Aomoto, K.
Osaka J. Math.
32 (1995), 35–39

ON A THETA PRODUCT FORMULA FOR THE SYMMETRIC A-TYPE CONNECTION FUNCTION

KAZUHIKO AOMOTO

(Received June 2, 1993)

1. Introduction

In this note we are concerned about a formula which gives a product expression for a sum of theta rational functions. This sum has already appeared in the connection formulae among symmetric A-type Jackson integrals (See [1], [2]).

Let $q\in\boldsymbol{C}$, $|q|<1$ be the elliptic modulus. We shall use frequently the Jacobi elliptic theta function $\theta(u)=(u)_\infty(q/u)_\infty(q)_\infty$, where $(u)_\infty=\Pi_{\nu=0}^{\infty}(1-q^\nu u)$. Let $\alpha_1,\alpha_2,\cdots,\alpha_n$, β, γ and γ' be complex numbers such that $\alpha_j=\alpha_1+(j-1)(\gamma'-\gamma)$ and $\gamma+\gamma'=1$. The symmetric group of n-th degree $\mathscr{G}_n$ acts on a function $f(t)$ on the n dimensional algebraic torus $(\boldsymbol{C}^*)^n$ as $\sigma f(t)=f(\sigma^{-1}(t))=f(t_{\sigma(1)},\cdots,t_{\sigma(n)})$ for $t=(t_1,\cdots,t_n)\in(\boldsymbol{C}^*)^n$.

Let $\{U_\sigma(t)\}_{\sigma\in\mathscr{G}_n}$ be the theta rational functions on $(\boldsymbol{C}^*)^n$ defined as follows,

$$U_\sigma(t)=\Pi_{\substack{1\leq i<j\leq n\\ \sigma^{-1}(i)>\sigma^{-1}(j)}}\left(\frac{t_j}{t_i}\right)^{\gamma-\gamma'}\frac{\theta(q^\gamma t_j/t_i)}{\theta(q^{\gamma'}t_j/t_i)}. \tag{1.1}$$

These are pseudo-constants and define one-cocycle of $\mathscr{G}_n$ with values in $\boldsymbol{C}^*$,

$$U_{\sigma\sigma'}(t)=U_{\sigma'}(t)\cdot\sigma U_{\sigma'}(t) \qquad \text{and} \qquad U_e(t)=1 \tag{1.2}$$

for all $\sigma,\sigma'\in\mathscr{G}_n$ (e denotes the identity).

Let $\varphi(x), x=(x_1,\cdots,x_n)\in(\boldsymbol{C}^*)^n$, be the theta rational function

$$\varphi(x)=\Pi_{j=1}^{n}x_j^{\alpha_j}\frac{\theta(q^{\alpha_j+\cdots+\alpha_n+\gamma+1}x_j/x_{j-1})}{\theta(q^{\gamma+1}x_j/x_{j-1})} \tag{1.3}$$

for $x_0=q^\gamma$. Consider the generalized alternating sum with the weight $\{U_\sigma^{-1}(x)\}_{\sigma\in\mathscr{G}_n}$ as follows,

$$\tilde{\varphi}(x)=\sum_{\sigma\in\mathscr{G}_n}\sigma\varphi(x)\cdot\operatorname{sgn}(\sigma)\cdot U_\sigma(x)^{-1}. \tag{1.4}$$

It has the equivariant property

(1.5) $$\sigma\tilde{\varphi}(x) = U_\sigma(x)\cdot\tilde{\varphi}(x)\cdot \operatorname{sgn}\sigma \quad \text{for} \quad \sigma\in\mathscr{G}_n.$$

We want to show that $\tilde{\varphi}(x)$ can be expressed as a product of theta monomials. More precisely we can prove the following Theorem.

Theorem.

(1.6) $$\tilde{\varphi}(x)=\Pi_{j=1}^{n} q^{-(j-1)^2\gamma}\frac{\theta(q^{\alpha_j+\cdots+\alpha_n+1})}{\theta(q^{\alpha_1+1-(n+j-2)\gamma})}$$
$$\cdot\Pi_{j=1}^{n} x_j^{\alpha_1-2(j-1)\gamma}\frac{\theta(q^{\alpha_1+1-(n-1)\gamma}x_j)}{\theta(qx_j)}\cdot\Pi_{1\le i<j\le n}\frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma}x_j/x_i)}.$$

This formula has been stated as a conjecture and has been proved in case of $n=2$ and 3 in [3]. It can be regarded as an elliptic version of the one concerning Hall-Littlewood polynomials stated in [9], p 104. We shall give elsewhere an application of it to establising the explicit connection formulae for general symmetric A-type Jackson integrals relevant to Yang-Baxter equation (See for relevant subjects [2], [10], [11], and [12]).

