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ON SPECTRA OF RANDOM SCHRÖDINGER OPERATORS WITH MAGNETIC FIELDS

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1. Introduction

In this paper, we investigate properties of spectra of random Schrödinger operators with magnetic fields. In particular we study the asymptotics of the density of states. On a probability space (Ω, P), we consider a 1-form valued random field $b=\sum_{j=1}^{d} b_{\omega}^{j}(x)dx^{j}$ $(\omega\in\Omega, x\in\boldsymbol{R}^{d})$ and a real valued random field $V=V_{\omega}(x)$ on $\boldsymbol{R}^{d}$. We assume that the pair $(db.(x), V.(x))_{x\in\boldsymbol{R}^{d}}$ is stationary and ergodic on $\boldsymbol{R}^{d}$. We assume further conditions on b and V later. On the space $L^{2}(\boldsymbol{R}^{d})$ of complex square integrable functions on $\boldsymbol{R}^{d}$, we consider the operator formally written as follows:

$$L(b_{\omega}, V_{\omega}) = -\frac{1}{2}\sum_{j=1}^{d}\left(\frac{\partial}{\partial x^{j}}-ib_{\omega}^{j}(x)\right)^{2}+V_{\omega}(x) \qquad (i=\sqrt{-1}).$$

Under the assumptions, as same as in the case of $b=0$ (cf. [1]), it is easily seen that the spectra of $L(b_{\omega}, V_{\omega})$ are independent of ω except for the elements of a P-measure null set. Our purpose is to show that several properties of random Schrödinger operators without magnetic fields are extended to our case. In particular we consider the asymptotics of the density of states at the infimum of its support. As same as in cases of Pastur [14] and Nakao [13] (cf. Chapter VI of [3]), the problem can be reduced to study the asymptotics of $t\to\infty$ of the Laplace transform of the density of states, i.e.,

$$\tag{1.1} \int_{-\infty}^{\infty} e^{-\lambda t}n(d\lambda) = \left(\frac{1}{2\pi t}\right)^{d/2} E^{W\times P}\left[\exp\left(-i\int_{0}^{t} db(*) - \int_{0}^{t} V(w(s))ds\right)\middle| w(t)=0\right],$$

where w is a d-dimensional Wiener process starting at 0, $\int_{0}^{t} db(*)$ is a stochastic integral of the 2-form db (for the exact form, see (3.2) below) and $E^{W\times P}$ is the expectation with respect to the product measure of P and the Wiener measure. One of difficulties of these problems comes from the fact that the right hand side of (1.1) is an oscillatory integral (cf. [7]).

In this paper, we assume that V is a Gaussian random field. Then, for the asymptotics, the effect of V is much stronger than that of b. Therefore, we can show that the leading term of the asymptotics is same as the result of Pastur [14] (see Theorem 4.2 below).

The organization of this paper is as follows. In Section 2, we give a setting to consider random Schrödinger operators with magnetic fields and discuss fundamental results: we show that the spectral structure is determined with probability one. Furthermore we give examples which have been studied. In Section 3, we show the existence of the density of states and a few properties of this. In Section 4, we investigate asymptotic properties of the integrated density of states.

2. Random Schrödinger operators with magnetic fields

In this section, following [3], we give a setting to consider spectra of random Schrödinger operators with magnetic fields and discuss fundametal results. Furthermore we give examples which have been studied.

Let $\Omega_1=\Gamma^1(\boldsymbol{R}^d, T^*\boldsymbol{R}^d)$ be the space of all C^1-differential forms endowed with the topology of uniform convergence in the wider sense, $\mathcal{F}_1=\mathcal{B}(\Omega_1)$ the topological σ-field of Ω_1, and $\mathcal{G}$ the sub σ-field of $\mathcal{F}_1$ defined by

$$\mathcal{G}=\{B\in\mathcal{F}_1;\ b\in B,\ db=db'\Rightarrow b'\in B\}.$$

Let $L(\boldsymbol{R}^d)$ be the set of all real valued measurable functions on $\boldsymbol{R}^d$. We take a measurable space $(\Omega_2, \mathcal{F}_2)$ satisfying the following two conditions: i) there is a map V from Ω_2 to $L(\boldsymbol{R}^d)$ such that a map $\Omega_2\times\boldsymbol{R}^d\ni(\omega, x)\mapsto V(\omega, x)=: V_\omega(x)\in\boldsymbol{R}$ is $\mathcal{F}_2\times\mathcal{B}(\boldsymbol{R}^d)$-measurable, ii) for any $x\in\boldsymbol{R}^d$, there is a measurable transformation T_x on Ω_2 such that $V_{T_x\omega}(y)=V_\omega(x+y)$. An element V of $L(\boldsymbol{R}^d)$ is said to be in K_d, when $d\geq 3$,

