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ON REGULAR RINGS WHOSE MAXIMAL RIGHT QUOTIENT RINGS ARE TYPE I_f

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Introduction

This paper is written about the property (DF) on regular rings whose maximal right quotient rings are Type I_f. Hereafter regular rings whose maximal right quotient rings are Type I_f are said to satisfy (∗). The property (DF) is very important property when we study on regular rings satisfying (∗), and it was treated in the paper [5] written by the first author, where (DF) for a ring R is defined as that if the direct sum of any two directly finite projective R-modules is always directly finite. In the above paper, the equivalent condition that a regular ring R of bounded index satisfies (DF) was discovered and called (♯). Stillmore, we proved that the condition (DF) is equivalent to (♯) for regular rings whose primitive factor rings are artinian in the paper [6]. Then we have the problem that (DF) is equivalent (♯) for regular rings satisfying (∗) or not, where the condition (∗) is weaker than one that primitive factor rings are artinian.

In § 2, we shall prove Theorem 2.4. This is important, and using this, Theorem 2.5 (i.e. if R is a regular ring satisfying (∗) and k is any positive integer, then kP is directly finite for every directly finite projective R-module P) is proved. Moreover, we shall solve the above problem in Theorem 2.11.

In § 3, we shall consider some applications of Theorem 2.11. We prove Theorem 3.3 that if R is a regular ring satisfying (∗) whose maximal right quotient ring of R satisfies (DF), then so does R. Though it is clear that a regular rings satisfying (∗) which has a nonzero essential socle satisfies (DF), we can prove that, for regular rings satisfying (∗), the condition having a nonzero essential socle is not equivalent to (♯) in Example 3.4. Next, we shall consider that $(\Pi_1^\infty R)/(\oplus R)$ satisfies (DF) or not for a regular ring R satisfying (∗). This problem is a generalization of Example 3.4, and we prove that, for a regular ring R of bounded index, $(\Pi_1^\infty R)/(\oplus R)$ satisfies (DF) (Theorem 3.9).

Throughout this paper, R is a ring with identity and R-modules are unitary right R-modules.

1. Definitions and notations

DEFINITION 1. A ring R is (Von Neumann) *regular* provided that for every $x \in R$ there exists $y \in R$ such that $xyx = x$.

NOTE. Every projective modules over regular rings have the exchange property.

DEFINITION 2. A module M is *directly finite* provided that M is not isomorphic to a proper direct summand of itself. If M is not directly finite, then M is said to be *directly infinite.* A ring R is said to be *directly finite* (resp. *directly infinite*) if so is R as an R-module.

DEFINITION 3. The *index* of a nilpotent element x in a ring R is the least positive integer such that $x^n = 0$ (In particular, 0 is nilpotent of index 1). The *index* of a two-sided ideal J of R is the supremum of the indices of all nilpotent elements of J.
If this supremum is finite, then J is said to have *bounded index.* If J does not have bounded index, J is said to be *index* ∞.

NOTE. Let R be a regular ring with index ∞. Then using [3, the proof of Lemma 2], there exists a family $\{A_n\}_{n=1}^{\infty}$ of independent right ideals of R such that A_n contains a direct sum of n nonzero pairwise isomorphic right ideals. Therefore R has a family $\{e_{ij}\}_{i,j=1,2,\cdots}$ of idempotents such that

$$e_{21}R \simeq e_{22}R$$
$$e_{31}R \simeq e_{32}R \simeq e_{33}R$$
$$\cdots\cdots$$

, where $e_{ij} = 0$ $(i<j)$, and $\{e_{i1}, \cdots, e_{ii}\}$ are orthogonal for all i.

DEFINITION 4. A ring R has (DF) if the direct sum of two directly finite projective R-modules is directly finite.

DEFINITION 5. A regular ring R is *abelian* provided all idempotents in R are central.

DEFINITION 6. A ring R satisfies $(*)$ if every nonzero two-sided ideal of R contains a nonzero two-sided ideal of bounded index.

DEFINITION 7. A ring R is *unit-regular* provided that for each $x \in R$ there is a unit $u \in R$ such that $xux = x$.

NOTE. Every finitely generated projective module over a unit-regular ring has the cancellation property ([2, Theorem 4.14]).

DEFINITION 8. Let e be an idempotent in a regular ring R. Then e is called an *abelian idempotent* (of R) whenever the ring eRe is abelian.