2. Proof of Theorem.

We denote by $\varphi^*(x)$ the function of the right hand side of (1.6). If $n=1$, $\tilde{\varphi}(x)$ reduces to $x_1^{\alpha_1}\frac{\theta(q^{\alpha_1+1}x_1)}{\theta(qx_1)}$ which coincides with $\varphi^*(x)$. So the Theorem holds. We assume now $n\geq 2$. Suppose that the formula (1.6) is true for $n\leq N-1$. We must prove it for $n=N$. We denote by σ_r the permutation: $(t_1,\cdots,t_n)\to(t_2,\cdots,t_r,t_1,t_{r+1},\cdots,t_n)$ so that $\sigma^{-1}(1)=2,\cdots,\sigma^{-1}(r-1)=r$, $\sigma^{-1}(r)=1$, $\sigma^{-1}(j)=j$ for $j\geq r+1$. Then $\tilde{\varphi}(x)$ can be described as

(2.1) $$\tilde{\varphi}(x)=\sum_{r=1}^{N}\sum_{\sigma'\in\mathscr{G}_{N-1}}\sigma_r\sigma'\varphi(x)\cdot\operatorname{sgn}(\sigma_r\sigma')\cdot U_{\sigma_r\sigma'}^{-1}(x)$$
$$=\sum_{r=1}^{N}(-1)^{r-1}U_{\sigma_r}^{-1}(x)\tilde{\varphi}_r(x),$$
$$\tilde{\varphi}_r(x)=\sum_{\sigma'\in\mathscr{G}_{N-1}}\sigma'\varphi(\sigma_r^{-1}(x))\cdot\operatorname{sgn}(\sigma')\cdot U_{\sigma'}^{-1}(\sigma_r^{-1}(x)),$$

where $\mathscr{G}_{N-1}$ denotes the symmetric group of $(N-1)$th degree consisting of the permutations which fix the argument 1. $\tilde{\varphi}_r(x)$ equals

$$\tilde{\varphi}_r(x)=x_r^{\alpha_1}\frac{\theta(q^{\alpha_1+\cdots+\alpha_N+1}x_r)}{\theta(qx_r)}\cdot\sum_{\sigma'\in\mathfrak{S}_N}\operatorname{sgn}(\sigma')U_{\sigma'}^{-1}(x')$$
$$\sigma'\{\Pi_{j=2}^N x_j'^{\alpha_j}\frac{\theta(q^{\alpha_j+\cdots+\alpha_N+\gamma+1}x_j'/x_1')}{\theta(q^{\gamma+1}x_j'/x_1')}\},\tag{2.2}$$

for $x_1'=x_r,x_2'=x_1,\cdots,x_r'=x_{r-1},x_j'=x_j$ for $j\geq r+1$. We can now apply the formula (1.6) for $n=N-1$ by replacing $\alpha_1,\cdots,\alpha_{N-1}$ and $x_1,\cdots,x_{N-1}$ by $\alpha_2,\cdots,\alpha_N$ and $q^\gamma x_2'/x_1',\cdots,q^\gamma x_N'/x_1'$ respectively (Remark that $\alpha_j=\alpha_2+(j-2)(\gamma'-\gamma)$). Hence we have

$$\tilde{\varphi}_r(x)=x_r^{\alpha_1}\frac{\theta(q^{\alpha_1+\cdots+\alpha_N+1}x_r)}{\theta(qx_r)}\tag{2.3}$$
$$\cdot\Pi_{j=1}^{r-1}(q^{-\gamma}x_r)^{j-1}x_j^{\alpha_2-2(j-1)\gamma}\frac{\theta(q^{\alpha_2+1-(N-2)\gamma+\gamma}x_j/x_r)}{\theta(q^{1+\gamma}x_j/x_r)}$$
$$\cdot\Pi_{j=r+1}^N(q^{-\gamma}x_r)^{j-2}x_j^{\alpha_2-2(j-2)\gamma}\frac{\theta(q^{\alpha_2+1-(N-2)\gamma+\gamma}x_j/x_r)}{\theta(q^{1+\gamma}x_j/x_r)}$$
$$\cdot\Pi_{j=2}^N q^{-(j-2)^2\gamma}\frac{\theta(q^{\alpha_j+\cdots+\alpha_N+1})}{\theta(q^{\alpha_2+1-(N+j-4)\gamma})}\cdot\Pi_{\substack{1\leq i<j\leq N\\ i,j\neq r}}\frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma}x_j/x_i)}.$$