$$\lim_{a\searrow 0}\sup_{x\in\boldsymbol{R}^d}\int_{|x-y|\leq a}|x-y|^{2-d}|V(y)|\,dy=0\,.$$

when $d=2$,

$$\lim_{a\searrow 0}\sup_{x\in\boldsymbol{R}^d}\int_{|x-y|\leq a}\log(1/|x-y|)|V(y)|\,dy=0\,.$$

V is said to be in K_d^{loc} when $V\chi_D$ is in K_d for all bounded domains D. We say V is in $\boldsymbol{K}_d$ when the positive part $V_+=\max\{V, 0\}$ is in K_d^{loc} and the negative part $V_-=\max\{-V, 0\}$ is in K_d (see [16]). For any $(b, V)\in\Omega_1\times\boldsymbol{K}_d$, we define a selfadjoint operator $L(b, V)$ on $L^2(\boldsymbol{R}^d)$ by the unique self adjoint extension of

$$L(b, V)f=-\frac{1}{2}\sum_{j=1}^{d}\left(\frac{\partial}{\partial x^j}-ib^j\right)^2 f+Vf\quad\text{on } C_0^\infty(\boldsymbol{R}^d)$$

(cf. [16] Theorem B.12.1). Setting $\Omega := \Omega_1 \times \Omega_2$, we consider a family of operators $\{L(b, V_\omega); (b, \omega) \in \Omega\}$ under a probability measure P on $(\Omega, \mathcal{G} \times \mathcal{F}_2)$ satisfying the following conditions;

(A.1) (stationarity) $P(T_x B) = P(B)$ for any $B \in \mathcal{G} \times \mathcal{F}_2$ and $x \in \boldsymbol{R}^d$.
(A.2) (ergodicity) If $B \in \mathcal{G} \times \mathcal{F}_2$ satisfies $P(T_x B \ominus B) = 0$ for any $x \in \boldsymbol{R}^d$, then $P(B) = 0$ or 1.
(A.3) $P(V \in \boldsymbol{K}_d \cap L^2_{\mathrm{loc}}) = 1$.

In (A.1) and (A.2), T_x acts on Ω_1 by $(T_x b)(y) = b(x+y)$. For any operator L, let $\Sigma(L)$, $\Sigma_{pp}(L)$, $\Sigma_{ac}(L)$ and $\Sigma_{sc}(L)$ be the spectrum, the point spectrum, the absolutely continuous spectrum and the singular continuous spectrum, respectively, of L.

Then we obtain the following (cf. [3] Proposition V.2.4):

Theorem 2.1. *If a probability measure P on $(\Omega, \mathcal{G} \times \mathcal{F}_2)$ satisfies* (A.1), (A.2) *and* (A.3), *then there exist closed subsets Σ, Σ_{pp}, Σ_{ac} and Σ_{sc} of $\boldsymbol{R}$ such that for P-almost all (b, ω) in Ω, we have $\Sigma(L(b, V_\omega)) = \Sigma$, $\Sigma_{pp}(L(b, V_\omega)) = \Sigma_{pp}$, $\Sigma_{ac}(L(b, V_\omega)) = \Sigma_{ac}$ and $\Sigma_{sc}(L(b, V_\omega)) = \Sigma_{sc}$.*

Proof. Let $E(\Lambda, b, V)$, $\Lambda \in \mathcal{B}(\boldsymbol{R})$, be the resolution of the identity of the operator $L(b, V)$ and

$$E(\cdot, b, V) = E_{pp}(\cdot, b, V) + E_{ac}(\cdot, b, V) + E_{sc}(\cdot, b, V)$$

its Lebesgue decomposition. To prove the theorem, we have only to prove that for any Λ, $A \in \mathcal{B}(\boldsymbol{R})$ and $\sharp = pp$, ac and sc,

$$\{(b, \omega) \in \Omega;\ \mathrm{tr}\, E_\sharp(\Lambda, b, V_\omega) \in A\} =: B$$

is a T_x-invariant element of $\mathcal{G} \times \mathcal{F}_2$. But, by the well-known method for the $b=0$ case, we can prove that this B is a T_x-invariant element of $\mathcal{F}_1 \times \mathcal{F}_2$ (cf. §§V.2 and V.3 of [3]). Then, by the gauge invariance of the operator $L(b, V)$, we see that this B is an element of $\mathcal{G} \times \mathcal{F}_2$. □