DEFINITION 9. Let e be an idempotent in a regular right self-injective ring R. Then e is *faithful* (in R) if 0 is the only central idempotent of R which is orthogonal to e. A regular right self-injective ring R is said to be *Type I* provided that it contains a faithful abelian idempotent, and R is *Type I_f* if R is Type I and directly finite.

NOTE. It is well-known from [4, Theorem 2] and [2, Lemma 7.17] that a regular ring R satisfies (*) if and only if the maximal right quotient ring of R is Type I_f.

NOTE. Let R be a regular ring satisfying (*). If P is a finitely generated projective R-module, then $\mathrm{End}_R(P)$ is a regular ring satisfying (*).

Proof. Choose a positive integer n and an idempotent matrix $e \in M_n(R)$ such that $e(nR_R) \simeq P$. Then $\mathrm{End}_R(P) \simeq eM_n(R)e$. Using [2, Corollary 10.5], we see that $eM_n(Q(R))e \simeq Q(eM_n(R)e)$ is Type I_f, where $Q(R)$ is the maximal right quotient of R. Since $eM_n(R)e \leqq_e Q(eM_n(R)e)$ as an $eM_n(R)e$-module, we have that $eM_n(R)e$ satisfies (*), and so has $\mathrm{End}_R(P)$.

NOTATIONS. Let A, B and A_i $(i \in I)$ be R-modules, and k be a positive integer. Take $x \in \Pi A_i$. Then we have some notations as following.

$A < B$; A is a submodule of B.
$A \lesssim B$; B has a submodule isomorphic to A.
$A < \oplus B$; A is a direct summand of B.
$A \lesssim \oplus B$; B has a direct summand isomorphic to A.
$A <_e B$; A is an essential submodule of B.
$A \lesssim_e B$; B has an essential submodule isomorphic to A.
kA ; the k-copies of A.
$x(i)$; the i-th component of x.
$Q(R)$; the maximal right quotient ring of R.

2. The property (DF) for regular rings satisfying (*)

Lemma 2.1 ([2, Theorem 6.6]). *Let R be a regular ring whose primitive factor rings are artinian. Then R satisfies* (*).

Lemma 2.2. *Let R be a regular ring satisfying $(*)$. Then there exist abelian regular rings $\{S_t\}_{t\in T}$ and orthogonal central idempotents $\{e_t\}_{t\in T}$ of R such that $R_R \lesssim_e [\Pi M_{n(t)}(S_t)]_R$, $\oplus M_{n(t)}(S_t) \lesssim R$* and $e_tR = M_{n(t)}(S_t)$. *Therefore $Q(R) \simeq \Pi M_{n(t)}(Q(S_t))$.*

Proof. This theorem follows from [2, Lemma 7.17 and the proof of Theorem 7.18].

Lemma 2.3. *Let R be a regular ring of bounded index and P be a finitely generated projective R-module. Then P can not contain a family $\{A_1, A_2, \cdots\}$ of nonzero finitely generated submodules such that $A_i \gtrsim A_{i+1}$ and $iA_i \lesssim P$ for each $i=1,2,\cdots$.*

Proof. By [2, Corollary 7.13], we see that $\mathrm{End}_R(P)$ has bounded index. Note the claim in the proof of [2, Theorem 6.6], and applying [5, Lemma 5] to $\mathrm{End}_R(P)$, we see that this lemma holds.

Theorem 2.4. *Let R be a regular ring satisfying $(*)$, and P be a projective R-module with a cyclic decomposition $P = \oplus_{i\in I} P_i$. Then the following conditions* (a)$\sim$(d) *are equivalent:*

(a) *P is directly infinite.*

(b) *There exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim P$.*

(c) *There exists a nonzero cyclic projective R-module X such that $X \lesssim \oplus_{i\in I-\{i_1,\cdots,i_n\}} P_i$ for any finite subset $\{i_1,\cdots,i_n\}$ of I.*

(d) *There exists a nonzero cyclic projective R-module X such that $\aleph_0 X \lesssim \oplus P$.*

