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MINIMAL THICKNESS AND UNIQUENESS OF KERNEL FUNCTIONS FOR THE HEAT EQUATION IN SEVERAL VARIABLES

Dedicated to Professor Mitsuru Nakai on his 60th birthday

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1. Introduction

Let $\boldsymbol{R}^{n+1}=\boldsymbol{R}^n\times\boldsymbol{R}$ be the $(n+1)$-dimensional Euclidean spase $(n\geq 1)$. We consider the heat equation

$$Lu:=\frac{\partial u}{\partial t}-\Delta u=0$$

and its nonnegative solutions (called parabolic functions). For an unbounded domain Ω in $\boldsymbol{R}^{n+1}$, a nonnegative parabolic function u in Ω is called a kernel function at infinity (resp. at a point $(y,s)\in\partial_p\Omega$) if u is not identically equal to zero and if u vanishes continuously on $\partial_p\Omega$ (resp. on $\partial_p\Omega\backslash\{(y,s)\}$), where $\partial_p\Omega$ denotes the parabolic boundary of Ω

We study the existence and uniqueness of kernel functions for the domains of the following form:

$$\Omega_\alpha(D)=\{(x,t)\in\boldsymbol{R}^n\times\boldsymbol{R};\ t<0,\ (-t)^{-\alpha}x\in D\},$$

where $\alpha\in\boldsymbol{R}$ and D is a bounded starlike Lipschitz domain in $\boldsymbol{R}^n$ with center 0, that is, D is starlike with center 0 and for every point $x_0\in\partial D,D$ is defined by a Lipschitz graph in some neighborhood of x_0 such that the ray $x_0 0$ is its axis(see [3,p. 513]).

J.T. Kemper [5] has studied kernel functions at finite boundary points, but our concern is ones at infinity, as discussed in [7], [8] and [4]. It has been shown that $\Omega_\alpha(D)$ has a unique kernel function at infinity if $n=1$, $\alpha<1$ ([8])and if $n\geq 1$, $\alpha\leq 1/2$ ([7]). Here we use the convention

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that a kernel function at a point is "unique" if any two kernel functions at the point differ only by a multiplicative constant.

The aim of this paper is to show the following theorem, which completes the above assertion.

Theorem A. *$\Omega_\alpha(D)$ has a unique kernel function at infinity if and only if $\alpha<1$.*

Remark that if $\alpha\geq 1$, then $\Omega_\alpha(D)$ has infinitely many kernel functions at infinity which are not proportional each other (cf. [8]).

Now we consider the Appell transformation $\mathscr{A}$: Put

$$\mathscr{A}(x,t)=(-t^{-1}x,-t^{-1})$$

for $(x,t)\in \boldsymbol{R}^n\times(-\infty,0)$ and

$$\mathscr{A}u(x,t)=(4\pi t)^{-n/2}\ \exp(-|x|^2(4t)^{-1})u(t^{-1}x,-t^{-1})$$

for a function u on a domain $\Omega\subset \boldsymbol{R}^n\times(-\infty,0)$ and for $(x,t)\in\mathscr{A}(\Omega):=\{\mathscr{A}(x,t);\ (x,t)\in\Omega\}$. Then $\mathscr{A}(\Omega_\alpha(D))=\{(x,t);t>0,\ t^{\alpha-1}x\in D\}$ and $\mathscr{A}u$ is the kernel function on $\mathscr{A}(\Omega_\alpha(D))$ at the origin if u is a kernel function on $\Omega_\alpha(D)$ at infinity (see [1, p.283]). Therefore Theorem A is easily deduced from the following

Theorem B. *Put $\Omega^\beta(D)=\{(x,t);t>0,t^{-\beta}x\in D\}$. Then $\Omega^\beta(D)$ has a unique kernel function at the origin if $\beta>0$.*

To prove the existence of kernel functions we prepare a kind of the boundary Harnack principle in §3. In §4 we show that a certain nontangential set is "minimally thick", i.e., it is not thin with respect to minimal kernel functions, which plays an important role to examine the uniqueness of kernel functions. This idea was first used in Hunt and Wheeden [3] for harmonic functions. Theorem B is proved in §5. Some comments on the boundedness of kernel functions at infinity are made in §6.

2. Preliminaries

For a domain Ω in $\boldsymbol{R}^{n+1}$ we denote by $\partial_p\Omega$ the set of $(y,s)\in\partial\Omega$ (=the boundary of Ω) satisfying $V\cap\Omega\cap \boldsymbol{R}^n\times(s,\infty)\neq\emptyset$ for every neighborhood V of (y,s). This is called the parabolic boundary of Ω. For $(x,t)\in\Omega$, we denote by $\omega_\Omega^{(x,t)}$ the parabolic measure at (x,t) with respect to Ω. The parabolic measure $\omega_\Omega^{(x,t)}$ is supported by $\partial_p\Omega\cap \boldsymbol{R}^n\times(-\infty,t]$ and for any bounded continuous function f on $\partial_p\Omega$, the function $\int f d\omega_\Omega^{(x,t)}$ of (x,t) is the solution of the Dirichlet problem with boundary value f. For the detail of potential theory for the heat equation, see [1] or [10].

