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LAGRANGEAN CONTACT STRUCTURES ON PROJECTIVE COTANGENT BUNDLES

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Introduction

Let (M,D) be a contact manifold of dimension $2n-1$, $n\geq 2$, and (E,E') a pair of subbundles of D. We say that $(D; E,E')$ is a Lagrangean contact structure on M if for each point $x\in M$ the fibres E_x and E'_x are transversal Lagrangean subspaces of D_x with respect to the natural conformal symplectic structure of D_x.

An example of Lagrangean contact structure is given on the projective cotangent bundle $P(T^*M)$ of a manifold M of dimension n in the following way. Let D be the canonical contact structure on $P(T^*M)$. Suppose that a projective structure Q on M is given. For $[\lambda]\in P(T^*M)$, we define $E'_{[\lambda]}$ to be the space of vertical vectors in $T_{[\lambda]}(P(T^*M))$ for the projection ϖ: $P(T^*M)\rightarrow M$. Furthermore, choosing a local torsionfree connection η belonging to Q defined over a neighbourhood of $x=\varpi([\lambda])\in M$, we define $E_{[\lambda]}$ to be the space of horizontal lifts to $[\lambda]$ of vectors $X\in T_xM$ with $\lambda(X)=0$. It is determined by Q independently on the choice of η. These subspaces $E_{[\lambda]}$, $E'_{[\lambda]}$ of $T_{[\lambda]}(P(T^*M))$, $[\lambda]\in P(T^*M)$, constitute subbundles E, E' of D such that $(D; E,E')$ becomes a Lagrangean contact structure on $P(T^*M)$ (Theorem 4.2).

A typical one is the Lagrangean contact structure $(D_0; E_0,E'_0)$ on the projective cotangent bundle of n-projective space P^n associated to the flat projective structure Q_0 on P^n. A Lagrangean contact structure is said to be flat if it is locally isomorphic to $(D_0; E_0,E'_0)$. The purpose of the present note is to prove:

The Lagrangean contact structure on $P(T^*M)$ associated to a projective structure Q on M is flat if and only if Q is projectively flat.

A conformal analogue to our theorem in the following form was proved by Miyaoka [2], Sato-Yamaguchi [3]: The Lie contact structure on the tangential sphere bundle $S(TM)$ associated to a conformal structure C on a manifold M is flat if and only if C is conformally flat, provided $\dim M\geq 3$.

The proof of our theorem is based on the theory of Tanaka [5] of G-structures associated to simple graded Lie algebras as in [2], [3]. First, we show that the Lagrangean contact structures are in bijective correspondence with the $\tilde{G}$-structures of type $\mathfrak{m}$ associated to $\mathfrak{sl}(n+1)$ endowed with gradation of contact type in the sense of Tanaka[5] (Theorem 5.1). Next, we construct a normal Cartan connection ω associated to the $\tilde{G}$-structure of type $\mathfrak{m}$ which corresponds to our Lagrangean contact structure on $P(T^*M)$, making use of the normal Cartan connection for the projective structure Q (Theorem 6.4). It turns out that the curvature of ω vanishes if and only if the projective curvature of Q vanishes. This implies our theorem.

1. Lagrangean pairs

In this paper we work in C^∞-category though all the arguments are valid also in complex analytic category, replacing the real number field $\boldsymbol{R}$ by the complex number field $\boldsymbol{C}$.

Let (W,A) be a symplectic vector space over $\boldsymbol{R}$ of dimension $2n$. A subspace E of W is said to be *Lagrangean* with respect to A (or with respect to the conformal symplectic structure determined by A) if $\dim E=n$ and $A(E,E)=\{0\}$. A pair (E,E') of subspaces of W is called a *Lagrangean pair* if E and E' are Lagrangean subspaces of (W,A) such that $E\cap E'=\{0\}$. A symplectic basis $\{e_1,\cdots,e_{2n}\}$ of (W,A) with $A(e_i,e_{n+j})=\delta_{ij}$ is said to be *adapted to* (E,E') if $E=[e_1,\cdots,e_n]$ and $E'=[e_{n+1},\cdots,e_{2n}]$, where $[*]$ denotes the subspace spanned by $*$. Any Lagrangean pair admits an adapted symplectic basis. The Lagrangean pairs are congugate to each other under the symplectic automorphisms or the conformal symplectic automorphisms of (W,A).

Now let us recall the notion of torsionfree connection in order to give a geometric example of Lagrangean pair. Let M be a manifold of dimension n and fix a vector space V over $\boldsymbol{R}$ of dimension n. Let $\pi\colon F(M)\to M$ be the frame bundle of M, with structure group $GL(V)$. Denote by θ the canonical form on $F(M)$, which is a V-valued 1-form on $F(M)$. A connection η in $F(M)$ is said to be *torsionfree* if

$$d\theta+[\eta,\theta]=0.$$

It is also described in the following way (see Kobayashi[1]). Let $\pi^2\colon F^2(M)\to M$ be the second order frame bundle of M, with structure group $G^2(V)$. We may consider $GL(V)$ as a subgroup of $G^2(V)$ through the natural monomorphism $GL(V)\to G^2(V)$. Then the natural projection $\pi^2_1\colon F^2(M)\to F(M)$ is $GL(V)$-equivariant. Denote by θ^2 the second canonical form on $F^2(M)$, which is a $V+\mathfrak{gl}(V)$-valued 1-form on $F^2(M)$. We decompose it to the sum

$$\theta^2=\Theta_{-1}+\Theta_0$$

of the V-component Θ_{-1} and the $\mathfrak{gl}(V)$-component Θ_0. Then the torsionfree connections η are in bijective correspondence with the $GL(V)$-equivariant sections s: $F(M)\to F^2(M)$ of π_1^2: $F^2(M)\to F(M)$ in such a way that $s^*\Theta_0=\eta$. The section s corresponding to η is constructed as follows. For a given $u\in F(M)$ a local diffeomorphism f: $(V,0)\to M$ is defined by $f(v)=\mathrm{Exp}^\eta u(v)$, where Exp^η denotes the exponential map for the linear connection in the tangent bundle TM induced by η. Then the correspondence $u\mapsto j_0^2(f)$, the second jet of f at 0, provides the required section s.

Let η be a connection in $F(M)$ and ∇ the linear connection in the cotangent bundle p: $T^*M\to M$ induced by η. For given $\lambda\in T_x^*M$ and $X\in T_xM$, we denote by $X_\lambda^H\in T_\lambda(T^*M)$ the horizontal lift of X to T^*M with respect to ∇. It may be also described as follows. Identify T^*M with the associated bundle $F(M)\times_{GL(V)}V^*$ with respect to the natural (contragredient) action $(\mathrm{id})^*$ of $GL(V)$ on the dual space V^* of V, and denote the projection $F(M)\times V^*\to T^*M$ by $(u,\xi)\mapsto u\cdot\xi$. For a fixed $\xi\in V^*$, the differential $T(F(M))\to T(T^*M)$ of the map $F(M)\to T^*M$ defined by $u\mapsto u\cdot\xi$ will be denoted by $X\mapsto X\cdot\xi$. Then we have

$$X_\lambda^H=X_u^*\cdot\xi \qquad \text{for } \lambda=u\cdot\xi,$$

where $X_u^*\in T_u(F(M))$ is the horizontal lift of X to $F(M)$ with respect to η.

EXAMPLE 1.1. Let η be a torsionfree connection in $F(M)$. For a given $\lambda\in T_x^*M$ we define subspaces E_λ and E'_λ of $T_\lambda(T^*M)$ by

$$E_\lambda=\{X_\lambda^H;\ X\in T_xM\}, \qquad E'_\lambda=\{\mu_\lambda^V;\ \mu\in T_x^*M\},$$

where $\mu\mapsto\mu_\lambda^V$ denotes the identification $T_x^*M=T_\lambda(T_x^*M)$. Further, we define a 1-form α on T^*M by

$$\alpha(X)=\lambda(p_*X) \qquad \text{for } X\in T_\lambda(T^*M),$$

whose exterior differential $d\alpha$ is known to be a symplectic form on each $T_\lambda(T^*M)$. Then (E_λ,E'_λ) is a Lagrangean pair of $(T_\lambda(T^*M),d\alpha)$. More precisely, we have that

$$d\alpha(E_\lambda,E_\lambda)=d\alpha(E'_\lambda,E'_\lambda)=\{0\},$$

$$d\alpha(\mu_\lambda^V, X_\lambda^H) = \frac{1}{2}\mu(X) \qquad \text{for } \mu \in T_x^*M, \quad X \in T_xM.$$

2. Lagrangean contact structures

In this section we assume that $n \geq 2$. A graded Lie algebra (abbreviated to GLA) over $\boldsymbol{R}$

$$\mathfrak{m} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}, \qquad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$$

is called a *fundamental GLA of contact type* of degree n, if $\dim \mathfrak{g}_{-2} = 1$, $\dim \mathfrak{g}_{-1} = 2n-2$, and $[\mathfrak{g}_{-1}, X] = \{0\}$ implies $X = 0$ for $X \in \mathfrak{g}_{-1}$. Such a GLA is unique up to GLA-isomorphism. If we take an $e_0 \in \mathfrak{g}_{-2}$ with $e_0 \neq 0$, a symplectic form A_0 on $\mathfrak{g}_{-1}$ is defined by

$$[X, Y] = A_0(X, Y)e_0 \qquad \text{for } X, Y \in \mathfrak{g}_{-1},$$

whose conformal class is determined by $\mathfrak{m}$ independently on the choice of e_0. We define $C(\mathfrak{m})$ to be the subgroup of $GL(\mathfrak{m})$ consisting of $a \in GL(\mathfrak{m})$ such that $a\mathfrak{g}_{-1} = \mathfrak{g}_{-1}$ and that the graded linear automorphism $\bar{a}$ of $\mathfrak{m}$ induced by a is a GLA-automorphism.