Similarly, we have the following theorem by the same method to prove the corresponding theorem for the $b=0$ case ([3] Proposition V.2.8):

Theorem 2.2. *Let P be a probability measure on $(\Omega, \mathcal{G} \times \mathcal{F}_2)$ satisfying* (A.1), (A.2) *and* (A.3). *Then:*

i) $\Sigma_{disc}(L(b, V)) = \phi$, *$P$-a.s.*

ii) *If the spectral multiplicity of the operator $L(b, V)$ is P-almost surely finite, then, for each λ in $\boldsymbol{R}$, λ is P-almost surely not an eigenvalue of $L(b, V)$.*

Before closing this section, we give examples which have been studied:

EXAMPLE 2.1. We assume that db is a deterministic constant magnetic field and the scalar potential V is a stationary ergodic random field. This case satisfies our conditions (A.1)-(A.3). When $d=2$, this is the object of the quantum Hall effect and there are many works concerning about this ([1], [11] and so on). We also have results on the structure of a density of states ([10], [18]).

EXAMPLE 2.2. We assume that $db(x)$ and $V(x)$ are deterministic periodic functions in x with same period. As is well known, these can be regarded as stationary ergodic random fields (cf. Example 1 of Chapter V.3.1 of [3]). Therefore this case also satisfies our conditions. In this case, the structure of the spectra were studied by serval authors (for example [4], [5], [6]).

3. Density of states

In this section, following [13], we introduce the density of states and discuss some properties of this. Let $W=\{w: [0, \infty)\to \boldsymbol{R}^d:$ continuous, $w(0)=0\}$ endowed with the topology of uniform convergence in the wider sense and P^W the Wiener measure on W. We consider a probability measure P on $(\Omega, \mathcal{G}\times\mathcal{F}_2)$ satisfying (A.1), (A.2) and the following:

$$\text{(A.4)}\quad E^{W\times P}\left[\exp\left(r\int_0^t V_-(w(s))ds\right)\right]<\infty$$

for some $r>2$ and any $t>0$,

$$\text{(A.5)}\quad P(db\colon C^1,\ V\in \boldsymbol{K}_d\cup C(\boldsymbol{R}^d))=1\,,$$

where $E^{W\times P}$ is the expectation with respect to the product measure $P^W\times P$.

For each rectangular domain D of the form $\Pi(-a_j, a_j)$ with $a_j>0$, let $(L(b, V)_D, D(L(b, V)_D))$ be the Friedrichs extension of the symmetric operator $(L(b, V), C_0^\infty(D))$. Let $e^{-tL(b,V)_D}(x, y)$, $(t, x, y)\in(0, \infty)\times D\times D$ be the integral kernel of the semigroup $e^{-tL(b,V)_D}$ generated by $L(b, V)_D$: for any $f\in L^2(D)$,

$$(e^{-tL(b,V)_D}f)(x)=\int_D e^{-tL(b,V)_D}(x, y)f(y)dy\,.$$

Then, by using the arguments in [15] and [16], we obtain the following:

Lemma 3.1. *$e^{-tL(b,V)_D}(x, y)$ is expressed as*

$$\begin{aligned}
&e^{-tL(b,V)_D}(x, y)\\
(3.1)\qquad &=E^W\Big[\exp\Big(-i\sum_{j=1}^d\int_0^t b^j(x+w(s))\circ dw^j(s)-\int_0^t V(x+w(s))ds\Big)\\
&\qquad\times \chi_{\{\tau_D(x)>t\}}\Big| x+w(t)=y\Big]\Big(\frac{1}{2\pi t}\Big)^{d/2}\exp\Big(-\frac{|x-y|^2}{2t}\Big),
\end{aligned}$$

where $\tau_D(x)=\inf\{s>0: x+w(s)\notin D\}$. $e^{-tL(b,V)_D}(x, y)$ is jointly continuous in

$(t, x, y) \in (0, \infty) \times D \times D$.

