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CONWAY POLYNOMIALS OF PERIODIC LINKS

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Introduction

Geometric properties of knots and links in a 3-sphere S^3 often have effect on their polynomial invariants. Periodicity of knots and links is one of them. Therefore to study periodic knots and links, it is significant to investigate their polynomial invariants. In this paper, we consider the following situation: Let $L=K_1\cup\cdots\cup K_\mu$, $\mu\geq 1$ be an oriented link and B be a trivial knot with $B\cap L=\emptyset$. We consider the p-fold cyclic cover q_p: $S^3\to S^3$ branched over B, where $p\geq 2$. We denote the preimage of L and K_i by $\tilde{L}$ and $\tilde{K}_i$, respectively and call them the covering links of L and K_i, respectively. Let $\tilde{K}_i=K_{i1}\cup\cdots\cup K_{i\nu_i}$ be a ν_i-component link. We give K_{ij} the orientation inherited from K_i. Then $\tilde{L}=\tilde{K}_1\cup\cdots\cup\tilde{K}_\mu=K_{11}\cup\cdots\cup K_{1\nu_1}\cup\cdots\cup K_{\mu 1}\cup\cdots\cup K_{\mu\nu_\mu}$ has the unique orientation. In 1971, Murasugi [7] showed a relationship between the Alexander polynomials of L and $\tilde{L}$ for the case $\mu\nu_\mu=1$. Later, Hillman [4], Sakuma [10] and Turaev [12] extended Murasugi's result to the general case. Our goal in this paper is to give a relation between the Conway polynomials of L and $\tilde{L}$ using their results (Theorem 2). To do this, we sharpen their formulas by expressing in terms of the Conway potential function [2] whose existence is shown by Hartley [3] (Theorem 1). Although the Alexander polynomial is usually defined with the difference by a unit of a polynomial ring, the Conway potential function is uniquely defined as an element of a polynomial ring.

We denote the Conway potential functions of $L, B\cup L$ and $\tilde{L}$ by $\Omega_L(t_1,\cdots,t_\mu)$, $\Omega_{B\cup L}(s,t_1,\cdots,t_\mu)$ and $\Omega_{\tilde{L}}(t_{11},\cdots,t_{1\nu_1},\cdots,t_{\mu 1},\cdots,t_{\mu\nu_\mu})$, respectively, where t_i, s and t_{ij} correspond to K_i, B and K_{ij}, respectively. Then concerning the Conway potential functions of L and $\tilde{L}$, the following formula holds:

Theorem 1. *Let $L=K_1\cup\cdots\cup K_\mu$, $\mu\geq 1$ be an oriented μ-component link and B be a trivial knot with $B\cap L=\emptyset$. Let $\tilde{L}$ be the p-fold covering link of L, where $p\geq 2$. Then*

$$\Omega_{\tilde{L}}(t_1,\cdots,t_1,\cdots,t_\mu,\cdots,t_\mu)$$
$$=\sqrt{-1}^{(p-1)(1-lk(B,L))}\,\Omega_L(t_1,\cdots,t_\mu)\prod_{j=1}^{p-1}\Omega_{B\cup L}(\xi^j,t_1,\cdots,t_\mu)\,,$$

where $lk(B, L)$ is the sum of the linking numbers of B and K_i, $1\leq i\leq\mu$, that is, $lk(B, L)=\sum_{i=1}^{\mu} lk(B, K_i)$ and $\xi=\exp(\sqrt{-1}\,\pi/p)$.

Using the 2-variable Conway polynomial [5], we have the following relation between $\nabla_L(z)$ and $\nabla_{\tilde{L}}(z)$, the Conway polynomials of L and $\tilde{L}$.

Theorem 2. *Let $L=K_1\cup\cdots\cup K_\mu$, $\mu\geq 1$ be an oriented μ-component link and B be a trivial knot with $B\cap L=\emptyset$. Let $\tilde{L}$ be the p-fold covering link of L, where $p\geq 2$. Then*

$$\nabla_{\tilde{L}}(z)=\sqrt{-1}^{(p-1)(1-lk(B,L))}\,\nabla_L(z)\prod_{j=1}^{p-1}\Psi_{B\cup L}(2\sqrt{-1}\sin(j\pi/p), z),$$

where $\Psi_{B\cup L}(z_1, z_2)$ is the 2-variable Conway polynomial such that z_1 and z_2 correspond to B and L, respectively.