Proof. It is clear that (a) $\to$ (b) and (c) $\to$ (d) $\to$ (a) hold, hence we shall prove that (b) $\to$ (c) holds. We may assume $\oplus M_{n(t)}(S_t) < R_R <_e [\Pi M_{n(t)}(S_t)]_R$ for some set of abelian regular rings $\{S_t\}_{t\in T}$ by Lemma 2.2. Now we assume that (b) holds, hence there exists a nonzero principal right ideal X of R such that $\aleph_0 X \lesssim P$. Let $\{i_1,\cdots,i_n\}$ be a subset of I and set $I' = I - \{i_1,\cdots,i_n\}$. Since $\oplus M_{n(t)}(S_t) < R_R <_e [\Pi M_{n(t)}(S_t)]_R$, there exists $t' \in T$ such that $Y = [(\Pi_{t\neq t'} 0) \times M_{n(t')}(S_t)] \cap X \neq 0$. By the property of regular ring, it is clear that Y is a principal right ideal of R. Then $\aleph_0 Y \lesssim P$, hence $Y \lesssim \oplus P$. Thus for each $i \in I$, we have decompositions $P_i = P_i^1 \oplus P_i^{(1)}$ and $Y \simeq P_{i_1}^1 \oplus \cdots \oplus P_{i_n}^1 \oplus (\oplus_{i\in I'} P_i')$. Set $(\Pi_{t\neq t'} 0) \times M_{n(t')}(S_{t'}) = S$, and then there exists a central idempotent e in R such that $eR = S$.
Note that S is a regular ring of bounded index. It is clear that

$$Y \otimes_R S_S \simeq (P_{i_1}^1 \otimes_R S) \oplus \cdots \oplus (P_{i_n}^1 \otimes_R S) \oplus [\oplus_{i\in I'} (P_i^1 \otimes_R S_s)]$$

and $2Y \otimes_R S_S \lesssim P \otimes_R S_S$. Since S is unit-regular, $Y \otimes_R S_S$ has the cancellation property. Hence

$$Y\otimes_R S_S \lesssim \oplus(P_{i_1}^{(1)}\otimes_R S)\oplus\cdots\oplus(P_{i_n}^{(1)}\otimes_R S)\oplus[\oplus_{i\in I'}(P_i^{(1)}\otimes_R S_s)]$$

Thus for each i, we obtain that $P_i^{(1)}\otimes_R S_S=\bar{P}_i^2\oplus\bar{P}_i^{(2)}$ for each $i\in I$ and

$$Y\otimes_R S_S\simeq\bar{P}_{i_1}^2\oplus\cdots\oplus\bar{P}_{i_n}^2\oplus(\oplus_{i\in I'}\bar{P}_i^2).$$

Continuing this procedure, we have that $\bar{P}_i^{(m)}=\bar{P}_i^{m+1}\oplus\bar{P}_i^{(m+1)}$ and

$$Y\otimes_R S_S\simeq\bar{P}_{i_1}^{m+1}\oplus\cdots\oplus\bar{P}_{i_n}^{m+1}\oplus(\oplus_{i\in I'}\bar{P}_i^{m+1})$$

for each $i\in I$ and each positive integer m.

Now we set $A_m=\bar{P}_{i_1}^m\oplus\cdots\oplus\bar{P}_{i_n}^m$, where $A_1=(P_{i_1}^1\otimes_R S)\oplus\cdots\oplus(P_{i_n}^1\otimes_R S)$. Then $A_1\lesssim\oplus A_2\oplus(\oplus_{i\in I'}\bar{P}_i^2)$, hence there exist a direct summand B_2 of A_2 and a direct summand Q_i^2 of $\bar{P}_i^2$ such that $A_1\simeq B_2\oplus(\oplus_{i\in I'}Q_i^2)$. Continuing this procedure, we obtain a family $\{B_1,B_2,\cdots\}$ $(A_1=B_1)$ of finitely generated projective S-submodules of $(P_{i_1}\oplus\cdots\oplus P_{i_n})\otimes_R S$ such that $B_m\gtrsim B_{m+1}$ and $mB_m\lesssim nS$ for all m. By Lemma 2.3, there exists a positive integer k such that $B_m=0$ for all m $(\geqq k)$. Thus we have that $A_1\simeq(\oplus_{i\in I'}Q_i^2)\oplus\cdots\oplus(\oplus_{i\in I'}Q_i^k)$ and $Y\otimes_R S_S\simeq(\oplus_{i\in I'}Q_i^1)\oplus\cdots\oplus(\oplus_{i\in I'}Q_i^k)$. Noting that $0\neq Y<S$, we have that $Y_R\lesssim\oplus_{i\in I'}P_i$.