From the condition (A.4) and (3.1), we see that $e^{-tL(b,V)_D}$ is a Hilbert-Schmidt operator on $L^2(D)$ for P-almost all $(b, \omega) \in \Omega$. Thus

$$\Sigma(L(b, V)_D) = \Sigma_{disc}(L(b, V)_D) =: \{\lambda_{D,b,V,1} \leq \lambda_{D,b,V,2} \leq \cdots \nearrow \infty\}.$$

We set

$$N_{D,b,V}(\lambda) := \frac{1}{|D|} \#\{j ; \lambda_{D,b,V,j} \leq \lambda\}$$

for $\lambda \in \boldsymbol{R}$. This is an increasing function of λ. Let $n_{D,b,V}(d\lambda)$ be the measure determined from $N_{D,b,V}(\lambda)$. The Laplace transform $\mathcal{L}(n_{D,b,V}, t)$ of the measure $n_{D,b,V}$, is

$$\mathcal{L}(n_{D,b,V}, t) = \int_{\boldsymbol{R}} e^{-\lambda t} n_{D,b,V}(d\lambda) = \frac{1}{|D|} \mathrm{tr}\,[e^{-tL(b,V)_D}]\,.$$

By using Lemma 3.1 and Mercer's expansion theorem, we have

$$\mathcal{L}(n_{D,b,V}, t) = \frac{1}{|D|} \int_D e^{-tL(b,V)_D}(x, x) dx\,.$$

Furthermore we use (3.1) and the stochastic version of Stokes' theorem (cf. [8], [17]). Then we obtain the following:

$$\begin{aligned}\mathcal{L}(n_{D,b,V}, t) = \frac{1}{|D|}\left(\frac{1}{2\pi t}\right)^{d/2} \int_D E^W\Big[\exp\Big(-i \int_0^t db(x+*) \\ -\int_0^t V(x+w(s))ds\Big)\chi_{\{\tau_D(x)>t\}}\Big| w(t)=0\Big]dx\,,\end{aligned} \tag{3.2}$$

where

$$\int_0^t db(x+*) = \sum_{i<k} \int_0^t \left(\int_0^1 (db)_{j,k}(x+uw(s))2udu\right) \circ dS_{j,k}(s)$$

and

$$S_{j,k}(t) = \frac{1}{2} \int_0^t (w^j(s) \circ dw^k(s) - w^k(s) \circ dw^j(s))$$

([8], [17]). Now we remark that (db, V) is a stationary and ergodic random field. Therefore, by the same argument with the case of $db=0$, we obtain the following existence result:

Theorem 3.1. *If a probability measure P on $(\Omega, \mathcal{G} \times \mathcal{F}_2)$ satisfies* (A.1), (A.2), (A.4) *and* (A.5), *then for P-almost all (b, ω) in Ω, the measures $n_{D,b,V}(d\lambda)$*

converge vaguely to a deterministic measure $n(d\lambda)$ *as* $D \to \boldsymbol{R}^d$ *(i.e. all the* a_j *diverge to* $+\infty$ *simultaneously). Moreover the Laplace transform of* $n(d\lambda)$ *is given by*

$$(3.3)\quad \mathcal{L}(n, t) = \left(\frac{1}{2\pi t}\right)^{d/2} E^{W\times P}\left[\exp\left(-i\int_0^t db(*) - \int_0^t V(w(s))\,ds\right)\middle|\, w(t) = 0\right].$$

We call the measure $n(d\lambda)$ the *density of states*. Moreover we call the increasing function $N(\lambda) := n((-\infty, \lambda])$ the *integrated density of states*.

In the following, we assume the condition (A.3). For any positive smooth function f on $\boldsymbol{R}^d$ with compact support such that $\int f(x)^2 dx = 1$ and for any bounded Borel set A in $\boldsymbol{R}$, let $E_f(A, b, V)$ be the operator defined by

$$(E_f(A, b, V)g)(x) = f(x)\cdot(E(A, b, V)(f\cdot g))(x)$$

for any $g \in L^2(\boldsymbol{R}^d)$. Then as same as in the case of $b=0$ (cf. [3] Proposition VI. 1.3), we have the following:

Proposition 3.1. *Let* P *be a probability measure on* $(\Omega, \mathcal{G}\times\mathcal{F}_2)$ *satisfying* (A.1), (A.2), (A.3), (A.4) *and* (A.5). *Then*

(i) $E_f(A, b, V)$ *is a trace class operator for* P*-almost all* (b, ω).