A boundary point $(y,s)\in\partial_p\Omega$ is said to be regular if for every bounded continuous function f on $\partial_p\Omega$,

$$\lim_{(x,t)\in\Omega\to(y,s)}\int f d\omega_\Omega^{(x,t)}=f(y,s).$$

For a nonempty open set B in $\boldsymbol{R}^n$, $(y,s)\in\boldsymbol{R}^{n+1}$ and for $r>0$, we define tusk cones with vertex (y,s) as follows:

$$T_{(y,s)}(B,r)=\{(x,t);0<t-s<r,(t-s)^{-1/2}(x-y)\in B\}$$

and

$$T^*_{(y,s)}(B,r)\{(x,t);0<s-t<r,(s-t)^{-1/2}(x-y)\in B\}.$$

It is well-known that a boundary point (y,s) of a domain Ω is regular if

$$T^*_{(y,s)}(B,r)\cap\Omega=\emptyset$$

with some open set $B\neq\emptyset$ and $r>0$ (cf. [2]).

DEFINITION 1. A domain Ω in $\boldsymbol{R}^{n+1}$ is called *uniformly regular* if there exist a nonempty open set B in $\boldsymbol{R}^n$ and $r>0$ such that for every point $(y,s)\in\partial_p\Omega$,

$$T^*_{(y,s)}(PB,r)\cap\Omega=\emptyset$$

with some orthogonal transformation P on $\boldsymbol{R}^n$.

We describe the assumptions for domains which will be considered in the later sections. These are adequate for domains $\Omega^\beta(D)$, $0<\beta<1/2$ (see the proof of Theorem 2 below).

DEFINITION 2. Let Ω be a domain in $\boldsymbol{R}^{n+1}_+=\boldsymbol{R}^n\times(0,\infty)$. We say that Ω satisfies *Condition* (*) if the following conditions are satisfied:

(1) There exist a constant $T>0$ and an upper semicontinuous function $\varphi\geq0$ on $\boldsymbol{R}^n$ such that

$$\Omega=\{(x,t);T>t>\varphi(x)\}.$$

(2) $\liminf_{x\to\infty}\varphi(x)>0$.

(3) $\limsup_{x\to0}|x|^{-2}\varphi(x)<\infty$.

(4) Ω is uniformly regular.

(5) There exist constants $r_2 > r_1 > 0$ such that for every $(y,s) \in \partial_p\Omega \setminus \{(0,0)\}$ with $0 < s < r_1$,

$$T_{(y,s)}(B(-|y|^{-1}r_2 y, r_1), r_2^{-2}|y|^2) \subset \Omega,$$

where $B(x,r)$ is the open ball in $\boldsymbol{R}^n$ with center x and radius $r > 0$.

(6) $\lim_{t \to 0} \sup\{|x|; \varphi(x) < t\} = 0$.

3. Boundary Harnack principle

Throughout this section Ω is a domain in $\boldsymbol{R}_+^{n+1}$ which satisfies Condition (*). Fix $0 < a < 1$ and take a constant r_0 with $r_0 > \limsup_{x \to 0} |x|^{-2}\varphi(x)$. For each $\tau > 0$, we put

$$A_\tau = (0, a\tau) \in \Omega,$$
$$E(\tau) = \{(x,t); \tau < t < (a+1)\tau, r_0|x|^2 < t\},$$
$$E'(\tau) = \{(x,t); 0 < t < a\tau, r_0|x|^2 < t + \tau\},$$
$$E''(\tau) = \{(x,t); r_0|x|^2 < a\tau - t < a\tau\}$$

and choose a number $t(\tau) > 0$ such that

$$\{(x,t) \in \Omega; t \leq t(\tau)\} \subset E''(\tau/2).$$

Then by the similar manner to the proof of Lemma 4 in [7] (see also [6, Lemma 5]) we have

$$\omega_{\Omega \cap \{t > t(\tau)\}}^{(x,t)} (B \times \{t(\tau)\}) \leq \text{Const } \omega_{\Omega \cap \{t > t(\tau)\}}^{A_\tau} (B \times \{t(\tau)\})$$

for $(x, t) \in \Omega \setminus E''(\tau)$ and for every open ball B in $\{x; (x, t(\tau)) \in \Omega\}$. This esttimate yields the following boundary Harnack principle (cf. [7, Lemma 5]).