Corollary 2.5. *Let R be a regular ring satisfying* (∗). *Then R contains no infinite direct sums of nonzero pairwise isomorphic right ideals. Hence R is directly finite.*

Proof. From Lemma 2.2, we may assume that, $R_R<_e\Pi M_{n(t)}(S_t)$ for some abelian regular rings $\{S_t\}_{t\in T}$. Set $T=\Pi M_{n(t)}(S_t)$. Now we assume that R contains a direct sum of nonzero pairwise isomophic right ideals, and so there exists a nonzero idempotent e of R such that $0\neq\aleph_0(eR)\lesssim R_R$. Then $\aleph_0(eR)\otimes_R T\lesssim R\otimes_R T$, and so $\aleph_0(eT)\lesssim T$, which contradicts to Theorem 2.4 because T is a directly finite regular ring satisfying (∗).

Theorem 2.6. *Let R be a regular ring satisfying* (∗) *and k be a positive integer. If P is a directly finite projective R-module, then so is kP.*

Proof. We may assume that $\oplus M_{n(t)}(S_t)<R_R<_e[\Pi M_{n(t)}(S_t)]_R$ for some abelian regular rings $\{S_t\}_{t\in T}$, and let $P=\oplus_{i\in I}P_i$ be a cyclic decomposition of P. It is sufficient to prove that this theorem holds in case $k=2$. Assume that $2P$ is directly infinite. Then Theorem 2.4 follows that there exists a nonzero principal right ideal X of R such that $X\lesssim\oplus_{i\in I-\{i_1,\cdots,i_n\}}2P_i$ for any finite subset $\{i_1,\cdots,i_n\}$ of I. By the proof of Theorem 2.4, we may assume that exists t' of T such that $X<(\Pi_{t\neq t'}0)\times M_{n(t')}(S_{t'})=S$. For any finite subset $\{i_1,\cdots,i_n\}$ of I, we have that $0\neq X\otimes_R S_S\lesssim\oplus_{i\in I-\{i_1,\cdots,i_n\}}(2P_i\otimes_R S)$. Since S is a regular ring of bounded index, we see that $2(P\otimes_R S)_S$ is directly infinite by Theorem 2.4 and so $(P\otimes_R S)_S$ is directly infinite by [5, Theorem 4]. Moreover, using Theorem 2.4 again, there exists a nonzero principal right ideal Y of S such that $Y \lesssim \oplus_{i\in I-\{i_1,\cdots,i_n\}}(P_i \otimes_R S_S)$ for any finite subset $\{i_1,\cdots,i_n\}$ of I. Considering Y as an R-module, $O \neq Y_R \lesssim \oplus_{I-\{i_1,\cdots,i_n\}} P_i$. Therefore P is directly infinite, and so this theorem is complete.

Corollary 2.7. *Let R be a regular ring satisfying $(*)$. Then every finitely generated projective R-module is directly finite.*

Proof. It is clear by Corollary 2.5 and Theorem 2.6.

Corollary 2.8. *Let R be a regular ring satisfying $(*)$.*

(a) *$M_n(R)$ is directly finite for all positive integer n, and so $M_n(R)$ contains no infinite direct sums of nonzero pairwise isomorphic right ideals.*

(b) *If P and Q are finitely generated projective R-modules, then $P \oplus Q$ is directly finite.*

Proof. (a) R is a regular ring satisfying $(*)$, and hence so is $M_n(R)$. Therefore Corollary 2.5 shows that (a) holds. (b) follows from Corollary 2.7.

NOTE. In [1], Chuang and Lee have shown that there exists a regular ring satisfying $(*)$ which is not unit-regular. Our Corollary 2.8 gives a partially solution for open problems 1 and 9 in Goodearl's book ([2]).

DEFINITION. Let R be a regular ring and P be a projective R-module. We call that P satisfies (#) provided that, for each nozero finitely generated submodule I of P and any family $\{A_1, B_1, \cdots\}$ of submodules of P with

$$I = A_1 \oplus B_1,$$
$$A_i = A_{2i} \oplus B_{2i},$$
$$B_i = A_{2i+1} \oplus B_{2i+1} \qquad \text{for each } i = 1, 2, \cdots,$$

there exists a nonzero projective R-module X such that $X \lesssim \oplus_{i=m}^{\infty} A_i$ or $X \lesssim \oplus_{i=m}^{\infty} B$ for any positive integer m.