(ii) $n(A) = E^P[\mathrm{tr}[E_f(A, b, V)f]]$.

(iii) $\mathrm{supp}\, n = \Sigma$.

Let Ω_1' be the set of all C^∞-differential forms b such that $(1+|x|)^{-k}\|\nabla^2 b(x)\|$ is bounded for some positive number k depending on b. Then, by the arguments in the proof of Theorem C.5.2 of [16], we obtain the following:

Lemma 3.1. *For any* $b \in \Omega_1'$, $V \in \boldsymbol{K}_d$ *and bounded Borel set* A *of* $\boldsymbol{R}$, *the spectral projection* $E(A, b, V)$ *of the operator* $L(b, V)$ *has a jointly continuous integral kernel* $E(A, b, V, x, y)$.

Let $\mathcal{G}'$ be the σ-field on Ω_1' defined by

$$\mathcal{G}' = \{B \cap \Omega_1';\, B \in \mathcal{G}\}.$$

By using Lemma 3.1 and Proposition 3.1, we obtain the following:

Proposition 3.2. *We assume that a probability measure* P *is defined on* $(\Omega_1' \times \Omega_2, \mathcal{G}'\times\mathcal{F}_2)$ *and satisfies* (A.1), (A.2), (A.3), (A.4), (A.5) *where* $(\Omega_1, \mathcal{G})$ *is replaced by* $(\Omega_1', \mathcal{G}')$. *Then any bounded Borel set* A, $n(A)$ *has the following expression:*

$$n(A) = E^P[E(A, b, V, 0, 0)].$$

4. Asymptotic properties of the integrated density of states

In this section, we consider the asymptotic properties of the integrated density of states $N(\lambda)$ introduced in the last section. First we consider the asymptotics of $N(\lambda)$ as $\lambda\nearrow\infty$ (cf. [3] Theorem VI.2.1, [13] Theorem 7.3).

Theorem 4.1. *If a probability measure P on $(\Omega, \mathcal{G}\times\mathcal{F}_2)$ satisfies* (A.1), (A.2), (A.4) *and* (A.5), *we have:*

$$\lim_{\lambda\nearrow\infty}\frac{N(\lambda)}{\lambda^{d/2}}=\frac{1}{\Gamma\left(\frac{d}{2}+1\right)(2\pi)^{d/2}}. \tag{4.1}$$

Proof. By a Tauberian argument, we have only to show that

$$\lim_{t\searrow 0}\mathcal{L}(n,t)(2\pi t)^{d/2}=1. \tag{4.2}$$

By using scaling property of the Wiener process in (3.1), we have

$$\mathcal{L}(n,t)(2\pi t)^{d/2}=E^{W\times P}[H(t,w,b,\omega)\,|\,w(1)=0], \tag{4.3}$$

where

$$H(t,w,b,\omega)=\exp\left(-it\int_0^1 db(\sqrt{t}*)-t\int_0^1 V(\sqrt{t}w(s))ds\right).$$

We can easily see that

$$H(t,w,b,\omega)\to 1 \quad \text{as } t\searrow 0$$

for $P^W\times P(\cdot\,|\,w(1)=0)$-almost all (w,b,ω). Furthermore we can show that $\{H(t), 0<t<1\}$ is uniformly integrable with respect to $P^W\times P(\cdot\,|\,w(1)=0)$. In fact, for any $\Lambda\in\mathcal{B}(W)$,

$$\begin{aligned}&E^{W\times P}[\,|H(t,w)|\,;\Lambda\,|\,w(1)=0]\\ &\leq E^{W\times P}\left[\exp\left(-\frac{r}{2}t\int_0^1 V(\sqrt{t}w(s))ds\right)\middle|\,w(1)=0\right]^{2/r}P^W(\Lambda\,|\,w(1)=0)^{1/q},\end{aligned}$$

where r is the number in (A.4) and q satisfies $2/r+1/q=1$. By Remark 3.2 of [13] we can estimate as follows:

$$\begin{aligned}&\sup_{0<t<1}E^{W\times P}\left[\exp\left(-\frac{r}{2}t\int_0^1 V(\sqrt{t}w(s))ds\right)\middle|\,w(1)=0\right]\\ &\leq 2^{d/2}E^{W\times P}\left[\exp\left(r\int_0^{1/2}V_-(w(s))ds\right)\right].\end{aligned}$$

