_{\xi\eta}$ is a $\gamma \times \gamma'$ matrix. In the same way, we define

$$(b_{J(j)\,\tau(J(h))}) := \begin{pmatrix} ((B_\tau^J)_{\xi\eta})^{(n-1-q)} & * \\ * & b_{nn} \end{pmatrix},$$

where $(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}\mathfrak{C} = \begin{pmatrix} * & \mathscr{B}^{(n)} \\ * & * \end{pmatrix}$ and $\mathscr{B} = (b_{jh})$.

Then we have

$$\begin{aligned} &d^* \sum_{\tau \in \mathfrak{S}_{n-1}} \mathrm{sgn}(\tau) \partial_\tau^J(q\,;\,(l_1, \cdots, l_{n-1}))(\delta) \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1}/\prod_{\gamma=1}^{n-1} \mathfrak{S}_\gamma^{l_\gamma}} \Big\{ \mathrm{sgn}(\sigma) \prod_{\xi=1}^{n-1-q} d^* \det((A_\sigma^J)_{\xi\xi})(\delta) \Big\}, \end{aligned}$$

by $d^* \det((A_\sigma^J)_{\xi\xi})(\delta) = (\gamma+1)!\, \delta_0 \det((B_\sigma^J)_{\xi\xi})$ $\quad (\sum_{\nu=1}^{\gamma-1} l_\nu + 1 \leq \xi \leq \sum_{\nu=1}^{\gamma} l_\nu)$,

$$\begin{aligned} &= \delta_0^{n-1-q} \prod_{\gamma=1}^{n-1} \{(\gamma+1)!\}^{l_\gamma} \sum_{\sigma \in \mathfrak{S}_{n-1}/\prod_{\gamma=1}^{n-1} \mathfrak{S}_\gamma^{l_\gamma}} \Big\{ \mathrm{sgn}(\sigma) \prod_{\xi=1}^{n-1-q} \det((B_\sigma^J)_{\xi\xi}) \Big\} \\ &= \delta_0^{n-1-q} \widetilde{b_{nn}} \prod_{\gamma=1}^{n-1} \{(\gamma+1)!\}^{l_\gamma}. \end{aligned}$$

Since the number of elements of Λ is $\left(\prod_{\gamma=1}^{n-1} \frac{1}{l_\gamma!}\right) \frac{(n-1)!}{(1!)^{l_1} \cdots ((n-1)!)^{l_{n-1}}}$, we obtain

$$(3.1)\qquad d^* \widetilde{\partial_{nn}}(\delta^{-k-s})=(-1)^{n-1}\sum_{q=0}^{n-2}\left\{a_{n-1}(q)\prod_{\mu=0}^{n-2-q}(2s+2k+2\mu)\right\}\delta_0^{-k-s}\,\widetilde{b_{nn}},$$

where

$$a_m(q)=(-1)^q 2^{-(m-q)} m! \sum_{(l_1,\cdots,l_m)\in L_m^q}\left(\prod_{\gamma=1}^{m}\frac{(\gamma+1)^{l_\gamma}}{l_\gamma!}\right)\quad (0<m,\ 0\leq q<m).$$

In the same way, we have

$$\begin{aligned}&\mathscr{D}(\det(\mathfrak{C}\mathfrak{Z}+\mathfrak{D})^{-k-s})\\&=(-1)^{n-1}\sum_{q=0}^{n-2}\left\{a_{n-1}(q)\prod_{\mu=0}^{n-2-q}(2s+2k+2\mu)\right\}\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-k-s}(\boldsymbol{t}_1\widetilde{\mathscr{B}}^t\boldsymbol{t}_2)\end{aligned}$$

and

$$\begin{aligned}\mathscr{D}(\det(\mathrm{Im}\mathfrak{Z})^s|_k g_{\hat{T}})&=(-1)^{n-1}\sum_{q=0}^{n-2}\left\{a_{n-1}(q)\prod_{\mu=0}^{n-2-q}(2s+2k+2\mu)\right\}\\&\quad\times\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-k}\det(\mathrm{Im}(g_{\hat{T}}\langle\mathfrak{Z}_0\rangle))^s(\boldsymbol{t}_1\widetilde{\mathscr{B}}^t\boldsymbol{t}_2).\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}&\det(\mathfrak{C}\mathfrak{Z}_0+\mathfrak{D})^{-k}\det(\mathrm{Im}(g_{\hat{T}}\langle\mathfrak{Z}_0\rangle))^s(\boldsymbol{t}_1\widetilde{\mathscr{B}}^t\boldsymbol{t}_2)\\&=\det(\mathrm{Im}(Z))^s\det(\mathrm{Im}(W))^s|\det(1_n-\hat{T}W\hat{T}Z)|^{-2s}\rho((1_n-\hat{T}W\hat{T}Z)^{-1})(\boldsymbol{t}_1\widetilde{\hat{T}}^t\boldsymbol{t}_2).\end{aligned}$$