Lemma 2.9 ([5, Lemma 6]). *Let P be a nonzero finitely generated projective module over a regular ring R, and set $T = \mathrm{End}_R(P)$. Then the following conditions are equivalent:*

(a) *P satisfies (#).*

(b) *T satisfies (#) as a T-module.*

Lemma 2.10 ([5, Lemma 7]). *Let P be a nonzero finitely generated projective* *module over a regular ring R, and set* $T=\mathrm{End}_R(P)$. *Then the following conditions are equivalent*:

(a) *R satisfies* (♯) *as an R-module.*
(b) *All nonzero finitely generated projective R-modules satisfy* (♯).
(c) *For any positive integer k, kR satisfies* (♯).
(d) *There exists a positive integer k such that kR satisfies* (♯).

Theorem 2.11. *Let R be a regular ring satisfying* (∗). *Then the following conditions are equivalent*:

(a) *R has* (DF).
(b) *R satisfies* (♯) *as an R-module.*
(c) *For any nonzero finitely generated projective R-module P*, $\mathrm{End}_R(P)$ *has* (DF).
(d) *For any positive integer k*, $M_k(R)$ *has* (DF).
(e) *There exists a positive integer k such that* $M_k(R)$ *satisfies* (DF).

Proof. Note that $\mathrm{End}_R(P)$ is a regular ring with (∗). [5, Theorem 8] was proved only using [5, Theorem 2]. Now [5, Theorem 2] holds on a regular ring satisfying (∗) by Theorem 2.4. Hence we see that this theorem holds under this condition using the similar proof of [5, Theorem 8] (Note that the unit-regularlity is not needed).

3. Some applications

Lemma 3.1. *Let R be a regular ring satisfying* (∗), *and let* $\{e_i\}$ *be a set of nonzero orthogonal central idempotents of R such that* $\oplus e_iR_R <_e R_R$. *Then R has* (DF) *if and only if* e_iR *has* (DF) *for all i.*

Proof. Note that e_iR is a ring direct summand of R. It is clear from Theorem 2.11 that "only if" part holds. We shall prove that "if" part holds. Let I be a nonzero direct summand of R, and so $e_iR\cap I\neq 0$ for some i. Setting $J=e_iR\cap I$, J is a principal right ideal of both R and e_iR. We consider decompositions

$$I=A_1\oplus B_1$$
$$A_j=A_{2j}\oplus B_{2j}$$
$$B_j=A_{2j+1}\oplus B_{2j+1} \qquad \text{for each } j=1,2,\cdots,$$

and so there exist decompositions of J such that

$$J=C_1\oplus D_1$$
$$C_j=C_{2j}\oplus D_{2j}$$

$$D_j = C_{2j+1} \oplus D_{2j+1}$$
$$C_j \lesssim \oplus A_j \quad \text{and} \quad D_j \lesssim \oplus B_j \qquad \text{for each } j=1,2,\cdots.$$

By the assumption, there exists a nonzero cyclic projective e_iR-module X such that $X \lesssim \oplus_{j=m}^{\infty} C_j$ or $X \lesssim \oplus_{j=m}^{\infty} D_j$ for each positive integer m. Hence $X \otimes_R e_iR \lesssim \oplus_{j=m}^{\infty}(C_j \otimes_R e_iR)$ or $X \otimes_R e_iR \lesssim \oplus_{j=m}^{\infty}(D_j \otimes_R e_iR)$. Note that $\oplus_{j=m}^{\infty}(C_j \otimes_R e_iR) \lesssim \oplus_{j=m}^{\infty} A_j$ and $\oplus_{j=m}^{\infty}(D_j \otimes_R e_iR) \lesssim \oplus_{j=m}^{\infty} B_j$. Therefore $X \otimes_R e_iR \lesssim \oplus_{j=m}^{\infty} A_j$ or $X \otimes_R e_iR \lesssim \oplus_{j=m}^{\infty} B_j$. Since $X \otimes_R e_iR \neq 0$, this lemma has proved by Theorem 2.11.