This is finite because of (A.4). □

Secondly we consider the asymptotics of $N(\lambda)$ as $\lambda\searrow-\infty$. The following theorem is an extension of Theorem III.6 of [14] (cf. Proposition VI.2.2 of [3], Theorem 7.1 of [13]).

Theorem 4.2. *We assume that a probability measure P on $(\Omega, \mathcal{G}\times\mathcal{F}_2)$ satisfies* (A.1), (A.2), (A.5) *and the following:*

(A.6) *V is a continuous Gaussian random field with nonzero covariance γ.*

(A.7) *Random fields db and V are independent, or there exists a positive number δ_0 such that for any positive δ smaller than δ_0, there exists $\lambda_\delta>0$ satisfying*

$$P(\inf \Sigma(L(b)_{\{|x|<\delta\}})<\lambda_\delta) = 1 .$$

Then we obtain

$$\lim_{\lambda\searrow-\infty} \frac{1}{\lambda^2} \log N(\lambda) = \frac{-1}{2\gamma(0)} . \tag{4.4}$$

Proof. By a Tauberian argument, we have only to show that

$$\lim_{t\nearrow\infty} \frac{1}{t^2} \log \mathcal{L}(n, t) = \frac{\gamma(0)}{2} .$$

The upper estimate is entirely same as in the case of $b=0$: if m is the mean of $V(x)$,

$$\begin{aligned}(2\pi t)^{d/2}\mathcal{L}(n, t) &\leq E^{W\times P}\left[\exp\left(-\int_0^t V(w(s))ds\right)\middle| w(t)=0\right] \\ &= E^W\left[\exp\left(-mt+\frac{1}{2}\int_0^t\int_0^t \gamma(w(s)-w(r))dsdr\right| w(t)=0\right] \\ &\leq \exp\left(-mt+\frac{t^2}{2}\gamma(0)\right).\end{aligned}$$

Hence, we have

$$\varlimsup_{t\nearrow\infty} \frac{1}{t^2} \log \mathcal{L}(n, t) \leq \frac{\gamma(0)}{2} . \tag{4.5}$$

For the lower estimate, we use methods of the functional analytic approach: let D_1 and D_2 be disjoint rectangular boxes of the form

$$D_i = \prod_{j=1}^{d} [a_j^i, b_j^i), \quad a_j^i<b_j^i .$$

Then, by using the min-max principle, we can see that

$$\mathrm{tr}[e^{-tL(b,V)_{D_1+D_2}}] \geq \mathrm{tr}[e^{-tL(b,V)_{D_1}}]+\mathrm{tr}[e^{-tL(b,V)_{D_2}}] ,$$

where, for any finite union B of rectangular boxes, $L(b, V)_B$ is the Friedrichs extension of $(L(b, V), C_0^\infty(B^\circ))$ and B° is the interior of B. Therefore, by Kirsch and Martinelli's argument [9] (cf. §VI.1.3 of [3]), we see that

$$\mathcal{L}(n, t) \geq E^P[\mathrm{tr}[e^{-tL(b,V)_D}]]$$

where $D=[-\frac{1}{2}, \frac{1}{2})^d$. We take any positive δ less than $\min\{\frac{1}{2}, \delta_0\}$. Then, by using the min-max principle furthermore, we have

$$\mathrm{tr}\,[e^{-tL(b,V)_D}] \geq \mathrm{tr}\,[e^{-tL(b,V)_{\{|x|<\delta\}}}] \geq \mathrm{tr}\,[e^{-tL(b)_{\{|x|<\delta\}}}]\,\exp\Big(-t \sup_{|x|<\delta} V\Big).$$

If we assume $P(\inf \Sigma(L(b)_{\{|x|<\delta\}}) < \lambda_\delta) = 1$, we have

$$\mathcal{L}(n, t) \geq e^{-t\lambda_\delta} E^P\Big[\exp\Big(-t \sup_{|x|<\delta} V\Big)\Big].$$