Since $U_{\sigma_r}(x)=\Pi_{1\leq i\leq r}(\frac{x_r}{x_i})^{\gamma-\gamma'}\frac{\theta(q^\gamma x_r/x_i)}{\theta(q^{\gamma'}x_r/x_i)}$, we have

$$U_{\sigma_r}^{-1}(x)\Pi_{\substack{1\leq j\leq N\\ j\neq r}}\frac{1}{\theta(q^{1+\gamma}x_j/x_r)}\Pi_{\substack{1\leq i<j\leq N\\ i,j\neq r}}\frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma}x_j/x_i)}\tag{2.4}$$
$$=(-1)^{r-1}\Pi_{i=1}^{r-1}(\frac{x_r}{x_i})^{-2\gamma}\Pi_{\substack{1\leq j\leq N\\ j\neq r}}\theta(qx_j/x_r)^{-1}\Pi_{1\leq i<j\leq N}\frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma}x_j/x_i)}.$$

Hence $\tilde{\varphi}(x)$ can be simplified to

$$\tilde{\varphi}(x)=-\{\Pi_{j=2}^N q^{-(j-2)^2\gamma}\frac{\theta(q^{\alpha_j+\cdots+\alpha_N+1})}{\theta(q^{1+\alpha_2-(N+j-4)\gamma})}\cdot\Pi_{j=1}^N x_j^{\alpha_1+1-2(j-1)\gamma}\tag{2.5}$$
$$\Pi_{1\leq i<j\leq N}\frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma}x_j/x_i)}\}\cdot\{\sum_{r=1}^N\frac{\theta(q^{\alpha_1+\cdots+\alpha_N+1}x_r)}{\theta(x_r)}(q^{-\gamma}x_r)^{\frac{(N-1)(N-2)}{2}}$$
$$\Pi_{\substack{1\leq j\leq N\\ j\neq r}}\frac{\theta(q^{\alpha_1+2-(N-1)\gamma}x_j/x_r)}{\theta(qx_j/x_r)}.$$

We now use the following Lemma.

Lemma. *We put* $f(u)=\Pi_{j=1}^{N}\theta(q^{\alpha_1+1-(N-1)\gamma}x_ju)$ *and* $\delta=\frac{(N-1)(N-2)}{2}$. *Then* $f(u)$ *can be described as an interpolation formula expressed by elliptic theta functions at the points* $u=q/x_r$,

$$f(u)=\sum_{r=1}^{N}\frac{\theta(q^{\alpha_1+\cdots+\alpha_N+1-\delta}x_ru)}{\theta(q^{\alpha_1+\cdots+\alpha_N+2-\delta})}f(qx_r^{-1})\Pi_{\substack{1\le j\le N\\ j\neq r}}\frac{\theta(x_ju)}{\theta(qx_j/x_r)}. \tag{2.6}$$

Proof. We denote by $f^*(u)$ the right hand side of (2.6). Remark that the theta polynomials $f(u)$ and $\theta(q^{\alpha_1+\cdots+\alpha_N+1-\delta}x_ru)\Pi_{\substack{1\le j\le N\\ j\neq r}}\theta(x_ju)$ both satisfy the same quasi-periodicity for the shift $u\to qu$: $f(qu)=(-1)^N\frac{q^{-N\alpha_1-N+N(N-1)\gamma}}{x_1\cdots x_N}u^{-N}f(u)$, since $\alpha_1+\cdots+\alpha_N=N\alpha_1+\frac{N(N-1)}{2}-N(N-1)\gamma$. Hence $f(u)$ and $f^*(u)$ have the same quasiperiodicity. On the other hand, we have $f(q/x_r)=f^*(q/x_r)$, $1\le r\le N$. Hence $f^*(u)-f(u)$ must be divided out by the product $\Pi_{j=1}^{N}\theta(x_ju)$: $f^*(u)-f(u)=g(u)\Pi_{j=1}^{N}\theta(x_ju)$, where $g(u)$ denotes a theta polynomial satisfying the multiplicative property with constant multiplier,

$$g(qu)=q^{-N\alpha_1-N+N(N-1)\gamma}g(u). \tag{2.7}$$

$g(u)$ having a Laurent expansion $g(u)=\sum_{-\infty}^{+\infty}c_mu^m$, (2.7) implies that the coefficients c_m vanish except probably for one, say c_k ($k\in\mathbf{Z}$) such that $k=-N\alpha_1-N+N(N-1)\gamma$. Since α_1 is a general complex number, this equality is impossible. Hence c_k also vanishes. $g(u)$ vanishes identically., i.e., $f^*(u)=f(u)$.