In §1, we state the definitions and properties of some polynomials which we use in this paper. In §2, we prove Theorems 1 and 2. We apply them to a judgement of some periodic links in §3.

Throughout this paper, all the links are oriented.

1. Preliminary

We use four polynomials to prove the theorems. We state their definitions and properties briefly. Let $L=K_1\cup\cdots\cup K_\mu$, be a μ-component link.

CONWAY POTENTIAL FUNCTION. Conway [2] introduced the potential function for links. Later its existence was shown by Hartley [3]. We call this the Conway potential function. It is shown by Traldi [11] that Conway's original potential function for L and Hartley's one always differ by a factor of $(-1)^{\mu-1}$. In this paper we use Hartley's version. We give some properties of the Conway potential function.

(1.1) For three links L_{++}, L_{--} and L_{00} which differ only in one place as shown in Fig. 1 (a) or alternatively (b),

$$\Omega(L_{++})+\Omega(L_{--})=(t_i\,t_j+t_i^{-1}\,t_j^{-1})\,\Omega(L_{00})\,;$$

$$\Omega(L_{++})+\Omega(L_{--})=(t_i\,t_j^{-1}+t_i^{-1}\,t_j)\,\Omega(L_{00})\,.$$

(1.2) $\Omega_L(1, t_2, \cdots, t_\mu)=(t_2^{\lambda_2}\cdots t_\mu^{\lambda_\mu}-t_2^{-\lambda_2}\cdots t_\mu^{-\lambda_\mu})\,\Omega_{L'}(t_2, \cdots, t_\mu)$,
where $\Omega_{L'}(t_2, \cdots, t_\mu)$ is the Conway potential function of the sublink $L'=K_2\cup\cdots\cup K_\mu$ and λ_i is the linking number of K_1 and K_i, $2\leq i\leq\mu$.

(1.3) $\Omega_L(t_1, \cdots, t_\mu)=(-1)^\mu\,\Omega_L(t_1^{-1}, \cdots, t_\mu^{-1})$.

2-VARIABLE CONWAY POTENTIAL FUNCTION. In the Conway potential function $\Omega_L(t_1, \cdots, t_\mu)$ of L, putting $t_i=t_1$ or t_2, $1\leq i\leq\mu$, we can obtain the 2-variable polynomial of L. We call this the 2-variable Conway potential function of

L[8] and denote it by $\Phi_L(t_1, t_2)$. From Hartley's definition of the Conway potential function, we can show the following formula.

Lemma 1. *For three links L_+, L_- and L_0 which differ only in one place as shown in Fig.* 2,

$$\Phi(L_+)-\Phi(L_-) = (t_i-t_i^{-1})\,\Phi(L_0)\,. \tag{1.4}$$

CONWAY POLYNOMIAL. The Conway polynomial of L is the polynomial $\nabla_L(z)\in \boldsymbol{Z}[z]$ defined by the following recursive formulas:
(1.5) For three links L_+, L_- and L_0 which differ only in one place as shown in Fig. 2,

$$\nabla(L_+)-\nabla(L_-) = z\nabla(L_0)\,.$$

(1.6) For a trivial knot K,

$$\nabla_K(z) = 1\,.$$

2-VARIABLE CONWAY POLYNOMIAL. In the 2-variable Conway potential function $\Phi_L(t_1, t_2)$ of L, we put $t_1-t_1^{-1}=z_1$ and $t_2-t_2^{-1}=z_2$, then we can write $\Phi_L(t_1, t_2)$ in a different form. We call this polynomial the 2-variable Conway polynomial and denote by $\Psi_L(z_1, z_2)$:

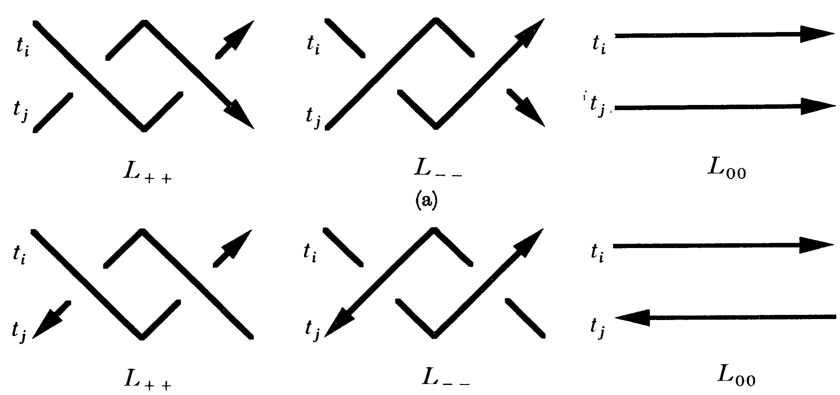

Fig. 1

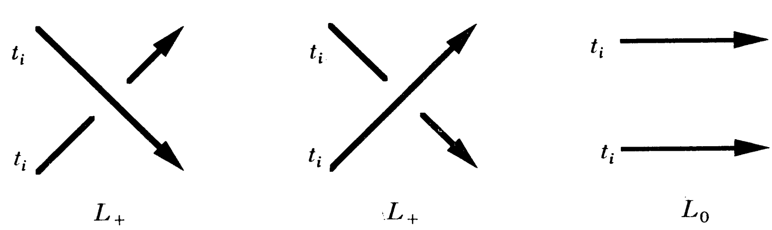

Fig. 2

$$\Psi_L(z_1, z_2) = \Psi_L(t_1 - t_1^{-1}, t_2 - t_2^{-1}) = \Phi_L(t_1, t_2)\,. \tag{1.7}$$

RELATIONS BETWEEN THE POLYNOMIALS. Next we state relations between the Conway potential function and other polynomials.

Let $\Delta_L(t_1, \cdots, t_\mu)$ be the Alexander polynomial of L. It is shown in [3] that

$$(t_1 - t_1^{-1})\,\Omega_L(t_1) \doteq \Delta_L(t_1^2) \quad \text{if} \quad \mu = 1\,; \tag{1.8}$$

$$\Omega_L(t_1, \cdots, t_\mu) \doteq \Delta_L(t_1^2, \cdots, t_\mu^2) \quad \text{if} \quad \mu > 1\,, \tag{1.9}$$

where $\doteq$ means equal up to unit $\pm t_1^{h_1} \cdots t_\mu^{h_\mu}$.

From [3], it is easy to see that

$$\nabla_L(z) = (t - t^{-1})\,\Omega_L(t, \cdots, t)\,, \tag{1.10}$$

where $z = t - t^{-1}$.

Let $\Omega_L(t_1, \cdots, t_\mu)$ be the Conway potential function of L and let $\Phi_L(t_1, t_2)$ be the 2-variable Conway potential function such that the first component is labeled by t_1 and other components are labeled by t_2. We have the next relation from the definition (1.7) of the 2-variable Conway polynomial.

$$\Psi_L(z_1, z_2) = \Phi_L(t_1, t_2) = \Omega_L(t_1, t_2, \cdots, t_2)\,, \tag{1.11}$$

where $z_i = t_i - t_i^{-1}$, $i = 1, 2$.

2. Proofs of Theorems

Proposition 1. *Let $L = K_1 \cup \cdots \cup K_\mu$, $\mu \geq 1$, be a link and B be a trivial knot such that $B \cap L = \emptyset$. Let $\tilde{L}$ be the p-fold covering link of L branched over B, where $p \geq 2$. Then*

$$\begin{aligned} \Omega_{\tilde{L}}(t_1, \cdots, t_1, \cdots, t_\mu, \cdots, t_\mu) \\ = \xi^\alpha\, t_1^{k_1} \cdots t_\mu^{k_\mu}\, \Omega_L(t_1, \cdots, t_\mu) \prod_{j=1}^{p-1} \Omega_{B \cup L}(\xi^j, t_1, \cdots, t_\mu)\,, \end{aligned} \tag{2.1}$$

where $\xi = \exp(\sqrt{-1}\,\pi/p)$ and $k_1, \cdots, k_\mu \in \boldsymbol{Z}$, and

$$\sum_{i=1}^{\mu} k_i = 0\,. \tag{2.2}$$

Proof. By [10], the Alexander polynomial of a periodic link L satisfy the formula:

$$\tilde{\Delta}_{\tilde{L}}(t_1, \cdots, t_\mu) \doteq \Delta_L(t_1, \cdots, t_\mu) \prod_{j=1}^{p-1} \Delta_{B \cup L}(\xi^{2j}, t_1, \cdots, t_\mu)\,, \tag{2.3}$$

where $\tilde{\Delta}_{\tilde{L}}(t_1, \cdots, t_\mu)$ is the reduced Alexander polynomial of which t_i, $1 \leq i \leq \mu$, corresponds to meridians of all the components of $q_p^{-1}(K_i)$, which is a ν_i-component link $\tilde{K}_i = K_{i1} \cup \cdots \cup K_{i\nu_i}$. By [9, Proposition 3], [1, Lemma 1.1] there is a relation between the Alexander polynomial $\Delta_{\tilde{L}}(t_{11}, \cdots, t_{1\nu_1}, \cdots, t_{\mu 1}, \cdots, t_{\mu\nu_\mu})$ of $\tilde{L}$ and $\tilde{\Delta}_{\tilde{L}}(t_1, \cdots, t_\mu)$:

$$(2.4) \quad \tilde{\Delta}_{\tilde{L}}(t_1, \cdots, t_\mu) \doteq \begin{cases} (t_1-1)\,\Delta_{\tilde{L}}(t_1, \cdots, t_1) & \text{if } \mu = 1 \text{ and } \nu_\mu \geq 2; \\ \Delta_{\tilde{L}}(t_1, \cdots, t_1, \cdots, t_\mu, \cdots, t_\mu) & \text{if otherwise.} \end{cases}$$

Using (1.9), (2.3) and (2.4), we have:

$$\begin{aligned} &\Omega_{\tilde{L}}(t_1, \cdots, t_1, \cdots, t_\mu, \cdots, t_\mu) \\ &\quad \doteq \Delta_{\tilde{L}}(t_1^2, \cdots, t_1^2, \cdots, t_\mu^2, \cdots, t_\mu^2) \\ &\quad \doteq \Delta_L(t_1^2, \cdots, t_\mu^2) \prod_{j=1}^{p-1} \Delta_{B\cup L}(\xi^{2j}, t_1^2, \cdots, t_\mu^2) \\ &\quad \doteq \xi^c\, \Omega_L(t_1, \cdots, t_\mu) \prod_{j=1}^{p-1} \Omega_{B\cup L}(\xi^j, t_1, \cdots, t_\mu) \end{aligned}$$

for some integer c. So we obtain the formula for the case $\mu \geq 2$. We can prove for the case $\mu=1$ in the same way as the case $\mu=2$.

Now we have only to prove (2.2). Let $f(\xi, t) = \prod_{j=1}^{p-1} \Omega_{B\cup L}(\xi^j, t, \cdots, t)$. By (1.9), there exist two integers l and m such that

$$\Omega_{B\cup L}(s, t, \cdots, t) = \pm s^l t^m \Delta_{B\cup L}(s^2, t^2, \cdots, t^2)\,.$$

Thus we obtain:

$$\Omega_{B\cup L}(-\xi^j, t^{-1}, \cdots, t^{-1}) = \pm\Omega_{B\cup L}(\xi^j, t^{-1}, \cdots, t^{-1})\,.$$

Hence we have:

$$(2.5) \qquad \begin{aligned} f(\xi^{-1}, t^{-1}) &= \prod_{j=1}^{p-1} \Omega_{B\cup L}(\xi^{-j}, t^{-1}, \cdots, t^{-1}) \\ &= \prod_{j=1}^{p-1} \Omega_{B\cup L}(-\xi^j, t^{-1}, \cdots, t^{-1}) \\ &= \pm \prod_{j=1}^{p-1} \Omega_{B\cup L}(\xi^j, t^{-1}, \cdots, t^{-1}) \\ &= \pm f(\xi, t^{-1})\,. \end{aligned}$$

On the other hand by (1.3) we have:

$$(2.6) \qquad \begin{aligned} f(\xi^{-1}, t^{-1}) &= \prod_{j=1}^{p-1} \Omega_{B\cup L}(\xi^{-j}, t^{-1}, \cdots, t^{-1}) \\ &= \prod_{j=1}^{p-1} \{(-1)^{\mu+1}\, \Omega_{B\cup L}(\xi^j, t, \cdots, t)\} \\ &= (-1)^{(p-1)(\mu+1)} \prod_{j=1}^{p-1} \Omega_{B\cup L}(\xi^j, t, \cdots, t) \\ &= (-1)^{(p-1)(\mu+1)} f(\xi, t)\,. \end{aligned}$$