Lemma 3.2 ([6, Proposition 2.1]). *Let R be an abelian regular ring. If $Q(R)$ has* (DF), *then so has R.*

Theorem 3.3. *Let R be a regular ring satisfying* $(*)$. *If $Q(R)$ has* (DF), *then so does R.*

Proof. By Lemma 2.2, we may assume that there exists a set $\{S_t\}$ of abelian regular rings such that $R_R <_e [\Pi M_{n(t)}(S_t)]$. Then $Q(R) = \Pi M_{n(t)} Q(S_t)$. Assume that $Q(R)$ has (DF), then so does $M_{n(t)}(Q(S_t))$ for all t by Lemma 3.1. Moreover, Theorem 2.11 shows that $Q(S_t)$ also has (DF), hence so has S_t by Lemma 3.2. Thus $M_{n(t)}(S_t)$ also has (DF) by Theorem 2.11. There exists the set $\{e_t\}$ of orthogonal central idempotents of R such that $e_tR = M_{n(t)}(S_t) \times [\Pi_{t \neq t'} 0]$ and $\oplus e_tR <_e R_R$. Therefore R has (DF) by Lemma 3.1.

Now we shall give an example of a regular ring with a zero socle satisfying $(*)$ which has (DF), as following.

EXAMPLE 3.4. Let F be a field, and set $R = \Pi_{i=1}^{\infty} F_i (F_i = F)$ and $\bar{R} = R/\mathrm{soc}(R)$. Then $\bar{R}$ is a regular ring satisfying $(*)$ which has (DF).

Proof. Since it is clear that $\bar{R}$ is a regular ring satisfying $(*)$, we shall prove that $\bar{R}$ has (DF) using Theorem 2.11. Let Ψ be the natural map from R to $\bar{R}$, and let I be a nonzero direct summand of $\bar{R}$ with following decompositions:

$$I = A_1 \oplus B_1$$
$$A_i = A_{2i} \oplus B_{2i}$$
$$B_i = A_{2i+1} \oplus B_{2i+1} \qquad \text{for } i=1,2,\cdots.$$

Now assume that there does not exist $\{C_j\}$ ($C_j = A_j$ for some i) which is an infinite subset of $\{A_i\}_{i=1}^{\infty}$ such that $C_j > C_{j+1}$ and $C_j \neq 0$ for all j. Let $\{D_p\}$ ($D_p = A_i$ for some i) be an infinite decreasing sequence of $\{A_i\}$, and so there exists a positive integer p' such that $D_p = 0$ ($p \leq p'$). Hence $0 = D_{p'} = A_{i_1}$ for some

i_1. Thus $B_{i_1}\neq 0$. Next, we take $\{E_q\}$ ($E_q=A_i$ for some i) which is an infinite decreasing sequence of $\{A_i\}$, where $E_q<B_{i_1}$ and $B_{k_q}<B_{i_1}$ ($A_{k_q}=E_q$) for all positive integer q. Similarly, there exists a positive integer q' such that $E_q=0$ ($q'\leqq q$). Hence there exists a positive integer i_2 ($i_2>i_1$) such that $E_{q'}=A_{i_2}=0$. Therefore $B_{i_2}\neq 0$ and $B_{i_1}>B_{i_2}$. Continuing this procedure, we can get an infinite set $\{B_{i_k}\}$ such that $\{B_i\}\supset\{B_{i_k}\}$ and $B_{i_k}\neq 0$ for all k. From the above, we may assume that there exists an infinite decreasing sequence $\{C_j\}$ such that $\{A_i\}\supset\{C_j\}$, $C_j>C_{j+1}$ and $C_j\neq 0$ for all j.
We have a set $\{e_j\}$ of idempotents of R such that $\Psi(e_jR)=C_j$ and $e_jR\geqq e_{j+1}R$ for all j. We take an idempotent $f_1(\in e_1R)$ with $\dim_F(f_1R)=1$. Next we take an idempotent $f_2(\in e_2R)$ such that $\dim_F(f_2R)=1$ and $f_1f_2=0$. Continuing this procedure, we can take a set $\{f_j\}$ of orthogonal idempotents of R. Set $e=\vee f_j$, and then $\Psi(e)\neq 0$. We have that $eR=J\oplus(eR\cap e_jR)$ and $J<\oplus F_i$ for some right ideal J. Noting that $J\otimes_R\bar{R}=0$, we have that

$$\begin{aligned}0\neq\Psi(e)\bar{R}&\simeq eR\otimes_R\bar{R}\\&\simeq[J\oplus(eR\cap e_jR)]\otimes_R\bar{R}\\&\lesssim e_jR\otimes_R\bar{R}\\&\simeq C_j\qquad\text{for all } j.\end{aligned}$$

Therefore $0\neq\Psi(e)\bar{R}\lesssim\oplus_{i=m}^{\infty}A_i$ for any positive integer m. Hence $\bar{R}$ has (DF) by Theorem 2.11.