Therefore, we obtain

$$\varliminf_{t \nearrow \infty} \frac{1}{t^2} \log \mathcal{L}(n, t) \geq \varliminf_{t \nearrow \infty} \frac{1}{t^2} \log E^P\Big[\exp\Big(-t \sup_{|x|<\delta} V\Big)\Big]. \tag{4.6}$$

If we assume that db and V are independent, we have

$$\begin{aligned}
&\varliminf_{t \nearrow \infty} \frac{1}{t^2} \log \mathcal{L}(n, t) \\
&\geq \varliminf_{t \nearrow \infty} \frac{1}{t^2} \log E^P[\mathrm{tr}[e^{-tL(b)_{\{|x|<\delta\}}}]] + \varliminf_{t \nearrow \infty} \frac{1}{t^2} \log E^P\Big[\exp\Big(-t \sup_{|x|<\delta} V\Big)\Big].
\end{aligned}$$

By using Jensen's inequality and Lebesgue's convergence theorem, we see that

$$\varliminf_{t \nearrow \infty} \frac{1}{t^2} \log E^P[\mathrm{tr}[e^{-tL(b)_{\{|x|<\delta\}}}]] \geq E^P\Big[\varliminf_{t \nearrow \infty} \frac{1}{t^2} \log \mathrm{tr}[e^{-tL(b)_{\{|x|<\delta\}}}]\Big] = 0$$

Therefore, we obtain (4.6) again. We estimate the right-hand side of (4.6). Now, we may assume that the mean of $V(x)$ is zero. We divide the random field V as follows:

$$V(x) = E[V(x) \mid V(0)] + (V(x) - E[V(x) \mid V(0)]).$$

$E[V(x) \mid V(0)]$ and $V(x) - E[V(x) \mid V(0)]$ are independent and

$$E[V(x) \mid V(0)] = \frac{\gamma(x)}{\gamma(0)} V(0).$$

Therefore, we can estimate as follows:

$$\begin{aligned}
&\varliminf_{t \nearrow \infty} \frac{1}{t^2} \log E\Big[\exp\Big(-t \sup_{|x|<\delta} V(x)\Big)\Big] \\
&\geq \varliminf_{t \nearrow \infty} \frac{1}{t^2} \log E\Big[\exp\Big(-t \sup_{|x|<\delta} \frac{\gamma(x)}{\gamma(0)} V(0)\Big)\Big] \\
&\quad + \varliminf_{t \nearrow \infty} \frac{1}{t^2} \log E\Big[\exp\Big(-t \sup_{|x|<\delta} (V(x) - E[V(x) \mid V(0)])\Big)\Big].
\end{aligned}$$

Now, since $V(0)$ is a 1-dimensional Gaussian random variable, we can easily see that

$$\varliminf_{t\nearrow\infty}\frac{1}{t^2}\log E\left[\exp\left(-t\sup_{|x|<\delta}\frac{\gamma(x)}{\gamma(0)}V(0)\right)\right]\geq\inf_{|x|<\delta}\gamma(x)^2/2\gamma(0)\,.$$

On the other hand, by using Jensen's inequality, we have

$$\varliminf_{t\nearrow\infty}\frac{1}{t^2}\log E\left[\exp\left(-t\sup_{|x|<\delta}(V(x)-E[V(x)\,|\,V(0)])\right)\right]$$
$$\geq-\varlimsup_{t\nearrow\infty}\frac{1}{t}E\left[\sup_{|x|<\delta}(V(x)-E[V(x)\,|\,V(0)])\right]$$

The right-hand side of this inequality is zero, because

$$\left|E\left[\sup_{|x|<\delta}(V(x)-E[V(x)\,|\,V(0)])\right]\right|\leq2E\left[\sup_{|x|<\delta}|V(x)|\right]<\infty\,.$$

By all this, we have

$$\varliminf_{t\nearrow\infty}\frac{1}{t^2}\log\mathcal{L}(n,t)\geq\inf_{|x|<\delta}\gamma(x)^2/2\gamma(0)\,.$$

Since δ is arbitrary, we have

$$\varliminf_{t\nearrow\infty}\frac{1}{t^2}\log\mathcal{L}(n,t)\geq\frac{\gamma(0)}{2}\,.\quad\Box\tag{4.7}$$

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