From (2.5) and (2.6) we obtain:

$$f(\xi, t^{-1}) = \pm f(\xi, t)\,. \tag{2.7}$$

From (2.1) we have:

$$\Omega_{\tilde{L}}(t, \cdots, t) = \xi^{\alpha} t^{k} \Omega_{L}(t, \cdots, t) f(\xi, t)\,, \tag{2.8}$$

$$\Omega_{\tilde{L}}(t^{-1}, \cdots, t^{-1}) = \xi^{\alpha} t^{-k} \Omega_{L}(t^{-1}, \cdots, t^{-1}) f(\xi, t^{-1})\,, \tag{2.9}$$

where $k=\sum_{i=1}^{\mu} k_i$. From (1.3), (2.7), (2.8) and (2.9) we have $t^k=\pm t^{-k}$. Thus $k=0$, completing the proof.

Lemma 2. *For any integers $p(>0)$ and y, we define:*

$$n = \begin{cases} \gcd(p, y)>0 & \text{if} \quad y\neq 0\,; \\ p & \text{if} \quad y=0\,, \end{cases}$$

where $\gcd(p, y)$ is the greatest common diviser of p and y. For the three numbers p, y and n, the next identity about X holds:

$$\prod_{j=0}^{p-1} (\xi^{jy} X - \xi^{-jy} X^{-1}) = \xi^{p(p-1)y/2} (X^{p/n} - X^{-p/n})^n\,.$$

Proof. It is easy to see that the next identities hold:

$$(X^{p/n} - X^{-p/n})^n = X^{-p} \prod_{j=1}^{2p/n} (X - \xi^{jn})^n\,,$$

$$\prod_{j=0}^{p-1} (\xi^{jy} X - \xi^{-jy} X^{-1}) = \xi^{p(p-1)y/2} X^{-p} \prod_{j=0}^{p-1} (X + \xi^{-jy})(X - \xi^{-jy})\,.$$

To prove the lemma we have only to check

$$\prod_{j=0}^{p-1} (X + \xi^{-jy})(X - \xi^{-jy}) = \prod_{j=1}^{2p/n} (X - \xi^{jn})^n\,.$$

We consider the sets S and T of the solutions of the equations $\prod_{j=0}^{p-1}(X+\xi^{-jy})(X-\xi^{-jy})=0$ and $\prod_{j=1}^{2p/n}(X-\xi^{jn})^n=0$. Then S consists of the elements $\pm\xi^{-jy}$, $0\leq j<p/n$, with multiplicity n, and T consists of the elements ξ^{jn}, $1\leq j\leq 2p/n$, with multiplicity n. Since

$$\{\xi^{jn} \mid 1\leq j \leq 2p/n\} = \{\pm\xi^{-jy} \mid 0\leq j \leq p/n-1\}\,,$$

we have $S=T$. This completes the proof.

Lemma 3. *Let $B\cup L$ be a torus link with $lk(B, L)=q$ as shown in Fig. 3, and let $\tilde{L}$ be the p-fold covering link of L branched over B, where $p\geq 2$. Then $\tilde{L}$ is a (p, q)-torus knot and*

$$\Omega_{B\cup L}(\xi^j, 1) = \frac{\sin(qj\pi/p)}{\sin(j\pi/p)}\,,$$

where $1 \leq j \leq p-1$.

Proof. We only prove for the case $q>0$. Let Ω_m be the Conway potential function of a $(2, 2m)$-torus link; $\Omega_{B\cup L}(t_1, t_2)=\Omega_q$. From a recursive formula (1.1), we have:

$$\Omega_{m+1}+\Omega_{m-1} = \gamma\,\Omega_m\,, \quad m\geq 1\,,$$
$$\Omega_0 = 0\,,$$
$$\Omega_1 = 1\,,$$

where $\gamma=t_1 t_2+t_1^{-1} t_2^{-1}$. Then we obtain:

$$\Omega_m = \left(\frac{1}{2}\right)^m \frac{1}{\sqrt{\gamma^2-4}} \{(\gamma+\sqrt{\gamma^2-4})^m-(\gamma-\sqrt{\gamma^2-4})^m\}\,.$$

Thus

$$\begin{aligned}\Omega_{B\cup L}(\xi^j, 1) &= \left(\frac{1}{2}\right)^q \frac{1}{\sqrt{(\xi^j-\xi^{-j})^2}} \{(2\xi^j)^q-(2\xi^{-j})^q\} \\ &= \frac{\xi^{jq}-\xi^{-jq}}{\xi^j-\xi^{-j}} \\ &= \frac{\sin(qj\pi/p)}{\sin(j\pi/p)}\,.\end{aligned}$$

This completes the proof.