Let M be a manifold of dimension $2n-1$ and D a subbundle of TM of codimension 1. Denote by κ: $TM \to TM/D$ the projection to the quotient line bundle TM/D. For a point $x \in M$ we define a GLA $\mathfrak{m}(x)$ as follows. Let $\mathfrak{g}_{-2}(x) = (TM/D)_x$, $\mathfrak{g}_{-1}(x) = D_x$, and $\mathfrak{m}(x) = \mathfrak{g}_{-2}(x) + \mathfrak{g}_{-1}(x)$. For $X, Y \in \mathfrak{g}_{-1}(x)$ we define

$$[X, Y] = \kappa[\tilde{X}, \tilde{Y}]_x \in \mathfrak{g}_{-2}(x),$$

taking local sections $\tilde{X}$ and $\tilde{Y}$ of D around x which extend X and Y, respectively. Further, we set $[\mathfrak{m}(x), \mathfrak{g}_{-2}(x)] = \{0\}$. If $\mathfrak{m}(x)$ is GLA-isomorphic to $\mathfrak{m}$ for every point $x \in M$, D is called a *contact structure* on M. Note that then D carries a natural conformal symplectic structure determined by the $\mathfrak{m}(x)$'s. A contact structure may be also defined by a *system of contact forms* $\{U_i, \gamma_i\}$, where $\{U_i\}$ is an open cover of M, and γ_i is a 1-form defined on U_i with $\gamma_i \wedge (d\gamma_i)^{n-1} \neq 0$ everywhere on U_i which satisfies $\gamma_i = f_{ij}\gamma_j$ on $U_i \cap U_j$ with a function f_{ij} on $U_i \cap U_j$. Then

$$D_x = \{X \in T_xM;\ \gamma_i(X) = 0\} \qquad \text{if } x \in U_i$$

defines a contact structure D. And every contact structure D is obtained in this way. Note that in this case the conformal symplectic structure on D is given by $(d\gamma_i)_x | D_x \times D_x$.

EXAMPLE 2.1. Let M be a manifold and $\varpi: P(T^*M) \to M$ the projective cotangent bundle of M. We set $\mathring{T}^*M = T^*M - \{\text{zero section}\}$ and denote by $q: \mathring{T}^*M \to P(T^*M)$ the natural projection $\lambda \mapsto [\lambda]$. Let α be the 1-form on T^*M defined in Example 1.1. If we take local sections $s_i: U_i \to \mathring{T}^*M$ of q and set $\gamma_i = s_i^*\alpha$, then $\{U_i, \gamma_i\}$ becomes a system of contact forms on $P(T^*M)$. The contact structure D determined by this system is called the *canonical contact structure* on $P(T^*M)$.

For a contact structure D on a manifold M of dimension $2n-1$, a frame $u: \mathfrak{m} \to T_xM$ at $x \in M$ is called a *contact frame* of (M,D) if $u\mathfrak{g}_{-1} = D_x$ and the graded linear isomorphism $\bar{u}: \mathfrak{m} \to \mathfrak{m}(x)$ induced by u is a GLA-isomorphism. Then the subset $F_D(M)$ of $F(M)$ consisting of the contact frames of (M,D) becomes a $C(\mathfrak{m})$-structure. Furthermore, $P = F_D(M)$ is a $C(\mathfrak{m})$-structure *of type* $\mathfrak{m}$ in the sense that

$$d\theta_{-2} + \frac{1}{2}[\theta_{-1}, \theta_{-1}] \equiv 0 \qquad \mathrm{mod}\, \theta_{-2},$$

where θ_{-2} and θ_{-1} denote the $\mathfrak{g}_{-2}$-component and the $\mathfrak{g}_{-1}$-component, respectively, of the restriction θ to P of the canonical form on $F(M)$. Conversely, for every $C(\mathfrak{m})$-structure P of type $\mathfrak{m}$ there exists uniquely a contact structure D such that $F_D(M) = P$.

Let D_i be a contact structure on M_i, $i = 1,2$. A diffeomorphism $\varphi: M_1 \to M_2$ is called a *contact isomorphism* of (M_1, D_1) to (M_2, D_2) if $\varphi_* D_1 = D_2$, which is equivalent to that φ_* induces a GLA-isomorphism of $\mathfrak{m}_1(x)$ to $\mathfrak{m}_2(\varphi(x))$ for each point $x \in M_1$, or to that φ is a $C(\mathfrak{m})$-structure isomorphism of $(M_1, F_{D_1}(M_1))$ to $(M_2, F_{D_2}(M_2))$, namely, the first prolongation $\varphi^{(1)}: F(M_1) \to F(M_2)$ of φ sends $F_{D_1}(M_1)$ onto $F_{D_2}(M_2)$.

EXAMPLE 2.2. Let D_i be the canonical contact structure on $P(T^*M_i)$, $i = 1,2$. A diffeomorphism $\varphi: M_1 \to M_2$ induces a diffeomorphism $\hat{\varphi}: P(T^*M_1) \to P(T^*M_2)$ such that the diagram

$$\begin{array}{ccc} \mathring{T}^*M_1 & \xrightarrow{(\varphi^*)^{-1}} & \mathring{T}^*M_2 \\ q_1 \downarrow & & q_2 \downarrow \\ P(T^*M_1) & \xrightarrow{\hat{\varphi}} & P(T^*M_2) \end{array}$$

is commutative, where q_i: $\mathring{T}^*M_i \to P(T^*M_i)$, $i=1,2$, are natural projections. Then $\hat{\phi}$ is a contact isomorphism of $(P(T^*M_1),D_1)$ to $(P(T^*M_2),D_2)$.

Let $\mathfrak{m}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}$ be a fundamental GLA of contact type of degree n. If a Lagrangean pair $(\mathfrak{e},\mathfrak{e}')$ of the symplectic vector space $(\mathfrak{g}_{-1},A_0)$ is given, the triple $(\mathfrak{m};\ \mathfrak{e},\mathfrak{e}')$ is called a *fundamental GLA of Lagrangean contact type* of degree n. Such a triple is unique up to isomorphism. Here for two such triples $(\mathfrak{m}_i;\ \mathfrak{e}_i,\mathfrak{e}'_i)$, $i=1,2$, a GLA-isomorphism φ of $\mathfrak{m}_1$ to $\mathfrak{m}_2$ such that $\varphi\mathfrak{e}_1=\mathfrak{e}_2$, $\varphi\mathfrak{e}'_1=\mathfrak{e}'_2$ is called an *isomorphism* of $(\mathfrak{m}_1;\ \mathfrak{e}_1,\mathfrak{e}'_1)$ to $(\mathfrak{m}_2;\ \mathfrak{e}_2,\mathfrak{e}'_2)$. For a fundamental GLA $(\mathfrak{m};\ \mathfrak{e},\mathfrak{e}')$ of Lagrangean contact type, a basis $\{e_0,e_1,\cdots,e_{2n-2}\}$ of $\mathfrak{m}$ is called a *Lagrangean contact basis* if $e_0\in\mathfrak{g}_{-2}$ and $\{e_1,\cdots,e_{2n-2}\}$ is a symplectic basis of $(\mathfrak{g}_{-1},A_0)$ adapted to $(\mathfrak{e},\mathfrak{e}')$. We define the *Lagrangean contact group* $C(\mathfrak{m};\ \mathfrak{e},\mathfrak{e}')$ to be the subgroup of $C(\mathfrak{m})$ consisting of $a\in C(\mathfrak{m})$ such that $a\mathfrak{e}=\mathfrak{e}, a\mathfrak{e}'=\mathfrak{e}'$. With respect to a Lagrangean contact basis, it is represented by

$$C(\mathfrak{m};\ \mathfrak{e},\mathfrak{e}')=\left\{\begin{pmatrix} c & 0 & 0 \\ b_1 & a & 0 \\ b_2 & 0 & c\,{}^t a^{-1} \end{pmatrix};\ c\in\boldsymbol{R}^*, b_1,b_2\in\boldsymbol{R}^{n-1}, a\in GL(n-1)\right\}.$$

Let D be a contact structure on a manifold M of dimension $2n-1$. Suppose that two subbundles E,E' of D are given. We say that $(D;\ E,E')$ is a *Lagrangean contact structure* if for every $x\in M,(E_x,E'_x)$ is a Lagrangean pair of D_x with respect to the natural conformal symplectic structure on D_x. A frame $u\colon \mathfrak{m}\to T_xM$ of M is called a *Lagrangean contact frame* of $(M,D;\ E,E')$ if it is a contact frame of (M,D) such that $u\mathfrak{e}=E_x, u\mathfrak{e}'=E'_x$. Then the subset $F_{(D;E,E')}(M)$ of $F(M)$ consisting of the Langrangean contact frames of $(M,D;\ E,E')$ becomes a $C(\mathfrak{m};\ \mathfrak{e},\mathfrak{e}')$-structure of type $\mathfrak{m}$. Conversely, for every $C(\mathfrak{m};\ \mathfrak{e},\mathfrak{e}')$-structure $\tilde{P}$ of type $\mathfrak{m}$ there exists uniquely a Lagrangean contact structure $(D;\ E,E')$ such that $F_{(D;E,E')}(M)=\tilde{P}$. Let $(D_i;\ E_i,E'_i)$ be a Lagrangean contact structure on $M_i, i=1,2$. A diffeomorphism $\varphi\colon M_1\to M_2$ is called a *Lagrangean contact isomorphism* if it is a contact isomorphism of (M_1,D_1) to (M_2,D_2) such that $\varphi_*E_1=E_2$, $\varphi_*E'_1=E'_2$, which is equivalent to that φ is a $C(\mathfrak{m};\ \mathfrak{e},\mathfrak{e}')$-structure isomorphism of $(M_1,F_{(D_1;E_1,E'_1)}(M_1))$ to $(M_2,F_{(D_2;E_2,E'_2)}(M_2))$.