Therefore we have only to prove

$$\sum_{q=0}^{n-2}\left\{a_{n-1}(q)\prod_{\mu=0}^{n-2-q}(2s+2k+2\mu)\right\}=\prod_{\mu=0}^{n-2}(2s+2k-\mu).$$

To prove the formula above, we put $x=2s+2k$ and $m=n-1$. Then we have to prove

$$(3.2)\qquad \sum_{q=0}^{m-1}\left\{a_m(q)\prod_{\mu=0}^{m-1-q}(x+2\mu)\right\}=\prod_{\mu=0}^{m-1}(x-\mu).$$

We put $a_m(q)=0$ if $q\geq m$, $0>q$ or $0\geq m$. We use induction on m. If $m=1$, the assertion is trivial. We suppose

$$\sum_{q=0}^{m'-1}\left\{a_{m'}(q)\prod_{\mu=0}^{m'-1-q}(x+2\mu)\right\}=\prod_{\mu=0}^{m'-1}(x-\mu).$$

for any $m'<m$. Then we have

$$\begin{aligned}\prod_{\mu=0}^{m-1}(x-\mu)&=\left\{\prod_{\mu=0}^{m-2}(x-\mu)\right\}(x-m+1)\\&=\left\{\sum_{q=0}^{m-2}\left\{a_{m-1}(q)\prod_{\mu=0}^{m-2-q}(x+2\mu)\right\}(x+(2m-2-2q)-(3m-3-2q))\right.\\&=\sum_{q=0}^{m-2}\left\{a_{m-1}(q)\prod_{\mu=0}^{m-1-q}(x+2\mu)\right\}\\&\quad-\sum_{q=1}^{m-1}\left\{(3m-2q-1)a_{m-1}(q-1)\prod_{\mu=0}^{m-1-q}(x+2\mu)\right\}\end{aligned}$$

$$=\sum_{q=0}^{m-1}\left\{(a_{m-1}(q)-(3m-2q-1)a_{m-1}(q-1))\prod_{\mu=0}^{m-1-q}(x+2\mu)\right\}.$$

If we note $3l_1+\cdots+(m+1)l_{m-1}=3m-2q-1$ in L_{m-1}^{q-1}, we have

$$\begin{aligned}
&a_{m-1}(q)-(3m-2q-1)a_{m-1}(q-1)\\
&=\frac{1}{m}(-1)^q 2^{-(m-q)}m!\sum_{L_{m-1}^q}\left\{(l_1+1)\frac{2^{l_1+1}}{(l_1+1)!}\left(\prod_{\gamma=2}^{m-1}\frac{(\gamma+1)^{l_\gamma}}{l_\gamma!}\right)\right\}\\
&\quad+\frac{1}{m}(-1)^q 2^{-(m-q)}m!\sum_{L_{m-1}^{q-1}}\left\{\left(\prod_{\gamma=1}^{m-1}\frac{(\gamma+1)^{l_\gamma}}{l_\gamma!}\right)\left(\sum_{\gamma=1}^{m-1}(\gamma+1)(l_{\gamma+1}+1)\frac{\gamma+2}{l_{\gamma+1}+1}\frac{l_\gamma}{\gamma+1}\right)\right\}\\
&=(-1)^q 2^{-(m-q)}m!\sum_{L_m^q}\left\{\left(\prod_{\gamma=1}^{m}\frac{(\gamma+1)^{l_\gamma}}{l_\gamma!}\right)\frac{1}{m}\sum_{\gamma=1}^{m}\gamma l_\gamma\right\}\\
&=a_m(q).
\end{aligned}$$

Thus we get (3.2). $\square$

REMARK. Under the notation above, we note that the formula

$$d^*\widetilde{\partial_{jh}}(\delta^{-k-s})=(-1)^{n-1}\prod_{\mu=0}^{n-2}(2s+2k-\mu)\delta_0^{-k-s}\widetilde{b_{jh}}$$

which is obtained from (3.1) and (3.2), and the formula

$$d^*\left(\det\left(\frac{\partial}{\partial U}\right)\right)(\delta^{-k-s})=(-1)^n\prod_{\mu=0}^{n-1}(2s+2k-\mu)\delta_0^{-k-s-1}\det(\widehat{T})$$

in [4, Satz 9], [5, Satz 3] have the same meaning.

For $\sum_{j=1}^n a_j t_{n+j}$, $\sum_{j=1}^n b_j t_{n+j}\in \mathrm{alt}^{n-1}(V_2)$, we define the inner product of them by

$$\left\langle\sum_{j=1}^n a_j t_{n+j},\ \sum_{j=1}^n b_j t_{n+j}\right\rangle:=\sum_{j=1}^n a_j\overline{b}_j.$$

Suppose f, $g\in M_k^n(\mathrm{alt}^{n-1}(V_2))^\infty$. The Petersson inner product of f and g is defined by

$$(f,g):=\int_{\Gamma^n\backslash\mathfrak{H}_n}\langle\rho'(\sqrt{\mathrm{Im}(W)})f(W),\ \rho'(\sqrt{\mathrm{Im}(W)})g(W)\rangle\det(\mathrm{Im}(W))^{-n-1}dXdY$$

if the right-hand side is convergent. Here $W=X+iY$ with real matrices $X=(x_{jh})$ and $Y=(y_{jh})$;

$$dX:=\prod_{j\le h}dx_{jh},\ dY:=\prod_{j\le h}dy_{jh};$$

the integral is taken over a fundamental domain of $\Gamma^n\backslash\mathfrak{H}_n$. We write $dW=dXdY$ when there is no fear of confudion.