By Example 3.4, we have a problem that, for any regular ring S, $R=(\Pi_1^{\infty}S)/(\oplus S)$ satisfies (DF) or not. Example 3.5 shows that, even if S satisfies (*), R does not satisfy (*). Therefore we shall give the necessary and sufficient condition for that R satisfies (*), and we solve the above problem under this condition.

EXAMPLE 3.5. Let F be a field and set $S=\Pi_{n=1}^{\infty}M_n(F)$, $\bar{S}=S/(\oplus M_n(F))$, $T=\Pi_{i=1}^{\infty}S_i$ ($S_i=S$) and $R=T/(\oplus S_i)$. Then S satisfies (*), but R does not satisfy (*).

Proof. It is clear that S satisfies (*). Therefore we shall show that R does not satisfy (*). Set a central idempotent e ($\in T$) as following;

$$e(n)=(\underbrace{0,\cdots,0}_{n-1},\underbrace{\begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix}}_{n},0,0,\cdots),$$

where $e(n)\in S_n$.
Let Φ be the natural map from S to $\bar{S}$, and ρ be the natural map from T to R. Set $\Psi=\rho|_{eT}$. Noting that $e(n)S_n\simeq M_n(F)$, we have $eT\simeq S$. Hence there exists a ring

isomorphism κ from eT to S. Now, we define a ring homomorphism α from $\Psi(e)R$ to $\bar{S}$ as following; for each x $(\in \Psi(e)R)$, we take any element y of $\Psi^{-1}(x)$ and set $\alpha(x)=\Phi\kappa(y)$.

$$\begin{array}{ccc} \Psi(e)R & \xrightarrow{\alpha} & \bar{S} \\ \uparrow^{\psi} & & \uparrow^{\Phi} \\ eT & \xrightarrow{\kappa} & S \end{array}$$

Similarly we define a ring homomorphiism β from $\bar{S}$ to $\Psi(e)R$. Then we have that $\beta\alpha=1_{\psi(e)R}$ and $\alpha\beta=1_{\bar{S}}$. Hence α and β are isomorphic. Therefore $\Psi(e)R\simeq\bar{S}$. Let I be a nonzero two-sided ideal of $\bar{S}$, and so $I=J/(\oplus S_i)$ for some nonzero two-sided ideal of S which contains $\oplus S_i$. There exists $0\neq x\in J-(\oplus S_i)$ with $x(i)\neq 0$ for almost all i. Since $S_ix(i)S_i=M_i(F)$ has index i, there exists a nonzero central idempotent $e(i)$ of $M_i(F)$ which $S_ix(i)S_i$ has index i. Therefore SxS does not have bounded index, and so does not $J/(\oplus S_i)$. Therefore $\bar{S}$ does not satisfy (*), and hence so does not $\Psi(e)R$. Thus R does not satisfy (*).

Lemma 3.6. *Let R be a ring, and e, f be idempotents of R. Then $eR\simeq fR$ if and only if there exist u and v of R such that $vu=e$ and $uv=f$.*

Lemma 3.7. *Let S be a regular ring which has index ∞, and set $R=(\Pi_{i=1}^{\infty}S_i)/(\oplus S_i)$ $(S_i=S)$. Then R has an infinite direct sum of nonzero pairwise isomorphic right ideals.*

Proof. Let Ψ be the natural map from $\Pi_{i=1}^{\infty}S_i$ to R. Since S has index ∞, there exists a set of idempotents $\{e_{ij}\}_{i,j=1,2,\cdots}$ as following:

$$\begin{array}{l} e_{11}S \\ e_{21}S\simeq e_{22}S \\ e_{31}S\simeq e_{32}S\simeq e_{33}S \\ \quad\cdots\cdots \end{array}$$