3. Projective structures

Let W be a vector space over $\boldsymbol{R}$ of dimension $n+1$, $n\geq 1$, and $P^n=P(W)$ the projective space associated to W. We denote by L the group of projective transformations of P^n, which is isomorphic to the quotient group $GL(W)/C$ of $GL(W)$ by its center C. The Lie algebra $\mathfrak{l}=\mathrm{Lie}\,L$ of L is identified with $\mathfrak{sl}(W)$, and L may be considered as a subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{l})$ of $\mathfrak{l}$ through the adjoint representation. We define an L-invariant nondegenerate symmetric bilinear form B on $\mathfrak{l}$ by

$$B(X, Y)=\mathrm{tr}(XY) \qquad \text{for } X, Y\in \mathfrak{l}.$$

In the following we fix a basis $\{w_0, w_1, \cdots, w_n\}$ of W, and denote by $\{\zeta^0, \zeta^1, \cdots, \zeta^n\}$ the basis of W^* dual to this. Thus we have identifications: $L=PL(n+1)=GL(n+1)/C$ where $C=\boldsymbol{R}^* 1_{n+1}$, $\mathfrak{l}=\mathfrak{sl}(n+1)$, and $P^n=P(\boldsymbol{R}^{n+1})$. We set

$$\bar{E}=\frac{1}{n+1}\begin{pmatrix} n & 0 \\ 0 & -1_n \end{pmatrix}\in \mathfrak{l},$$

$$\mathfrak{l}_p=\{X\in\mathfrak{l};\ [\bar{E}, X]=pX\} \qquad p=-1,0,1,$$

which determines a GLA-structure on $\mathfrak{l}$:

$$\mathfrak{l}=\mathfrak{l}_{-1}+\mathfrak{l}_0+\mathfrak{l}_1, \qquad [\mathfrak{l}_p, \mathfrak{l}_q]\subset \mathfrak{l}_{p+q}.$$

We set $V=\mathfrak{l}_{-1}$. Then, since $B|\mathfrak{l}_{-1}\times\mathfrak{l}_1$ is nondegenerate, $\mathfrak{l}_1$ is identified with V^* through B. These subspaces $\mathfrak{l}_p$ are explicitly given as follows.

$$\mathfrak{l}_{-1}=\left\{\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix};\ v\in\boldsymbol{R}^n\right\}, \quad \mathfrak{l}_1=\left\{\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix};\ {}^t\xi\in\boldsymbol{R}^n\right\},$$

$$\mathfrak{l}_0=\left\{\begin{pmatrix} \alpha & 0 \\ 0 & A \end{pmatrix};\ \alpha\in\boldsymbol{R}, A\in\mathfrak{gl}(n), \mathrm{tr}\,A=-\alpha\right\}.$$

We set

$$\mathfrak{l}'=\mathfrak{l}_0+\mathfrak{l}_1,$$

which is a subalgebra of $\mathfrak{l}$ with $\mathfrak{l}=\mathfrak{l}_{-1}+\mathfrak{l}'$ (direct sum as vector space). Let L_0 denote the group of GLA-automorphisms of $\mathfrak{l}$. It is given by

$$L_0\cong\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix};\ a\in\boldsymbol{R}^*, b\in GL(n)\right\}\Big/C,$$

and thus $L_0\subset L$ and $\mathrm{Lie}\,L_0=\mathfrak{l}_0$. Further, we have that $L=L_0\,\mathrm{Inn}(\mathfrak{l})$, where $\mathrm{Inn}(\mathfrak{l})$ denotes the group of inner automorphisms of $\mathfrak{l}$. We define $L_1=\exp\mathfrak{l}_1$ and $L'=N_L(\mathfrak{l}')$, the normalizer of $\mathfrak{l}'$ in L, whose Lie algebras are $\mathfrak{l}_1$ and $\mathfrak{l}'$, respectively. The subgroup L' has a semidirect decomposition $L'=L_0L_1$ and is identical with the isotropy subgroup in L at the point $[w_0]\in P^n$. Therefore, we have an identification

$$L/L'=P^n,$$

which implies an identification $T_{[w_0]}P^n=\mathfrak{l}_{-1}$. Let $\bar{\rho}$: $L'\rightarrow GL(\mathfrak{l}_{-1})=GL(V)$ be the linear isotropy representation at $[w_0]$. Then we have that Kernel $\bar{\rho}=L_1$ and Image $\bar{\rho}=GL(V)$, and hence, $\bar{\rho}$ maps L_0 isomorphically onto $GL(V)$. We shall identify L_0 with $GL(V)$ through $\bar{\rho}$, and also $\mathfrak{l}_0$ with $\mathfrak{gl}(V)$ through $\bar{\rho}_*$. We define

$$\bar{e}_i=E_{i+1,1}, \quad \bar{e}_i^*=E_{1,i+1} \qquad \text{for } 1\leq i\leq n,$$

where the E_{ij}'s denote the standard matrix units in $\mathfrak{gl}(n+1)$. Then $\{\bar{e}_i\}$ is an orthonormal basis of $\mathfrak{l}_{-1}$ with respect to the inner product

$$(X, Y)=\operatorname{tr}({}^tXY) \qquad \text{for } X, Y\in\mathfrak{l}$$

on $\mathfrak{l}$. (In complex analytic category, one should replace $\operatorname{tr}({}^tXY)$ by $\operatorname{tr}({}^tX\bar{Y})$.) Furthermore, $\{\bar{e}_i^*\}$ is the basis of $\mathfrak{l}_1$ dual to $\{\bar{e}_i\}$ under the previous identification $\mathfrak{l}_1=V^*$. We may identify $\mathfrak{l}_0=\mathfrak{gl}(V)$ with $\mathfrak{gl}(n)$ through the basis $\{\bar{e}_i\}$ of V. It is easy to see the following.

Lemma 3.1. *Under the identification above, for*

$$v={}^t(v^1,\cdots,v^n)=\sum_i v^i\bar{e}_i\in V=\mathfrak{l}_{-1},$$

$$\xi=(\xi_1,\cdots,\xi_n)=\sum_i \xi_i\bar{e}_i^*\in V^*=\mathfrak{l}_1,$$

$[v,\xi]\in\mathfrak{gl}(n)=\mathfrak{l}_0$ *is given by*

$$[v,\xi]=v\xi+(\xi v)1_n.$$

Next, we embed $V=\mathfrak{l}_{-1}$ into P^n as an open set containing $[w_0]$ by the map $v\mapsto(\exp v)[w_0]$, and so every $a\in L$ determines a local diffeomorphism a: $(V,0)\rightarrow P^n$. We define a map $\iota: L\rightarrow F^2(P^n)$ by

$$\iota(a)=j_0^2(a) \qquad \text{for } a\in L.$$

Then ι is an embedding which induces a monomorphism $\iota: L'\rightarrow G^2(V)$ such that $\pi_1^2\circ\iota=\bar{\rho}$, where π_1^2: $G^2(V)\rightarrow GL(V)$ is the natural projection. In the following we shall consider L' as a subgroup of $G^2(V)$ through the monomorphism ι.

Now let M be a manifold of dimension n. An L'-subbundle $Q \subset F^2(M)$ of the second order frame bundle π^2: $F^2(M) \to M$ is called a *projective structure* on M. For example, let η be a torsionfree connection in $F(M)$, with the corresponding $GL(V)$-equivariant section s: $F(M) \to F^2(M)$ of π_1^2: $F^2(M) \to F(M)$. Then $Q_\eta = s(F(M)) \cdot L_1 \subset F^2(M)$ is a projective structure on M, which we call the *projective structure associated to* η. Let η and η' be torsionfree connections in $F(M)$. They are said to be *projectively equivalent* if $Q_\eta = Q_{\eta'}$, which is equivalent to that there exists an $\mathfrak{l}_1 = V^*$-valued function p on $F(M)$ of type $\mathrm{Ad} = (\mathrm{id})^*$ such that

$$\eta - \eta' = [\theta, p],$$

which is the case that $s' = s \cdot \exp p$ for the corresponding sections s, s': $F(M) \to F^2(M)$. Let $Q \subset F^2(M)$ be a projective structure and $U \subset M$ an open set. A torsionfree connection η in $F(M)|U = \pi^{-1}(U)$ is called a *local torsionfree connection belonging to* Q if $Q_\eta = Q|U$. For any projective structure $Q \subset F^2(M)$ there exists a family $\{U_i, \eta_i\}$ of local torsionfree connections belonging to Q, where (*): $\{U_i\}$ is an open cover of M; η_i is a torsionfree connection in $F(M)|U_i$; η_i and η_j are projectively equivalent over $U_i \cap U_j$. Conversely, for any family $\{U_i, \eta_i\}$ with (*), there exists uniquely a projective structure $Q \subset F^2(M)$ such that each η_i belongs to Q.