Lemma 1. *There is a constant $C_1 > 0$ such that for every parabolic function $u \geq 0$ on Ω vanishing continuously on $\partial_p\Omega \cap \{t > t(\tau)\}$, we have*

$$u(x,t) \leq C_1 u(A_\tau)\omega_\Omega^{(x,t)}(E'(\tau) \cap \partial\Omega) \quad \text{on } \Omega \setminus E''(\tau).$$

By Lemma 1 and the standard argument (see the proof of Proposition 2 in [7]), we can see the following

Lemma 2. *There is a kernel function on Ω at the origin.*

4. Minimal thickness

A nonnegative parabolic function u on a domain Ω is called minimal if every parabolic function v satisfying $0 \leq v \leq u$ is a constant multiple of u. A nonnegative lower semicontinuous function u on Ω is called superparabolic if $Lu \geq 0$ in the distribution sense. For a superparabolic function u on Ω and an open set F in Ω, $R_F u$ denotes the reduced function of u on F, that is, the minimum superparabolic function on Ω with $R_F u = u$ on F.

The following assertion means "minimal thickness" of a nontangential set.

Lemma 3. *Let Ω be a domain in $\boldsymbol{R}^{n+1}$ satisfying Condition(*). For a decreasing sequence $\{\tau_m\}_{m=1}^{\infty}$ in $\boldsymbol{R}_+$ tending to 0, we put*

$$F = F(\{\tau_m\}_{m=1}^{\infty}) := \bigcup_{m=1}^{\infty} E(\tau_m).$$

If u is a minimal kernel function on Ω at the origin, then $R_F u = u$ on Ω. Here $E(\tau)$ is the set defined in §3.

Proof. Let $(0, t_0) \in \Omega$. We may assume that $u(0,t_0) = 1$. For each $m \in \boldsymbol{N}$, put

$$F_m = \bigcup_{j=m}^{\infty} E(\tau_j).$$

Then $R_{F_m} u$ decreases to some parabolic function $v \geq 0$ as $m \to \infty$ because F_m decreases to the empty set. Since $v \leq u$ and u is minimal, $v = ku$ for some constant $k \geq 0$. By the (parabolic) Harnack inequality, for $0 < \tau < t_0$,

$$R_{E(\tau)} u = u \geq Cu(A_\tau) \text{ on } \Omega \cap E(\tau)$$

with some constant $C > 0$, so that

$$R_{E(\tau)} u(x, t+\tau) \geq Cu(A_\tau)\omega_\Omega^{(x,t)}(\partial\Omega \cap E'(\tau)).$$

On the other hand, Lemma 1 shows

$$1 = u(0,t_0) \leq C_1 u(A_\tau)\omega_\Omega^{(x,t_0)}(\partial\Omega \cap E'(\tau)).$$

Hence

$$v(0,t_0) = \lim_{m\to\infty} R_{F_m} u(0, t_0 + \tau_m)$$

$$\geq \liminf_{m\to\infty} R_{E(\tau_m)}u(0,t_0+\tau_m) \geq CC_1^{-1} > 0,$$

which implies $k>0$. Since $R_F u + R_F v - v = u$ on F, we see

$$R_F u + R_F v - v \geq R_F u \text{ on } \Omega.$$

This shows $R_F v = v$, so that

$$R_F u = R_F(v/k) = (R_F v)/k = v/k = u$$

on Ω. The lemma is proved.

5. Uniqueness of kernel functions

We begin with the following

Theorem 1. *Let Ω be a domain in $\boldsymbol{R}_+^{n+1}$ satisfying Condition (*) and let $(0, t_0) \in \Omega$. Then there exists a unique kernel function u on Ω at the origin with $u(0, t_0)=1$.*

Proof. The existence of the kernel functions at the origin is stated in Lemma 2. To show the uniqueness, we denote by $H(\Omega)$ the space of all parabolic functions on Ω endowed with the topology of uniform convergence on compact sets and set

$$H_0^+(\Omega) = \{u \in H(\Omega); u \geq 0, \text{ continuously vanishes on } \partial_p\Omega \setminus \{(0,0)\} \text{ and } u(0,t_0)=1\}.$$

Then with the aid of the Harnack inequality and Lemma 1, we see that $H_0^+(\Omega)$ is a compact convex set in $H(\Omega)$. Hence by the Krein-Milman theorem, it is sufficient to show the uniqueness of the minimal kernel functions at the origin. Let $u_i, i=1,2$, be minimal kernel functions at the origin with $u_i(0,t_0)=1$. We may assume that there is a decreasing sequence $\{\tau_m\}_{m=1}^{\infty}$ tending to 0 such that

$$u_1(A_{\tau_m}) \geq u_2(A_{\tau_m})$$

for each integer $m \geq 1$. By the Harnack inequality and Lemma 1 again, there is a constant $C>0$ such that

$$u_1 \geq Cu_2 \text{ on } F(\{\tau_m\}_{m=1}^{\infty}).$$

Thus Lemma 3 leads to

$$u_1 \geq Cu_2 \text{ on } \Omega.$$

Since u_1 minimal, we have $u_1 = C'u_2$ for some constant $C' \geq 0$, which implies $u_1 = u_2$. This completes the proof.