Corollary. *When we put* $u=1$, *then*

$$\sum_{r=1}^{N}\frac{\theta(q^{\alpha_1+\cdots+\alpha_N+1-\delta}x_r)\theta(q^{\alpha_1+2-(N-1)\gamma})}{\theta(q^{\alpha_1+\cdots+\alpha_N+2-\delta})}\Pi_{\substack{1\le j\le N\\ j\neq r}}\{\theta(q^{\alpha_1+2-(N-1)\gamma}x_j/x_r)$$
$$\Pi_{\substack{1\le j\le N\\ j\neq r}}\frac{\theta(x_j)}{\theta(qx_j/x_r)}\}=\Pi_{j=1}^{N}\theta(q^{\alpha_1+1-(N-1)\gamma}x_j)\qquad \textit{for}\qquad \delta=\frac{(N-1)(N-2)}{2}, \tag{2.8}$$

or equivalently

$$\sum_{r=1}^{N}(q^{-1}x_r)^{\delta}\frac{\theta(q^{\alpha_1+\cdots+\alpha_N+1}x_r)\theta(q^{\alpha_1+2-(N-1)\gamma})}{\theta(q^{\alpha_1+\cdots+\alpha_N+2})\theta(x_r)}$$
$$\Pi_{\substack{1\le j\le N\\ j\neq r}}\frac{\theta(q^{\alpha_1+2-(N-1)\gamma}x_j/x_r)}{\theta(qx_j/x_r)}=\Pi_{j=1}^{N}\frac{\theta(q^{\alpha_1+1-(N-1)\gamma}x_j)}{\theta(x_j)}. \tag{2.9}$$

We now return to the proof of the Theorem. By applying the formula (2.9) to the RHS of (2.5), we have finally

$$\tilde{\varphi}(x) = -\frac{\theta(q^{\alpha_1+\cdots+\alpha_N+2})}{\theta(q^{\alpha_1+2-(N-1)\gamma})} q^{(1-\gamma)\delta} \cdot \Pi_{j=2}^{N} \{ q^{-(j-2)^2\gamma} \frac{\theta(q^{\alpha_j+\cdots+\alpha_N+1})}{\theta(q^{\alpha_1+2-(N+j-2)\gamma})} \} \cdot \Pi_{j=1}^{N} x_j^{\alpha_1+1-2(j-1)\gamma} \frac{\theta(q^{\alpha_1+1-(N-1)\gamma}x_j)}{\theta(x_j)} \Pi_{1\le i<j\le N} \frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma}x_j/x_i)} = \varphi^*(x). \tag{2.10}$$

Q.E.D.

References

[1] K. Aomoto: *On connection coefficients for q-difference system of A-type Jackson integral* (dedicated to Prof. R. Askey on occasion of his 60th birthday), SIAM J. Math. Anal., **25** (1994), 256–273.

[2] K. Aomoto and Y. Kato: *Connection formula of symmetric A-type Jackson integrals*, Duke Math. J., **74** (1994), 129–143.

[3] K. Aomoto: 2 *conjectural formulae for symmetric A-type Jackson integrals*, preprint, 1992.

[4] R. Askey: *Some basic hypergeometric extensions of integrals of Selberg and Andrews*, SIAM J., Math. Anal., **11** (1980), 938–951.

[5] R. Evans: *Multidimensional q-beta integrals*, ibid., **23** (1992), 758–765.

[6] L. Habsieger: *Une q-intégrale de Selberg et Askey*, ibid., **19** (1988), 1475–1489.

[7] H. Hancock: Lectures on the theory of elliptic functions, Dover, 1958.

[8] K. Kadell: *A proof of Askey's conjectured q-analogue of Selberg's integral and a conjecture of Morris*, SIAM J., Math. Anal., **19** (1989), 969–986.

[9] I. G. Macdonald: Symmetric functions and Hall polynomials, Clarendon Press, Oxford, 1979.

[10] A. Matsuo: *Quantum algebra structure of certain Jackson integrals*, Commu. Math. Phys., **157** (1993), 479–498.

[11] K. Mimachi: *Connection problem in holonomic q-difference system associated with a Jackson integral of Jordan Pochhammer type*, Nagoya Math. J., **116** (1989), 149–161.

[12] A. Varchenko: *Quantized Knizhnik-Zamolodchikov equations, quantum Yang-Baxter, and difference equations for q-hypergeometric functions*, preprint, 1993.

Dept. of Math.
Nagoya University
Furo-cho 1, Chikusa-ku
Nagoya, Japan