Theorem 5. *Let k be an even integer, n an odd integer and $2k\ge n>2$. If*

$f\in S_k^n(\mathrm{alt}^{n-1}(V_2))$ *is an eigenform,*

$$\left(f,\ (\mathcal{D}\boldsymbol{E}_k^{2n})\left(\begin{pmatrix}-\overline{Z}^{(n)} & 0\\ 0 & *\end{pmatrix},\ \frac{\overline{s}+n}{2}\right)\right)$$
$$=2\pi^{nk-\frac{1}{2}(n-1)^2}i^{nk+n-1}\gamma(s)\Lambda(s,\ f,\ \underline{\mathrm{St}})(\iota^{-1}(f))(Z).$$

If Theorem 5 is proved, the functional equation of $\Lambda(s,\ f,\ \underline{\mathrm{St}})$is obtained from that of $\boldsymbol{E}_k^{2n}(\mathfrak{Z},\ s)$. Since it follows from Theorem 3 that the location of poles of $E_k^{2n}(\mathfrak{Z},\ s)$ is invariant under the operation of $\mathcal{D}$, its holomorphy is proved in the same way as that by Mizumoto [19, Theorem 1] (cf. Weissauer [24]). Thus we get Theorem 1.

Proof of Theorem 5. It follows from Theorem 3 that $\left(f,\ (\mathcal{D}E_k^{2n})\left(\begin{pmatrix}-\overline{Z} & 0\\ 0 & *\end{pmatrix},\ \overline{s}\right)\right)$ converges absolutely and locally uniformly for $k+2\mathrm{Re}(s)>2n+1$. We note that $\mathcal{R}(Z,\ W,\ s)$ is orthogonal to $S_k^n(\mathrm{alt}^{n-1}(V_2))$ in the variable W by the same reason as that in Klingen [15, Satz 2]. Since the Hecke operators are Hermitian operators and f is an eigenform, we have

$$\left(f,\ (\mathcal{D}E_k^{2n})\left(\begin{pmatrix}-\overline{Z} & 0\\ 0 & *\end{pmatrix},\ \overline{s}\right)\right)$$
$$=\frac{\Gamma(2k+2s+1)}{\Gamma(2k+2s-n+2)}D(k+2s,\ f)(f,\ \mathcal{P}(-\overline{Z},\ *,\ \overline{s}))$$

by the definition (1.2). If we compute the integral $(f,\ \mathcal{P}(-\overline{Z},\ *,\ \overline{s}))$ according to Klingen [14, § 1] (see also [5], [7], [23]), we obtain

$$(f,\ \mathcal{P}(-\overline{Z},\ *,\ \overline{s}))=2^{n(n-2s-k)+2}i^{nk+n-1}\psi(\iota^{-1}(f))(Z)$$

and

$$\psi=\int_{S^n}\det(1_n-S\overline{S})^{k+s-n-1}\left(\widetilde{(1_n-S\overline{S})}[{}^t\boldsymbol{p}_n]\right)dS,$$

where $\boldsymbol{p}_n^{(1,n)}:=(0,\ \cdots,\ 0,\ 1)$ and $\boldsymbol{S}^n:=\{S\in\boldsymbol{C}^{(n)}|S={}^tS,\ 1_n-\overline{S}S>0\}$. Moreover, by Hua [10, § 2.3] (see also [5], [7], [14], [23]), we get

$$\psi=\pi^{\frac{n(n+1)}{2}}\left(\frac{2k+2s-n+1}{2}\right)\frac{\Gamma(k+s-n)}{\Gamma(k+s+1)}\prod_{j=1}^{n-1}\frac{\Gamma(2k+2s-2n+1+2j)}{\Gamma(2k+2s-n+1+j)}.$$

Thus, by (1.3), we obtain

$$\left(f,\ (\mathcal{D}E_k^{2n})\left(\begin{pmatrix}-\overline{Z} & 0\\ 0 & *\end{pmatrix},\ \frac{\overline{s}+n-k}{2}\right)\right)$$
$$=2^{n(1-s)+2}i^{nk+n-1}\pi^{\frac{n(n+1)}{2}}\zeta(s+n)^{-1}\prod_{j=1}^{n}\zeta(2s+2n-2j)^{-1}$$

$$\times \frac{\Gamma(s+k)}{\Gamma(s+k-n+1)} \prod_{j=1}^{n} \frac{\Gamma(s+k-n-2+2j)}{\Gamma(s+k-1+j)} L(s, f, \underline{\mathrm{St}})(\iota^{-1}(f))(Z)$$

and Theorem 5 is proved. $\square$

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