Now we turn to Theorem B. According to the result in [7], only the case $0 < \beta < 1/2$ must be handled, but this case follows from the following

Theorem 2. *Let D be a bounded starlike Lipschitz domain in $\boldsymbol{R}^n$ with center 0 and let ψ be a lower semicontinuous and increasing function on an open interval$(0,\tau_0)$. If* $\liminf_{t\to 0} t^{-1/2}\psi(t) > 0$, *then $\Omega(\psi,D)$ has a unique kernel function at the origin, where*

$$\Omega(\psi,D) = \{(x,t); 0 < t < \tau_0, \psi(t)^{-1}x \in D\}.$$

Proof. If $\lim_{t\to 0}\psi(t) > 0$, the assertion is verified easily. In case $\lim_{t\to 0}\psi(t) = 0$, by Theorem 1, it is sufficient to check that the domain $\Omega(\psi,D)$ satisfies Condition(*). Because ψ is lower semicontinuous and increasing, $\Omega(\psi,D)$ can be written

$$\Omega(\psi,D) = \{(x,t); \tau_0 > t > \varphi(x)\}$$

for some upper semicontinuous function φ such that $\varphi(x) = \tau_0$ when $(\lim_{t\to\tau_0}\psi(t))^{-1}x \notin D$. This implies (1) and (2). The condition (3) follows from $\liminf_{t\to 0} t^{-1/2}\psi(t) > 0$. Since D is Lipschitz and ψ is increasing, we see easily (4) and (5). The remained condition (6) follows from $\lim_{t\to 0}\psi(t) = 0$.

6. Bounded kernel functions

In this section we consider the case that the origin is an irregular boundary point with respect to the adjoint heat equation (coirregular point). Then we can construct a kernel function at the origin in the following way.

For a domain Ω in $\boldsymbol{R}^{n+1}$ with $0 \in \partial\Omega$, put

$$u_\Omega(x,t) = W(x,t) - \int W d\omega_\Omega^{(x,t)},$$

where W is the fundamental solution of the heat equation, that is,

$$W(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\dfrac{|x|^2}{4t}\right) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}.$$

If Ω is regular, i. e., every parabolic boundary point is regular, then u_Ω is a kernel function at the origin provided that $u_\Omega \not\equiv 0$.

Remark. $u_\Omega \not\equiv 0$ if and only if the origin is a coirregular boundary point of Ω (cf. [9, Lemma 3]).

By Theorem 1 we have

Proposition 1. *Let Ω be a domain in $\boldsymbol{R}_+^{n+1}$* satisfying Condition (*). *If the origin is coirregular, then u_Ω is the unique kernel function at the origin.*

Applying the Appell transformation $\mathscr{A}$, we can give a criterion whether a domain has a bounded kernel functions at infinity. Note that $\mathscr{A}^{-1}u_\Omega = 1-\omega_\Omega^{(\cdot,\cdot)}(\partial\Omega)$ and that a kernel function at infinity is a barrier function at all parabolic boundary points.

Proposition 2. *A domain Ω in $\boldsymbol{R}^{n+1}$ has a bounded kernel function at infinity if and only if $\mathscr{A}(\Omega)\cap\boldsymbol{R}^n\times(-\infty,0))$ is regular and the origin is a coirregular point of $\mathscr{A}(\Omega)\cap\boldsymbol{R}^n\times(-\infty,0))$. In fact, $1-\omega_\Omega^{(x,t)}$ $(\partial\Omega)$ is a bounded kernel function on Ω at infinity.*

As an example, we now consider domains

$$\Omega=\{(x,t);t<0,\ |x|^2<-2nt\ \log(-t)\}$$

and for $k>0$

$$\Omega_{(k)}=\{(x,t);t<0,\ |x|^2<-kt\ \log\log(-t)\}.$$

Theorem 1 and the Appell transformation again show that these domains have a unique kernel function at infinity. Furthermore since the origin is a coirregular point of $\mathscr{A}(\Omega)$ and of $\mathscr{A}(\Omega_{(k)})$ for $k>4$ (see [1, pp. 338–340]), the present kernel function is bounded. On the other hand for $0<k\leq 4, \Omega_{(k)}$ does not have any bounded kernel function at infinity. This observation justifies the comments in [4, p. 869].

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