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REMARKS ON AN EXCLUSIVE EXTENSION GENERATED BY A SUPER-PRIMITIVE ELEMENT

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Throughout this paper, all fields, rings and algebras are assumed to be commutative with unity. Our special notations are indicated below, and our general reference for unexplained technical terms is [3].

Let R be a Noetherian domain and K its quotient field. Let α be an algebraic element over K with the minimal polynomial $\varphi_\alpha(X)=X^d+\eta_1X^{d-1}+\cdots+\eta_d$. In [6], we have shown that if $R[\alpha]\cap K=R$ then $\bigcap_{i=1}^{d-1}I_{\eta_i}\subseteq I_{\eta_d}$. Our objective of this paper is to show the converse of this result under certain assumptions, which will be established in Theorem 5.

In what follows, we use the following notations unless otherwise sepecified:

R: a Noetherian integral domain,
$K:=K(R)$: the quotient field of R,
L: an algebraic field extension of K,
α: a non-zero element of L,
$d=[K(\alpha):K]$,
$\varphi_\alpha(X)=X^d+\eta_1X^{d-1}+\cdots+\eta_d$, the minimal polynomial of α over K.
$I_{[\alpha]}:=\bigcap_{i=1}^{d}(R:_R\eta_i)$, which is an ideal of R.
$I_a:=R:_RaR$ for $a\in K$.

It is clear that for $a\in K$, $I_{[a]}=I_a$ from difinitions.

$J_{[\alpha]}:=I_{[\alpha]}(1,\eta_1,\cdots,\eta_d)$,
$\tilde{J}_{[\alpha]}:=I_{[\alpha]}(1,\eta_1,\cdots,\eta_{d-1})$.

We also use the following standard notation :

$Dp_1(R):=\{p\in Spec(R)|depth R_p=1\}$.

Let R be a Noetherian domain and K its quotient field. Take an element α in an field extension of K. When $R[\alpha]\cap K=R$, we say that α is an *exclusive* element over R and that $R[\alpha]$ is an *exclusive extension* of R. Let $\pi:R[X]\to R[\alpha]$ be the R-algebra homomorphism sending X to α. The element α is called an *anti-integral* element of degree d over R if $Ker\,\pi=I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an *anti-integral extension* of R. For $f(X)\in R[X]$, let $C(f(X))$ denote the ideal generated by the coefficients of $f(X)$, that is, the content ideal of $f(X)$. Note that $J_{[\alpha]}:=I_{[\alpha]}C(\varphi_{\alpha}(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called a *super-primitive* element of degree d over R if $J_{[\alpha]}\nsubseteq p$ for all primes p of depth one. It is known that a super-primitive element is an anti-integral element (See [5] for detail).

A non-zero element α in L is called *co-monic* if $1/\alpha$ is integral over R and a polynomial $f(X)\in R[X]$ is *co-monic* if the constant term of $f(X)$ is 1.

Proposition 1. *Assumu that α is anti-integral over R of degree d. Then the following conditions are equivalent:*

(i) *α is co-monic;*
(ii) $\eta_d I_{[\alpha]}=R$;
(iii) *there exists $a\in R$ such that $\eta_d=1/a$ and $I_{[\alpha]}=aR$;*
(iv) *there exists a co-monic polynomial $g(X)\in R[X]$ of degree d with $g(\alpha)=0$;*
(v) *$R[1/\alpha]$ is a free R-module of rank d.*

Proof. (i)$\Rightarrow$(ii): Take a co-monic polynomial $f(X)$ in $R[X]$ satisfying $f(\alpha)=0$. Then $f(X)\in I_{[\alpha]}\varphi_{\alpha}(X)R[X]$ and hence $f(0)=1\in\eta_d I_{[\alpha]}\subseteq R$. So we have $\eta_d I_{[\alpha]}=R$.
(ii)$\Rightarrow$(iii): There exists $a\in I_{[\alpha]}$ such that $\eta_d a=1$. Hence $\eta_d=1/a$ and $I_{[\alpha]}=aR$.
(iii)$\Rightarrow$(iv): Since $I_{[\alpha]}=aR$, $g(X):=a\varphi_{\alpha}(X)\in R[X]$ is a required polynomial.
(iv)$\Rightarrow$(v): The polynomial $f(Y):=X^{-d}g(X)$ in $R[Y]$ with $Y=1/X$ is monic in Y of degree d and satisfies $f(1/\alpha)=0$. Hence $R[1/\alpha]$ is a free R-module of rank d because $1/\alpha$ is degree d.
(v)$\Rightarrow$(i) is obvious. □

Recall the following result shown in [1, Theorem 7]:

Lemma 2. *Assume that α is anti-integral over R of degree d. Let $\Delta_{R[\alpha]/R}:=\{p\in Spec(R)|pR[\alpha]=R[\alpha]\}$ and $\Gamma_{J_{[\alpha]}}:=\{p\in Spec(R)|p+J_{[\alpha]}\}=R$. Then $\Delta_{R[\alpha]/R}=V(\tilde{J}_{[\alpha]})\cap\Gamma_{J_{[\alpha]}}$, where $V(\tilde{J}_{[\alpha]})$ denotes $\{p\in Spec(R)|\tilde{J}_{[\alpha]}\subseteq p\}$.*

REMARK 1. Under the assumptions in Lemma 2, if $\tilde{J}_{[\alpha]}=R$ or *grade* $\tilde{J}_{[\alpha]}>1$ then $pR[\alpha]\neq R[\alpha]$ for all $p\in Dp_1(R)$.

Lemma 3. *Assume that α is super-primitive over R. The following statements are equivalent:*

(i) *$\tilde{J}_{[\alpha]}=R$ or grade $\tilde{J}_{[\alpha]}>1$ (i.e., $\tilde{J}_{[\alpha]}$ contains an R-regular sequence of length >1; see [3] for the definition);*
(ii) $\bigcap_{i=1}^{d-1}I_{\eta_i}\subseteq I_{\eta_d}$.

Proof. (i) $\Rightarrow$ (ii): First consider the case $\tilde{J}_{[\alpha]} \neq R$. Suppose that there exists $a \in \bigcap_{i=1}^{d-1} I_{\eta_i}$ but $a \notin I_{\eta_d}$. Then $\beta := a\eta_d \notin R$. Since $\beta I_{[\alpha]} = a(\eta_d I_{[\alpha]}) \subseteq R$ and $\beta\eta_i I_{[\alpha]} = (a\eta_i)\eta_d I_{[\alpha]} \subseteq R$ $(i=1,\cdots,d-1)$, we heve $\beta I_{[\alpha]}(1,\eta_1,\cdots,\eta_{d-1}) \subseteq R$. Since *grade* $\tilde{J}_{[\alpha]} =$ *grade* $I_{[\alpha]}(1,\eta_1,\cdots,\eta_{d-1}) > 1$, $\beta R_p = \beta(I_{[\alpha]}(1,\eta_1,\cdots,\eta_{d-1}))_p \subseteq R_p$ for all $p \in Dp_1(R)$. Hence $\beta \in \bigcap_{Dp_1(R)} R_p = R$, which is a contradiction. Thus $\bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$. Second, suppose that $\tilde{J}_{[\alpha]} = R$. By the assumption, there is an equation $1 = a_0 + a_1\eta_1 + \cdots + a_{d-1}\eta_{d-1}$ with $a_i \in I_{[\alpha]}$. So $\eta_d = a_0\eta_d + (a_1\eta_d)\eta_1 + \cdots + (a_{d-1}\eta_d)\eta_{d-1}$. Take $x \in \bigcap_{i=1}^{d-1} I_{\eta_i}$. Then $x\eta_d = xa_0\eta_d + (a_1\eta_d)x\eta_1 + \cdots + (a_{d-1}\eta_d)x\eta_{d-1} \in R$, and hence $x \in I_{\eta_d}$. Hence $\bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$.

(ii) $\Rightarrow$ (i): We may assume that $\tilde{J}_{[\alpha]} = I_{[\alpha]}(1,\eta_1,\cdots,\eta_{d-1}) \neq R$. Suppose that *grade* $\tilde{J}_{[\alpha]}$ $=$ *grande* $I_{[\alpha]}(1,\eta_1,\cdots,\eta_{d-1}) = 1$. Then there exists $p \in Dp_1(R)$ such that $I_{[\alpha]}(1,\eta_1,\cdots,\eta_{d-1}) \subseteq p$. Since $p \in Dp_1(R)$, there exists $\beta \in K$ such that $\beta \notin R$ and $I_\beta = p$. Then $\beta I_{[\alpha]}(1,\eta_1,\cdots,\eta_{d-1}) \subseteq R$. Thus $(I_{[\alpha]}\beta)\eta_i \subseteq R$ for $i=1,\cdots,d-1$ and $I_{[\alpha]}\beta \subseteq R$, which yield $I_{[\alpha]}\beta \subseteq \bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$. Hence $I_{[\alpha]}\beta\eta_d \subseteq R$, which shows that $J_{[\alpha]}\beta \subseteq R$. But since α is super-primitive over R, $J_{[\alpha]} \nsubseteq p$ for all $p \in Dp_1(R)$. Thus $\beta \in R$, which is a contradiciton. So we conclude that *grade* $\tilde{J}_{[\alpha]} > 1$. □