EXAMPLE 3.2. Set $Q_{\mathrm{I}} = L$ and regard it as a submanifold of $F^2(P^n)$ through the embedding ι. Then $Q_{\mathrm{I}} \subset F^2(P^n)$ is a projective structure on P^n, which we call the *flat projective structure on* P^n.

Let $Q_i \subset F^2(M_i)$ be a projective structure on M_i, $i = 1, 2$. A diffeomorphism φ: $M_1 \to M_2$ is called a *projective isomorphism* of (M_1, Q_1) to (M_2, Q_2) if the second prolongation $\varphi^{(2)}$: $F^2(M_1) \to F^2(M_2)$ of φ sends Q_1 onto Q_2. A projective structure $Q \subset F^2(M)$ is said to be *projectively flat* if (M, Q) is locally projectively isomorphic to (P^n, Q_{I}), that is, for each point $x \in M$ there exist an open neighbourhood U of x and an open set U_0 of P^n such that $(U, Q|U)$ is projectively isomorphic to $(U_0, Q_{\mathrm{I}}|U_0)$.

Now we recall the theory of Cartan connections for projective structures following the formulation by Tanaka [4]. Let Q be a projective structure on a manifold M of dimension n. An $\mathfrak{l}$-valued 1-form $\bar{\omega}$ on Q is called a *Cartan connection* in Q of type L/L' if

(1) for each $z \in Q$, $\bar{\omega}$: $T_zQ \to \mathfrak{l}$ is a linear isomorphism;

(2) $R_a^*\bar{\omega} = \mathrm{Ad}\, a^{-1}\bar{\omega}$ for $a \in L'$; and

(3) $\bar{\omega}(A^*) = A$ for $A \in \mathfrak{l}'$,

where R_a denotes the action of $a \in L'$ on Q, and A^* the fundamental vector field on Q generated by $A \in \mathfrak{l}'$. Let

$$\bar{\omega} = \bar{\omega}_{-1} + \bar{\omega}_0 + \bar{\omega}_1$$

be the decomposition of $\bar{\omega}$ into the sum of $\mathfrak{l}_p$-components $\bar{\omega}_p$. We call

$$\bar{\Omega} = d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}]$$

the *curvature* of $\bar{\omega}$, which is semibasic in the sense that $\bar{\Omega}(X, Y) = 0$ if X or $Y \in T_z Q$ is tangent to the fibre of π^2. Thus there exists an $\mathfrak{l} \otimes \Lambda^2 \mathfrak{l}^*_{-1}$-valued function $\bar{K}$ on Q, called the *curvature function* of $\bar{\omega}$, such that

$$\bar{\Omega} = \frac{1}{2} \bar{K}(\bar{\omega}_{-1} \wedge \bar{\omega}_{-1}).$$

Let

$$\bar{K} = \bar{K}_{-1} + \bar{K}_0 + \bar{K}_1$$

be the decomposition of $\bar{K}$ into the sum of $\mathfrak{l}_p$-components $\bar{K}_p$. Recall that the second canonical form θ^2 on $F^2(M)$ is a $V + \mathfrak{gl}(V) = \mathfrak{l}_{-1} + \mathfrak{l}_0$-valued 1-form with decomposition into the sum

$$\theta^2 = \Theta_{-1} + \Theta_0$$

of $\mathfrak{l}_p$-components Θ_p. A Cartan connection $\bar{\omega}$ is said to be *normal* if it satisfies the following two conditions.

(1) The restrictions of Θ_{-1} and Θ_0 to Q are identical with $\bar{\omega}_{-1}$ and $\bar{\omega}_0$, respectively. (In this case $\bar{K}_{-1} = 0$.)

(2) If $\{\bar{e}_1, \cdots, \bar{e}_n\}$ is a basis of $\mathfrak{l}_{-1}$ with $(\bar{e}_i, \bar{e}_j) = \delta_{ij}$, and $\{\bar{e}^*_1, \cdots, \bar{e}^*_n\}$ the basis of $\mathfrak{l}_1$ dual to $\{\bar{e}_i\}$ with respect to B, then

$$(\bar{\partial}^* \bar{K})(X) = \sum_i [\bar{e}^*_i, \bar{K}(\bar{e}_i, X)] = 0 \qquad \text{for } X \in \mathfrak{l}_{-1}.$$

EXAMPLE 3.3. The Maurer-Cartan form $\bar{\omega}$ of $L = Q_{\mathfrak{l}}$ is a normal Cartan connection in $Q_{\mathfrak{l}} \subset F^2(P^n)$ of type L/L' with the curvature $\bar{\Omega} = 0$.

Theorem 3.4. (Tanaka [4]) *For any projective structure Q on a manifold M of dimension $n \geq 2$, there exists uniquely a normal Cartan connection $\bar{\omega}$ in Q of type L/L'.*

The following follows from Example 3.3 and Frobenius' theorem.

Corollary 3.5. *Q is projectively flat if and only if the curvature $\bar{\Omega}$ of $\bar{\omega}$ vanishes on Q, provided $n \geq 2$.*

4. Lagrangean contact structures on projective cotangent bundles

Let M be a manifold of dimension $n \geq 2$ and ϖ: $P(T^*M) \to M$ the projective cotangent bundle of M. We identify $P(T^*M)$ with the associated bundle $F(M) \times_{GL(V)} P(V^*)$ with respect to the natural projective action of $GL(V)$ on $P(V^*)$. In the same way as in Section 1, the projection $F(M) \times P(V^*) \to P(T^*M)$ will be denoted by $(u, [\xi]) \mapsto u \cdot [\xi]$), and for a fixed $[\xi] \in P(V^*)$ the differential of the map $F(M) \to P(T^*M)$ defined by $u \mapsto u \cdot [\xi]$ will be denoted by $X \mapsto X \cdot [\xi]$. Then we have

$$X \cdot [\xi] = q_*(X \cdot \xi) \qquad \text{for } \xi \in \mathring{V}^* = V^* - \{0\}, \quad X \in T(F(M)),$$

for the natural projection q: $\mathring{T}^*M \to P(T^*M)$. Let η be a connection in $F(M)$. For given $[\lambda] \in P(T^*M)$ and $X \in T_xM$ with $\varpi([\lambda]) = x$, the *horizontal lift* $X^{\bar{H}}_{[\lambda]} \in T_{[\lambda]}(\mathrm{P}(T^*M))$ of X to $P(T^*M)$ with respect to η is defined by

$$X^{\bar{H}}_{[\lambda]} = X^*_u \cdot [\xi] \qquad \text{for } [\lambda] = u \cdot [\xi],$$

where $X^*_u \in T_u(F(M))$ is the horizontal lift of X to $F(M)$ with respect to η. It is also described as follows. Choose an element $\lambda \in \mathring{T}^*_xM$ so that $q(\lambda) = [\lambda]$, and let $X^H_\lambda \in T_\lambda(T^*M)$ be the horizontal lift of X to T^*M with respect to the linear connection in T^*M induced by η. Then $q_* X^H_\lambda$ is independent on the choice of λ, and is equal to $X^{\bar{H}}_{[\lambda]}$.

Lemma 4.1. *Suppose that η and η' are torsionfree connections in $F(M)$ which are projectively equivalent. Let $[\lambda] \in P(T^*M)$ with $\varpi([\lambda]) = x$. Then for every X in*

$$[\lambda]^{\perp}_x = \{X \in T_xM;\ \lambda(X) = 0\},$$

the corresponding horizontal lifts are identical:

$$X^{\bar{H}}_{[\lambda]} = X^{\bar{H}'}_{[\lambda]}.$$

Proof. It follows from the assumption that there exists an $\mathrm{I}_1 = V^*$-valued function p on $F(M)$ such that $\eta - \eta' = [\theta, p]$. Take an element $\lambda = u \cdot \xi \in \mathring{T}^*_xM$ with $q(\lambda) = [\lambda]$. We shall show that

$$X_\lambda^H - X_\lambda^{H'} = p_u(\theta(X_u^*))\lambda_\lambda^V \in T_\lambda(T^*M).$$

Then, applying q_* to this we obtain the assertion. Indeed, since $\pi_* X_u^* = \pi_* X_u^{*\prime} = X$, there is an $A \in \mathfrak{gl}(V)$ such that $X_u^* - X_u^{*\prime} = A_u^*$. Then

$$\begin{aligned}(\eta - \eta')(X_u^*) &= -\eta'(X_u^*) = -\eta'(X_u^* - X_u^{*\prime}) \\ &= -\eta'(A_u^*) = -A.\end{aligned}$$

On the other hand, the lefthand side is equal to $[\theta, p](X_u^*) = [v, p_u]$ where $v = \theta(X_u^*)$, and hence $A = -[v, p_u]$. Therefore, we have

$$\begin{aligned}X_\lambda^H - X_\lambda^{H'} &= (X_u^* - X_u^{*\prime}) \cdot \xi = A_u^* \cdot \xi \\ &= u \cdot (A \cdot \xi) = -u \cdot ([v, p_u] \cdot \xi),\end{aligned}$$

under the identification $T_\xi V^* = V^*$, where $A \cdot \xi$ denotes the natural action of $\mathfrak{gl}(V)$ on V^*. Here for $v = {}^t(v^1, \cdots, v^n) \in V = \mathfrak{l}_{-1}, p_u = (p_1, \cdots, p_n) \in V^* = \mathfrak{l}_1$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathring{V}^*$, by Lemma 3.1 we have

$$\begin{aligned}[v, p_u] \cdot \xi &= (v p_u + (p_u v) 1_n) \cdot \xi = -\xi(v p_u + (p_u v) 1_n) \\ &= -\xi(v) p_u - p_u(v) \xi.\end{aligned}$$

Thus we obtain

$$X_\lambda^H - X_\lambda^{H'} = \xi(v) u \cdot p_u + p_u(v) u \cdot \xi = p_u(v) \lambda,$$

since $\xi(v) = (u \cdot \xi)(X) = \lambda(X) = 0$. This implies the required equality.