, where $e_{ij}=0$ $(i<j)$ and $\{e_{i1},\cdots,e_{ii}\}$ are nonzero orthogonal for all i. For all positive integer m, we take idempotents $\{f_m\}$ such that $f_m(k)=e_{km}$ for all positive integer k. Since $e_{k1}S\simeq e_{k2}S$ for all k, there exist u_k and v_k of S such that $u_kv_k=e_{k2}$ and $v_ku_k=e_{k1}$ by Lemma 3.6. Set u and v of $\Pi_{i=1}^{\infty}S_i$ such that $u(k)=u_k$ and $v(k)=v_k$. Then $uv=f_2$ and $vu=f_1-e$, where e is an idempotent with $e(1)=e_{11}$ and $e(k)=0$ $(k\neq 1)$. Hence $(f_1-e)(\Pi S_i)\simeq f_2(\Pi S_i)$ and $(f_1-e)(\Pi S_i)\cap f_2(\Pi S_i)=0$. Therefore we see from Lemma 3.6 that $\Psi(f_1-e)R\simeq\Psi(f_2)R$ and $\Psi(f_1-e)R\cap\Psi(f_2)R=0$. Since $\Psi(f_1-e)R=\Psi(f_1)R$, we have that $\Psi(f_1)R\simeq\Psi(f_2)R$ and $\Psi(f_1)R\cap\Psi(f_2)R$

$=0$. Continuing this produre, for all positive integers i and j, $\Psi(f_i)R \simeq \Psi(f_j)R$ and $\Psi(f_i) \cap \Psi(f_j)R = 0$ $(i \neq j)$. Thus R has an infinite direct sum of nonzero pairwise isomorphic right ideals.

Theorem 3.8. *Let S be a regular ring, and set $R = (\Pi_{i=1}^{\infty} S_i)/(\oplus S_i)$ $(S_i = S)$. Then the following conditions are equivalent:*

(a) *R satisfies* $(*)$.
(b) *R is a regular ring whose primitive factor rings are artinian.*
(c) *R has bounded index.*
(d) *R contains no infinite direct sums of nonzero pairwise isomorphic right ideals.*
(e) *S has bounded index.*

Proof. It is clear by Lemma 3.7 that (d) $\rightarrow$ (e) $\rightarrow$ (c) $\rightarrow$ (b) $\rightarrow$ (a) hold. (a) $\rightarrow$ (d) follows from Corollary 2.5. Therefore this theorem is complete.

Theorem 3.9. *Let S be a regular ring of bounded index. Set $R = (\Pi_{n=1}^{\infty} S_n)/(\oplus S_n)$ $(S_n = S)$. Then R has* (DF).

Proof. Set $\Pi_{n=1}^{\infty} S_n = T$, and let Ψ be the natural map from T to R. Let I be a nonzero direct summand of R with following decompositions:

$$I = A_1 \oplus B_1$$
$$A_i = A_{2i} \oplus B_{2i}$$
$$B_i = A_{2i+1} \oplus B_{2i+1} \qquad \text{for } i = 1, 2, \cdots.$$

Similarly to the proof of Example 3.4, we may assume that there exists an infinite subset $\{C_j\}$ of $\{A_i\}$ ($C_j = A_i$ for some i) such that $C_j > C_{j+1}$ and $C_j \neq 0$ for all positive integer j. We have the set of idempotents $\{e_j\}$ of T such that $\Psi(e_jT) = C_j$ and $e_jT > e_{j+1}T$. Set $J_n = S_n \times (\Pi_{i \neq n} 0)$. Then, $J_{n_1} \cap e_1 T \neq 0$ for some positive integer n_1. There exists a nonzero idempotent $f_1 \in T$ such that $f_1 T = J_{n_1} \cap e_1 T$. Next we have a nonzero idempotent $f_2 \in T$ for some n_2 $(> n_1)$ such that $f_2 R = J_{n_2} \cap e_2 R$. Continuing this procedure, we have the set $\{f_j\}$ of orthogonal idempotents of T. Now, we set an idempotent g of T as following;

$$g(n_j) = f_j(n_j) = e_j(n_j)$$
$$g(k) = 0 \ \ (k \notin \{n_j\}).$$

Put $K_j = f_1 T \oplus \cdots \oplus f_{j-1} T$ for all j. Then $gT = K_j \oplus (gT \cap e_j T)$. Noting $K_j \otimes_T R = 0$, we have that

$$
\begin{aligned}
0 \neq \Psi(g)R &\simeq gT \otimes_T R \\
&\simeq [K_j \oplus (gT \cap e_jT)] \otimes_T R \\
&\simeq (gT \cap e_jT) \otimes_T R \\
&\lesssim e_jT \otimes_T R \\
&\simeq C_j \qquad \text{for all } j.
\end{aligned}
$$

From the above, we have that $\Psi(g)R \lesssim \oplus_{i=m}^{\infty} A_i$ for any positive integer m. Therefore R has (DF) by Theorem 2.11.

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