Proposition 4. *Assume that α is super-primitive and co-monic over R. If $\bigcap_{i=1}^{d-1} I_{\eta_i} \subseteq I_{\eta_d}$, then $R[\alpha] \cap K = R$, i.e., α is exclusive.*

Proof. Suppose that $R[\alpha] \cap K \neq R$. Then there exists $\beta \in R[\alpha] \cap K \backslash R$. Write $\beta = c_0 + c_1\alpha + \cdots + c_n\alpha^n$ with $c_i \in R$. We shall show that $\alpha \in I_\beta R[\alpha]$. Since $\beta, c_0, \cdots, c_n \in K$ and α is an algebraic element of degree d over K, we have $n \geq d$. From this we have:

$$(1) \qquad (1/\alpha)^n\beta = c_0(1/\alpha)^n + c_1(1/\alpha)^{n-1} + \cdots + c_n \in R[1/\alpha].$$

Since α is co-monic, $R[1/\alpha]$ is a free R-module of rank d by Proposition 1. So we rewrite (1) and have:

$$(1/\alpha)^n\beta = b_0(1/\alpha)^{d-1} + b_1(1/\alpha)^{d-2} + \cdots + b_{d-1}$$

with $b_i \in R$. Hence we obtain:

$$(2) \qquad (1/\alpha)^n = \beta^{-1}b_0(1/\alpha)^{d-1} + \beta^{-1}b_1(1/\alpha)^{d-2} + \cdots + \beta^{-1}b_{d-1}.$$

On the other hand, since $(1/\alpha)^n \in R[1/\alpha]$, we have:

$$(3) \qquad (1/\alpha)^n = a_0(1/\alpha)^{d-1} + a_1(1/\alpha)^{d-2} + \cdots + a_{d-1} \text{ with } a_i \in R.$$

Comparing the coefficients of (2) with those of (3), we have $a_i = \beta^{-1}b_i$ $(0 \leq i \leq d-1)$, and hence $\beta a_i = b_i \in R$. Thus $(a_0, a_1, \cdots, a_{d-1})R \subseteq I_\beta$. It is obvious that $\alpha \in (a_0, a_1, \cdots, a_{d-1})R[\alpha] \subseteq I_\beta R[\alpha]$ by (3). Take $p \in Dp_1(R)$ such that $I_\beta \subseteq p$. By (3), $1/\alpha = a_0\alpha^{n-d} + a_1\alpha^{n-d+1} + \cdots + a_{d-1}\alpha^{n-1}$. So we get $1/\alpha \in pR[\alpha]$. Thus $pR[\alpha]$

contains α, $1/\alpha$. So we have $pR[\alpha]=R[\alpha]$, which is a contradiciton because of Remark 1 and Lemma 3. Therefore $R[\alpha]\cap K=R$. □

REMARK 2. In [6], we showed that if α is super-primitive over R with $\eta_d\in R$ then α is exclusive.

In the next theorem, the implication (i) ⇒ (ii) has been shown in [6, Propositon 1] without any assumptions.

Theorem 5. *Assume that R contains an infinite field k and that α is super-primitive over R. Then the following atatements are equivalent*:

(i) α *is exclusive over* R;
(ii) $\bigcap_{i=1}^{d-1} I_{\eta_i}\subseteq I_{\eta_d}$;
(iii) *grade* $\tilde{J}_{[\alpha]}>1$ *or* $\tilde{J}_{[\alpha]}=R$.

Proof. The implication (i) ⇒ (ii) was shown in [6, Propositon 1]. (ii) ⇔ (iii) follows from Lemma 3. So we have only to prove the implication (ii) ⇒ (i). Take $p\in Dp_1(R)$. If $I_{[\alpha]}\nsubseteq p$, then α is integral over R_p and hence $R[\alpha]\cap K\subseteq R_p[\alpha]\cap K=R_p$. Consider the case $I_{[\alpha]}\subseteq p$. Since α is super-primitive, $J_{[\alpha]}\nsubseteq p$. Since k is infinite, there exists $\lambda\in k$ such that $\alpha-\lambda$ is co-monic over R_p. Moreover $\alpha-\lambda$ is super-primitive by [5, (1.14)]. Hence by Propositon 4, we have $R[\alpha]\cap K=R[\alpha-\lambda]\cap K\subseteq R_p[\alpha-\lambda]\cap K=R_p[\alpha]\cap K=R_p$. Thus $R[\alpha]\cap K\subseteq\bigcap_{p\in Dp_1(R)}R_p=R$, and hence $R=R[\alpha]\cap K$, that is, $R[\alpha]$ is exclusive over R. □

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