Now suppose that a projective structure $Q \subset F^2(M)$ is given. Let D be the canonical contact structure on $P(T^*M)$. For a given $[\lambda] \in P(T^*M)$ with $\varpi([\lambda]) = x$, take a local torsionfree connection η belonging to Q defined over a neighbourhood of x, and set

$$E_{[\lambda]} = \{X_{[\lambda]}^{\bar{H}};\ X \in [\lambda]_x^\perp\},$$

$X_{[\lambda]}^{\bar{H}}$ being the horizontal lift of X to $P(T^*M)$ with respect to η. By Lemma 4.1 it is determined by Q independently on the choice of η. Further we set

$$E'_{[\lambda]} = \mathrm{Kernel}\, \varpi_*\colon\ T_{[\lambda]}(P(T^*M)) \to T_x M.$$

These determine subbundles E and E' of $T(P(T^*M))$.

Theorem 4.2. *The triple $(D;\ E, E')$ above is a Lagrangean contact* structure on $P(T^*M)$. *This will be called associated to* Q.

Proof. Recall that D is given by

$$D_{[\lambda]} = \mathrm{Kernel}(s^*\alpha)_{[\lambda]},$$

taking a local section s of $q\colon \mathring{T}^*M \to P(T^*M)$ around $[\lambda]$. We set $\lambda = s([\lambda])$. First note that then we have

$$s_* X^{\bar{H}}_{[\lambda]} - X^H_\lambda \subset \mathrm{Kernel}(q_*)_\lambda \in \mathrm{Kernel}(p_*)_\lambda,$$

because $q_* s_* X^{\bar{H}}_{[\lambda]} - q_* X^H_{[\lambda]} = X^{\bar{H}}_{[\lambda]} - X^{\bar{H}}_{[\lambda]} = 0$. Now for each $X \in [\lambda]^\perp_x$ we have that $\alpha(s_* X^{\bar{H}}_{[\lambda]} - X^{\bar{H}}_\lambda) = 0$ by the remark above, and hence

$$(s^*\alpha)(X^{\bar{H}}_{[\lambda]}) = \alpha(X^H_\lambda) = \lambda(p_* X^H_\lambda) = \lambda(X) = 0.$$

Thus we get $E_{[\lambda]} \subset D_{[\lambda]}$. Furthermore, for each $\bar{X} \in E'_{[\lambda]}$ we have

$$p_* s_* \bar{X} = \varpi_* q_* s_* \bar{X} = \varpi_* \bar{X} = 0,$$

and so

$$(s^*\alpha)(\bar{X}) = \alpha(s_* \bar{X}) = \lambda(p_* s_* \bar{X}) = 0.$$

Therefore, we have also $E'_{[\lambda]} \subset D_{[\lambda]}$.

Next, we shall show that $d(s^*\alpha)(X^{\bar{H}}_{[\lambda]}, Y^{\bar{H}}_{[\lambda]}) = 0$ holds for every $X, Y \in [\lambda]^\perp_x$. Indeed, by the remark above we can write

$$s_* X^{\bar{H}}_{[\lambda]} = X^H_\lambda + a\lambda^V_\lambda, \quad s_* Y^{\bar{H}}_{[\lambda]} = Y^H_\lambda + b\lambda^V_\lambda, \quad a, b \in \boldsymbol{R},$$

and so by Example 1.1 we have

$$\begin{aligned} d(s^*\alpha)(X^{\bar{H}}_{[\lambda]}, Y^{\bar{H}}_{[\lambda]}) &= d\alpha(X^H_\lambda + a\lambda^V_\lambda, Y^H_\lambda + b\lambda^V_\lambda) \\ &= \frac{1}{2}(a\lambda(Y) + b\lambda(X)) = 0. \end{aligned}$$

It remains to show that $d(s^*\alpha)(E'_{[\lambda]}, E'_{[\lambda]}) = \{0\}$. But this is clear since E' is an integrable subbundle of $T(P(T^*M))$.

5. Cartan connections associated to Lagrangean contact structures

In this section we assume that $n \geq 2$ and retain the notation in Section 3. Let $G = L = PL(n+1)$ and $\mathfrak{g} = \mathrm{Lie}\,G = \mathfrak{l} = \mathfrak{sl}(n+1)$ so that $G \subset \mathrm{Aut}(\mathfrak{g})$. We set

$$E=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\in\mathfrak{g},$$

$$\mathfrak{g}_p=\{X\in\mathfrak{g};\ [E,X]=pX\}\qquad p=-2,-1,0,1,2,$$

which determines a GLA-structure on $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_2,\qquad [\mathfrak{g}_p,\mathfrak{g}_q]\subset\mathfrak{g}_{p+q}.$$

We set

$$\mathfrak{m}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1},\qquad \mathfrak{m}^*=\mathfrak{g}_1+\mathfrak{g}_2,$$
$$\mathfrak{g}'=\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_2,$$

so that $\mathfrak{g}=\mathfrak{m}+\mathfrak{g}'$ (direct sum as vector space). Then $\mathfrak{m}$ becomes a fundamental GLA of contact type of degree n. Since $B|\mathfrak{m}\times\mathfrak{m}^*$ is nondegenerate, $\mathfrak{m}^*$ is identified with the dual space of $\mathfrak{m}$ through B. These subspaces $\mathfrak{g}_p$ are explicitly given as follows.

$$\mathfrak{g}_{-2}=\left\{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \alpha & 0 & 0\end{pmatrix};\ \alpha\in\boldsymbol{R}\right\},\quad \mathfrak{g}_2=\left\{\begin{pmatrix}0 & 0 & \alpha\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix};\ \alpha\in\boldsymbol{R}\right\},$$

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix}0 & 0 & 0\\ b_1 & 0 & 0\\ 0 & {}^tb_2 & 0\end{pmatrix};\ b_1,b_2\in\boldsymbol{R}^{n-1}\right\},\quad \mathfrak{g}_1=\left\{\begin{pmatrix}0 & {}^tb_1 & 0\\ 0 & 0 & b_2\\ 0 & 0 & 0\end{pmatrix};\ b_1,b_2\in\boldsymbol{R}^{n-1}\right\},$$

$$\mathfrak{g}_0=\left\{\begin{pmatrix}\alpha & 0 & 0\\ 0 & A & 0\\ 0 & 0 & \beta\end{pmatrix};\ \alpha,\beta\in\boldsymbol{R}, A\in\mathfrak{gl}(n-1),\ \mathrm{tr}\,A=-(\alpha+\beta)\right\}.$$

Let G_0 denote the group of GLA-automorphims of $\mathfrak{g}$. It is given by

$$G_0 \cong \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}; \ a,c \in \boldsymbol{R}^*, b \in GL(n-1) \right\} \Big/ C,$$

and thus $G_0 \subset G$ and Lie $G_0 = \mathfrak{g}_0$. It holds that $G = G_0 \mathrm{Inn}(\mathfrak{g})$. We define $G_2 = \exp \mathfrak{g}_2$ and $G' = N_G(\mathfrak{g}')$, whose Lie algebras are $\mathfrak{g}_2$ and $\mathfrak{g}'$, respectively. Note that $G' \subset L'$. Furthermore, $\mathfrak{g}_0$ may be considered as an algebra of GLA-derivations of $\mathfrak{m}$ through the adjoint action. It is known (Yamaguchi [6]) that then the GLA $\mathfrak{g}$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ in the sense of Tanaka[5].

Let $\mathscr{F}(W)$ be the manifold of flags of W of type $(1,n)$, that is, the manifold of all pairs $([w],\sigma) \in P_n \times G_n(W)$ with $[w] \subset \sigma$, where $G_n(W)$ denotes the Grassmann manifold of n-subspaces of W. Let $\tilde{\omega}_0 : \mathscr{F}(W) \to P^n$ denote the projection $([w],\sigma) \mapsto [w]$ to the first factor. For $[w] \in P^n$, since $T_{[w]}P^n$ is linearly isomorphic to $W/[w]$, $T^*_{[w]}(P^n)$ is linearly isomorphic to

$$[w]^{\perp} = \{\zeta \in W^*;\ \zeta(w) = 0\},$$

and hence we can identify $P(T^*_{[w]}P^n)$ with $P([w^{\perp}])$ in a canonical way. For $[\zeta] \in P([w]^{\perp}) = P(T^*_{[w]}P^n)$ we define an element $\sigma \in G_n(W)$ with $[w] \subset \sigma$ by $\sigma = \{v \in W;\ \zeta(v) = 0\}$. Then the correspondence $[\zeta] \mapsto ([w],\sigma)$ gives a diffeomorphism of $P(T^*P^n)$ onto $\mathscr{F}(W)$ by which the projection $\varpi_0 : P(T^*P^n) \to P^n$ corresponds to our projection ϖ_0. On the other hand, G acts transitively on $\mathscr{F}(W)$, where the isotropy subgroup in G at the standard flag $([w_0], [w_0, \cdots, w_{n-1}])$ is identical with G'. Therefore, we have an identification

$$G/G' = P(T^*P^n),$$

by which the origin G' of G/G' corresponds to the point $[\zeta^n] \in P(T^*_{[w_0]}P^n) \subset P(T^*P^n)$. Note that under our identification the action of G on G/G' corresponds to the natural action $\varphi \mapsto \hat{\varphi}$ of $G = PL(n+1)$ on $P(T^*P^n)$. This induces an identification $T_{[\zeta^n]}P(T^*P^n) = \mathfrak{m}$. Let $\rho : G' \to GL(\mathfrak{m})$ denote the linear isotropy representation at $[\zeta^n]$, and set

$$\tilde{G} = \rho(G') \subset GL(\mathfrak{m}).$$

Then we have that Kernel $\rho = G_2$ and $\tilde{G} = \rho(G_0)\rho(\exp \mathfrak{g}_1)$.

Next, we set

$$\mathfrak{e} = \mathfrak{g}_{-1} \cap \mathfrak{l}_{-1}, \qquad \mathfrak{e}' = \mathfrak{g}_{-1} \cap \mathfrak{l}_0.$$

Then $(\mathfrak{e},\mathfrak{e}')$ is a Lagrangean pair of $(\mathfrak{g}_{-1}, A_0)$, and hence, $(\mathfrak{m};\ \mathfrak{e},\mathfrak{e}')$ is a fundamental GLA of Lagrangean contact type. If we set

$$e_0 = -E_{n+1,1},$$

$$e_i = E_{i+1,1}, \qquad e_{n-1+i} = E_{n+1,i+1} \qquad \text{for } 1 \leq i \leq n-1,$$

we have $(e_i, e_j) = \delta_{ij}$, and $\{e_0, e_1, \cdots, e_{2n-2}\}$ is a Lagrangean contact basis of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{e}')$. With respect to this basis, ρ is given by

$$G_0 \ni \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \bmod C \mapsto \begin{pmatrix} a^{-1}c & 0 & 0 \\ 0 & a^{-1}b & 0 \\ 0 & 0 & c\,{}^t b^{-1} \end{pmatrix},$$

$$\exp \mathfrak{g}_1 \ni \exp \begin{pmatrix} 0 & {}^t b_1 & 0 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ b_2 & 1_{n-1} & 0 \\ -b_1 & 0 & 1_{n-1} \end{pmatrix},$$

and thus $\tilde{G} = \rho(G_0)\rho(\exp \mathfrak{g}_1)$ is represented by

$$\tilde{G} = \left\{ \begin{pmatrix} c & 0 & 0 \\ b_1 & a & 0 \\ b_2 & 0 & c\,{}^t a^{-1} \end{pmatrix} ;\ c \in \boldsymbol{R}^*, b_1, b_2 \in \boldsymbol{R}^{n-1}, a \in GL(n-1) \right\}.$$

Therefore, $\tilde{G}$ is identical with the Lagrangean contact group $C(\mathfrak{m}; \mathfrak{e}, \mathfrak{e}')$. Thus we have proved

Theorem 5.1. *The $\tilde{G}$-structures of type $\mathfrak{m}$ are in bijective correspondence with the Lagrangean contact structures.*

EXAMPLE 5.2. Let $Q_{\mathrm{I}} \subset F^2(P^n)$ be the flat projective structure on P^n and $(D_0; E_0, E'_0)$ the Lagrangean contact structure on $P(T^*P^n)$ associated to Q_{I}. Then the $\tilde{G}$-structure of type $\mathfrak{m}$ on $P(T^*P^n)$ corresponding to $(D_0; E_0, E'_0)$ is given as follows. We embed $\mathfrak{m}$ into $P(T^*P^n)$ as an open set containing $[\zeta^n]$ by the map $X \mapsto (\exp X)[\zeta^n]$, and so each $a \in G$ determines a local diffeomorphism $a: (\mathfrak{m}, 0) \to P(T^*P^n)$. We define a map $\hat{\rho}_0: G \to F(P(T^*P^n))$ by

$$\hat{\rho}_0(a) = j^1_0(a) \qquad \text{for } a \in G.$$

Then we have that $\hat{\rho}_0(z \cdot a) = \hat{\rho}_0(z) \cdot \rho(a)$ for $z \in G$ and $a \in G'$, and the image

$\tilde{P}_{\mathfrak{g}}=\hat{\rho}_0(G)\subset F(P(T^*P^n))$ is a $\tilde{G}$-structure such that $\tilde{P}_{\mathfrak{g}}=F_{(D_0;E_0,E_0')}(P(T^*P^n))$. We call $\tilde{P}_{\mathfrak{g}}$ the flat *$\tilde{G}$-structure of type* $\mathfrak{m}$ on $P(T^*P^n)$.

A $\tilde{G}$-structure $\tilde{P}\subset F(M)$ of type $\mathfrak{m}$ or the corresponding Lagrangean contact structure is said to be *flat* if $(M,\tilde{P})$ is locally isomorphic to $(P(T^*P^n),\tilde{P}_{\mathfrak{g}})$ as $\tilde{G}$-structure.

Now we recall the result of Tanaka on Cartan connections associated to $\tilde{G}$-structures of type $\mathfrak{m}$. Let P be a principal G'-bundle over a manifold M of dimension $2n-1$ and ω a Cartan connection in P of type G/G' in the same sense as in Section 3. Let

$$\omega=\omega_{-2}+\omega_{-1}+\omega_0+\omega_1+\omega_2$$

be the decomposition of ω into the sum of $\mathfrak{g}_p$-components ω_p. Then the curvature $\Omega=d\omega+\frac{1}{2}[\omega,\omega]$ of ω can be written

$$\Omega=\frac{1}{2}K((\omega_{-2}+\omega_{-1})\wedge(\omega_{-2}+\omega_{-1}))$$

by a $\mathfrak{g}\otimes\Lambda^2\mathfrak{m}^*$-valued function K on P. We say that ω is *normal* if it satisfies the following two conditions.

(1) The $\mathfrak{g}_{-2}\otimes\Lambda^2\mathfrak{g}^*_{-1}$-component of K vanishes on P.

(2) If $\{e_0,e_1,\cdots,e_{2n-2}\}$ is a basis of $\mathfrak{m}$ with $(e_i,e_j)=\delta_{ij}$, and $\{e_0^*,e_1^*,\cdots,e_{2n-2}^*\}$ the basis of $\mathfrak{m}^*$ dual to $\{e_i\}$ with respect to B, then

$$(\partial^*K)(X)=\sum_i[e_i^*,K(e_i,X)]+\frac{1}{2}\sum_i K([e_i^*,X]_{\mathfrak{m}},e_i)=0 \qquad \text{for } X\in\mathfrak{m},$$

where in general $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $X\in\mathfrak{g}$ with respect to the decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{g}'$.

Let P be a principal G'-bundle over M endowed with a Cartan connection ω of type G/G' and $\tilde{P}\subset F(M)$ a $\tilde{G}$-structure of type $\mathfrak{m}$ with the restriction θ to $\tilde{P}$ of the canonical form on $F(M)$. We say that (P,ω) is *associated to* $\tilde{P}$, if there exists a group reduction $\hat{\rho}\colon P\to\tilde{P}$ relative to ρ, namely, a bundle map $\hat{\rho}$ inducing the identity on M and satisfying $\hat{\rho}(z\cdot a)=\hat{\rho}(z)\cdot\rho(a)$ for $z\in P$ and $a\in G'$, such that

$$\hat{\rho}^*\theta=\omega_{-2}+\omega_{-1}.$$

EXAMPLE 5.3. Let $\tilde{P}_{\mathfrak{g}}$ be the flat $\tilde{G}$-structure of type $\mathfrak{m}$ on $P(T^*P^n)$. Set $P_{\mathfrak{g}}=G$, which is a principal G'-bundle over $P(T^*P^n)$. Then the Maurer-Cartan form ω of G is a normal Cartan connection in $P_{\mathfrak{g}}$ of type G/G' with the curvature $\Omega=0$ such that $(P_{\mathfrak{g}},\omega)$ is associated to $\tilde{P}_{\mathfrak{g}}$.

Theorem 5.4. (Tanaka [5]) *For any $\tilde{G}$-structure $\tilde{P}$ of type $\mathfrak{m}$ on a manifold M of dimension $2n-1$, there exist a principal G'-bundle P over M and a normal Cartan connection ω in P of type G/G' such that (P,ω) is associated to $\tilde{P}$. Our (P,ω) is unique in the following sense. Let (P,ω) and (P',ω') be associated to $\tilde{P}$ and $\tilde{P}'$ with canonical forms θ and θ' by group reductions $\hat{\rho}$ and $\hat{\rho}'$, respectively. Then*

(a) *for any G'-bundle isomorphism $\varphi\colon P\to P'$ with $\varphi^*\omega'=\omega$ there exists a G-bundle isomorphism $\tilde{\varphi}\colon \tilde{P}\to\tilde{P}'$ with $\tilde{\varphi}^*\theta'=\theta$ which is induced by φ in the sense that $\hat{\rho}'\circ\varphi=\tilde{\varphi}\circ\hat{\rho}$; and*

(b) *conversely, for any $\tilde{G}$-bundle isomorphism $\tilde{\varphi}\colon \tilde{P}\to\tilde{P}'$ with $\tilde{\varphi}^*\theta'=\theta$ there exists uniquely a G'-bundle isomorphism $\varphi\colon P\to P'$ with $\varphi^*\omega'=\omega$ which induces $\tilde{\varphi}$.*

In the same way as in Corollary 3.5 we get the following.

Corollary 5.5. *$\tilde{P}$ is flat if and only if the curvature Ω of ω vanishes on P.*

6. Cartan connections associated to projective cotangent bundles

Let M be a manifold of dimension ≥ 2 and $\tilde{\omega}\colon P(T^*M)\to M$ the projective cotangent bundle of M. Fix a projective structure $\bar{\pi}^2\colon Q\to M$ on M. Let $(D;E,E')$ be the Lagrangean contact structure on $P(T^*M)$ associated to Q, and $\pi\colon \tilde{P}\to P(T^*M)$ the $\tilde{G}$-structure of the Lagrangean contact frames of $(P(T^*M),D;E,E')$. We define maps $\hat{\rho}\colon Q\to F(P(T^*M))$ and $\hat{\pi}\colon Q\to P(T^*M)$ as follows. Let $z=j_0^2(f)\in Q$ where $f\colon (V,0)\to M$ is a local diffeomorphism. We embed V into P^n as an open set containing $[w_0]$ as in Section 3. Then f induces a local diffeomorphism $\hat{f}\colon (P(T^*P^n),[\zeta^n])\to P(T^*M)$, and the differential

$$(\hat{f}_*)_{[\zeta^n]}\colon \mathfrak{m}=T_{[\lambda^n]}P(T^*P^n)\to T_{[\lambda]}P(T^*M),$$

where $[\lambda]=\hat{f}([\zeta^n])$, is a linear isomorphism. We define

$$\hat{\rho}(z)=(\hat{f}_*)_{[\zeta^n]}\in F(P(T^*M)),$$

$$\hat{\pi}(z)=\hat{f}([\zeta^n])\in P(T^*M).$$

Then we have that $\pi\circ\hat{\rho}=\hat{\pi}$, $\tilde{\omega}\circ\hat{\pi}=\bar{\pi}^2$, and

$$(*)\qquad \hat{\rho}(z\cdot a)=\hat{\rho}(z)\cdot\rho(a)\qquad \text{for } z\in Q,\quad a\in G'.$$

Note that $\hat{\pi}$ is a surjective map.

Lemma 6.1. $\hat{\rho}(Q)=\tilde{P}$.

Proof. We may assume that f above is given by

$$f(v)=\mathrm{Exp}^{\eta}u(v) \qquad \text{for } v\in V,$$

where η is a local torsionfree connection belonging to Q defined over a neighbourhood of $x=\bar{\pi}^2(z)$ and $u=\bar{\pi}_1^2(z)\in F(M)$, $\bar{\pi}_1^2$ being the projection $F^2(M)\to F(M)$. We make use of the basis $\{\bar{e}_i\}$ of $\mathfrak{l}_{-1}$ in Section 3, its dual basis $\{\xi^i\}$, and the basis $\{e_i\}$ of $\mathfrak{m}$ in Section 5. Let q: $\mathring{T}^*M\to P(T^*M)$ and q_0: $\mathring{T}^*P^n\to P(T^*P^n)$ be the natural projections. Then we have the following commutative diagram.

$$\begin{array}{ccc} T_{\xi^n}(\mathring{T}^*P^n)=V+V^* & \xrightarrow{(f^*)_*^{-1}} & T_\lambda(\mathring{T}^*M) \\ q_{0*}\downarrow & & q_*\downarrow \\ T_{[\zeta^n]}(P(T^*P^n))=\mathfrak{m} & \xrightarrow{\hat{\rho}(z)} & T_{[\lambda]}(P(T^*M)), \end{array}$$

where $V=\mathfrak{l}_{-1}=T_{[w_0]}P^n$ and $\lambda=u\cdot\xi^n$. Here the decomposition $T_{\xi^n}(\mathring{T}^*P^n)=V+V^*$ is the one induced from the trivialization of T^*P^n around $[w_0]$ through the embedding $V\subset P^n$. For $1\le i\le n$ we have $q_{0*}(\bar{e}_i)=e_i$ and $(f^*)_*^{-1}(\bar{e}_i)=(X_i)_\lambda^H$, where $X_i=u(\bar{e}_i)\in[\lambda]_x^\perp\subset T_xM$, and hence $\hat{\rho}(z)e_i=(X_i)_{[\lambda]}^{\bar{H}}$. Furthermore, we have $q_{0*}(\xi^i)=e_{n-1+i}$ and $(f^*)_*^{-1}(\xi^i)=(\lambda^i)_\lambda^V$, where $\lambda^i=u\cdot\xi^i\in T_x^*M$, and hence $\hat{\rho}(z)e_{n-1+i}=q_*(\lambda^i)_\lambda^V$. Thus $\hat{\rho}(z)$ maps $\mathfrak{e}$ and $\mathfrak{e}'$ to $E_{[\lambda]}$ and $E'_{[\lambda]}$, respectively. Together with Example 2.2, we know that $\hat{\rho}(z)$ is a Lagrangean contact frame of $(P(T^*M),D;\ E,E')$, that is, $\hat{\rho}(z)\in\tilde{P}$. Furthermore, it follows from (*) that $\hat{\rho}(Q)$ is invariant under $\tilde{G}$. Thus we obtain the lemma.

Lemma 6.2. *For $z,z'\in Q$, we have $\hat{\pi}(z)=\hat{\pi}(z')$ if and only if there exists an element $a\in G'$ such that $z'=z\cdot a$. Therefore $\hat{\pi}$: $Q\to P(T^*M)$ is a principal G'-bundle over $P(T^*M)$.*

Proof. Let $z=j_0^2(f)$ and $z'=j_0^2(f')$. Suppose that $\hat{\pi}(z)=\hat{\pi}(z')$, that is, $\hat{f}([\zeta^n])=\hat{f}'([\zeta^n])$. Since then $\bar{\pi}^2(z)=\bar{\pi}^2(z')$, there exists an element $a\in L'$ such that $j_0^2(f')=j_0^2(f\circ a)$. This implies that $\hat{f}'([\zeta^n])=\hat{f}(\hat{a}[\zeta^n])$. Therefore, from the assumption we obtain $\hat{a}[\zeta^n]=[\zeta^n]$, which means that $a\in G'$. Thus we get $z'=z\cdot a$ where $a\in G'$. The converse is clear from (*) and $\pi\circ\hat{\rho}=\hat{\pi}$.

Let $\bar{\omega}$ be the normal Cartan connection in the L'-bundle $\bar{\pi}^2\colon Q \to M$ of type L/L' (see Theorem 3.4). Since $G' \subset L' \subset L = G$, we may regard $\bar{\omega}$ as a Cartan connection in the G'-bundle $\hat{\pi}\colon Q \to P(T^*M)$ of type G/G'. Let

$$\bar{\omega} = \omega_{-2} + \omega_{-1} + \omega_0 + \omega_1 + \omega_2$$

be the decomposition of $\bar{\omega}$ into the sum of $\mathfrak{g}$-components ω_p.

Lemma 6.3. *For the restriction θ to $\tilde{P}$ of the canonical form on $F(P(T^*M))$, we have*

$$\hat{\rho}^*\theta = \omega_{-2} + \omega_{-1}.$$

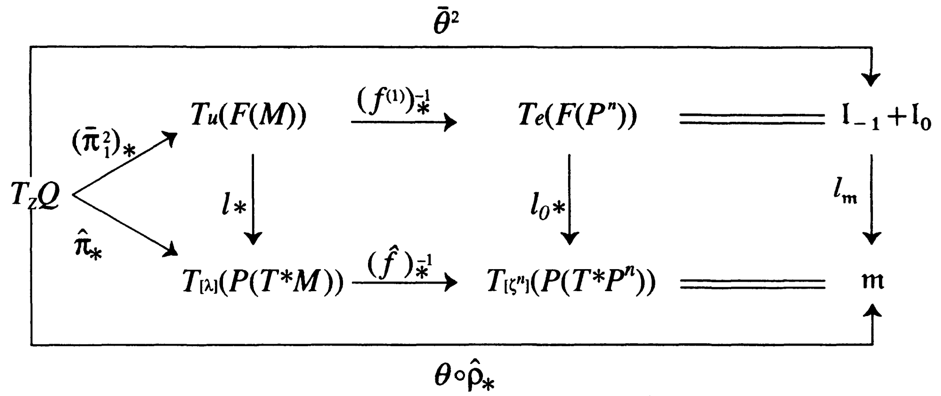
Proof. Since $\mathfrak{l}_{-1} = \mathfrak{e} + \mathfrak{g}_{-2}$, $\mathfrak{l}_0 = \mathfrak{e}' + (\mathfrak{l}_0 \cap \mathfrak{g}')$, and $\mathfrak{m} = \mathfrak{g}_{-2} + \mathfrak{e} + \mathfrak{e}'$, we get a decomposition

$$\mathfrak{l}_{-1} + \mathfrak{l}_0 = \mathfrak{m} + (\mathfrak{l}_0 \cap \mathfrak{g}') \quad \text{(direct sum as vector space).}$$

Denote by $l_{\mathfrak{m}}\colon \mathfrak{l}_{-1} + \mathfrak{l}_0 \to \mathfrak{m}$ the projection with respect to the decomposition above. Let $\bar{\theta}^2$ be the restriction to Q of the second canonical form on $F^2(M)$, which is an $\mathfrak{l}_{-1} + \mathfrak{l}_0 = V + \mathfrak{gl}(V)$-valued 1-form on Q. First, we shall show that

$$\hat{\rho}^*\theta = l_{\mathfrak{m}} \circ \bar{\theta}^2.$$

For that purpose we define a map $l\colon F(M) \to P(T^*M)$ by $u \mapsto u \cdot [\xi^n]$, where $\{\xi^i\}$ is the one in Lemma 6.1. The corresponding map for P^n will be denoted by $l_0\colon F(P^n) \to P(T^*P^n)$. Note that at the point $e = \mathrm{id}_V \in F(P^n)$ its differential $l_{0*}\colon T_e(F(P^n)) \to T_{[\xi^n]}(P(T^*P^n))$ corresponds to the projection $l_{\mathfrak{m}}$ under the identification $T_e(F(P^n)) = \mathfrak{l}_{-1} + \mathfrak{l}_0$ induced from the local trivialization $F(P^n)\,|\,V = V \times GL(V)$ through the embedding $V \subset P^n$. Now let $z = j_0^2(f) \in Q$ and set $u = \bar{\pi}_1^2(z)$, $[\lambda] = \hat{\pi}(z) = l(u)$. Then it follows from definitions that the following diagram is commutative.



This implies the required equality. Thus it suffices to show that

$$l_\mathfrak{m}\circ\bar{\theta}^2=\omega_{-2}+\omega_{-1}.$$

Let

$$\bar{\omega}=\bar{\omega}_{-1}+\bar{\omega}_0+\bar{\omega}_1$$

be the decomposition of $\bar{\omega}$ into the sum of $\mathfrak{l}_p$-components $\bar{\omega}_p$. Since $\bar{\omega}$ is normal, for any $X\in TQ$ we have $\bar{\theta}^2(X)=\bar{\omega}_{-1}(X)+\bar{\omega}_0(X)$, and hence $\bar{\omega}(X)=\bar{\theta}^2(X)+\bar{\omega}_1(X)$ with $\bar{\omega}_1(X)\in\mathfrak{l}_1\subset\mathfrak{g}'$. Therefore

$$\bar{\omega}(X)\equiv\bar{\theta}^2(X)\mod\mathfrak{g}'.$$

On the other hand we have

$$\bar{\omega}(X)\equiv\omega_{-2}(X)+\omega_{-1}(X)\mod\mathfrak{g}'.$$

These imply the required equality.

Theorem 6.4. *Let Q be a projective structure on a manifold M of dimension $n\geq 2$ and $\bar{\omega}$ the normal Cartan connection in Q of type L/L'. Let $\tilde{P}$ be the $\tilde{G}$-structure of type $\mathfrak{m}$ on $P(T^*M)$ corresponding to the Lagrangean contact structure on $P(T^*M)$ associated to Q. Then $\bar{\omega}$ is a normal Cartan connection of type G/G' in the principal G'-bundle $\hat{\pi}\colon Q\to P(T^*M)$ such that $(Q,\bar{\omega})$ is associated to $\tilde{P}$.*

Proof. It follows from Lemmas 6.1 and 6.3 that $(Q,\bar{\omega})$ is associated to $\tilde{P}$. The curvature $\bar{\Omega}=d\bar{\omega}+\frac{1}{2}[\bar{\omega},\bar{\omega}]$ of $\bar{\omega}$ is written in two ways:

$$\bar{\Omega}=\frac{1}{2}\bar{K}(\bar{\omega}_{-1}\wedge\bar{\omega}_{-1})=\frac{1}{2}K((\omega_{-2}+\omega_{-1})\wedge(\omega_{-2}+\omega_{-1})),$$

where $\bar{K}$ is the curvature function for the L'-bundle $\bar{\pi}^2\colon Q\to M$, and K the one for the G'-bundle $\hat{\pi}\colon Q\to P(T^*M)$. They are related as

$$K(X,Y)=\bar{K}(r(X),r(Y)) \qquad \text{for } X,Y\in\mathfrak{m},$$

where $r\colon\mathfrak{m}\to\mathfrak{l}_{-1}$ denotes the projection with respect to the decomoisition $\mathfrak{m}=\mathfrak{l}_{-1}+\mathfrak{e}'$. For any $X,Y\in\mathfrak{g}_{-1}$, from $\bar{K}_{-1}=0$ we have

$$K(X,Y)=\bar{K}(r(X),r(Y))=\bar{K}_0(r(X),r(Y))+\bar{K}_1(r(X),r(Y))$$
$$\in\mathfrak{l}_0+\mathfrak{l}_1\subset\mathfrak{g}_{-1}+\mathfrak{g}',$$

that is, the $\mathfrak{g}_{-2}$-component of $K(X,Y)$ is 0, and hence the $\mathfrak{g}_{-2}\otimes\Lambda^2\mathfrak{g}^*_{-1}$-component of K vanishes. Thus it remains to show that $\partial^*K=0$.

Let $\{\bar{e}_1,\cdots,\bar{e}_n\}$ be the basis of $\mathfrak{l}_{-1}$ with $(\bar{e}_i,\bar{e}_j)=\delta_{ij}$ and $\{\bar{e}^*_1,\cdots,\bar{e}^*_n\}$ the basis of $\mathfrak{l}_1$ with $B(\bar{e}_i,\bar{e}^*_j)=\delta_{ij}$, defined in Section 3. Let $\{e_0,e_1,\cdots,e_{2n-2}\}$ be the basis of $\mathfrak{m}$ with $(e_i,e_j)=\delta_{ij}$ defined in Section 5, and further define a basis $\{e^*_0,e^*_1,\cdots,e^*_{2n-2}\}$ of $\mathfrak{m}^*$ with $B(e_i,e^*_j)=\delta_{ij}$ by

$$e^*_0=-E_{1,n+1},$$
$$e^*_i=E_{1,i+1},\quad e^*_{n-1+i}=E_{i+1,n+1} \qquad \text{for } 1\le i\le n-1.$$

Note that $e_0=-\bar{e}_n, e^*_0=-\bar{e}^*_n$, and $e_i=\bar{e}_i, e^*_i=\bar{e}^*_i$ for $1\le i\le n-1$, and that $r(e_0)=-\bar{e}_n$ and $r(e_i)=\bar{e}_i, r(e_{n-1+i})=0$ for $1\le i\le n-1$. Recall that for $X\in\mathfrak{m}$ we have

$$(\partial^*K)(X)=\sum_{i=0}^{2n-2}[e^*_i,K(e_i,X)]+\frac{1}{2}\sum_{i=0}^{2n-2}K([e^*_i,X]_{\mathfrak{m}},e_i).$$

Now we have

$$[e^*_i,K(e_i,X)]=[e^*_i,\bar{K}(r(e_i),r(X))]$$
$$=\begin{cases}[\bar{e}^*_n,\bar{K}(\bar{e}_n,r(X))] & i=0,\\ [\bar{e}^*_i,\bar{K}(\bar{e}_i,r(X))] & 1\le i\le n-1,\\ 0 & n\le i\le 2n-2,\end{cases}$$

and hence

$$\sum_{i=0}^{2n-2} [e_i^*, K(e_i, X)] = \sum_{i=1}^{n} [\bar{e}_i^*, \bar{K}(\bar{e}_i, r(X))] = (\bar{\partial}^* \bar{K})(r(X)).$$

Furthermore, $[e_0^*, X] \in [\mathfrak{g}_2, \mathfrak{g}_{-2} + \mathfrak{g}_{-1}] \subset \mathfrak{g}_0 + \mathfrak{g}_1 \subset \mathfrak{g}'$, and so

$$K([e_0^*, X]_{\mathfrak{m}}, e_0) = 0.$$

For $1 \leq i \leq n-1$, since $[e_i^*, \mathfrak{m}] \subset \mathfrak{l}_0$ we have $[e_i^*, \mathfrak{m}]_{\mathfrak{m}} \subset \mathfrak{e}'$, and hence $r([e_i^*, X]_{\mathfrak{m}}) = 0$. Therefore

$$K([e_i^*, X]_{\mathfrak{m}}, e_i) = \bar{K}(r([e_i^*, X]_{\mathfrak{m}}), r(e_i)) = 0.$$

For $n \leq i \leq 2n-2$, since $r(e_i) = 0$ we have

$$K([e_i^*, X]_{\mathfrak{m}}, e_i) = \bar{K}(r([e_i^*, X]_{\mathfrak{m}}), r(e_i)) = 0.$$

Consequently we get

$$(\partial^* K)(X) = (\bar{\partial}^* \bar{K}(r(X)) \qquad \text{for } X \in \mathfrak{m}.$$

Since $\bar{\partial}^* \bar{K} = 0$ by normality for $\bar{K}$, we obtain $\partial^* K = 0$.

Now Corollaries 3.5 and 5.5 imply the following.

Corollary 6.5. *The Lagrangean contact structure on $P(T^*M)$ associated to a projective structure Q on M is flat if and only if Q is projectively flat, provided* $\dim M \geq 2$.

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