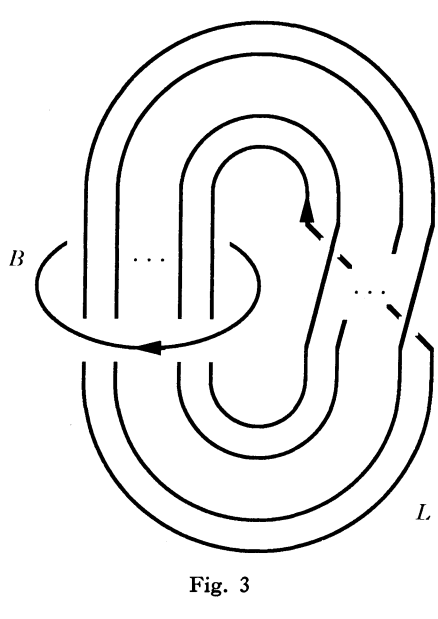

Fig. 3

Lemma 4. *Let p and q be integers such that* (i) $p>0$, (ii) $\gcd(p, q)=1$. *Then*

$$\xi^{p(p-1)(q-1)/2}=\sqrt{-1}^{(p-1)(q-1)}=\prod_{j=1}^{p-1}\frac{\sin(qj\pi/p)}{\sin(j\pi/p)}.$$

Proof. We have only to prove the second identity. It is to see that

$$\left|\prod_{j=1}^{p-1}\frac{\sin(qj\pi/p)}{\sin(j\pi/p)}\right|=1.$$

We consider three cases:

(1) p is odd and q is odd;

(2) p is even and q is odd;

(3) p is odd and q is even.

First note that $\prod_{j=1}^{p-1}\sin(j\pi/p)>0$ in any case.

Case (1). Since $\sin(qj\pi/p)=\sin(q(p-j)\pi/p)$, $1\leq j\leq(p-1)/2$, $\prod_{j=1}^{p-1}\sin(qj\pi/p)=\prod_{j=1}^{(p-1)/2}\sin^2(qj\pi/p)>0$. Thus the right-hand side is equal to 1. On the other hand, $\sqrt{-1}^{(p-1)(q-1)}=1$.

Case (2). Since $\sin(qj\pi/p)=\sin(q(p-j)\pi/p)$, $\prod_{j=1}^{p-1}\sin(qj\pi/p)=\sin(q\pi/2)\prod_{j=1}^{(p/2-1)}\sin^2(qj\pi/p)$. Furthermore since $\sin(q\pi/2)=(-1)^{(q-1)/2}$, the right-hand side is equal to $(-1)^{(q-1)/2}$. On the other hand, the left-hand side equals to $\sqrt{-1}^{(p-1)(q-1)}=(-1)^{(q-1)/2}$.

Case (3). Since $\sin(q(p-j)\pi/p)=-\sin(qj\pi/p)$, $\prod_{j=1}^{p-1}\sin(qj\pi/p)=(-1)^{(p-1)/2}\prod_{j=1}^{(p-1)/2}\sin^2(qj\pi/p)$. So the right-hand side is equal to $(-1)^{(p-1)/2}$. We also find that the left-hand side equals to $(-1)^{(p-1)/2}$. This completes the proof.

Proof of Theorem 1. We will prove that $\alpha=p(p-1)(1-lk(B, L))/2$ and $k_i=0$, $1\leq i\leq\mu$, in Proposition 1.

Step 1. The case where $\tilde{L}$ is a (p, q)-torus knot, $p>0$.

We may assume an image L of $\tilde{L}$ is a trivial knot as shown in Fig. 3. By Proposition 1 and (1.10), we have:

$$\nabla_{\tilde{L}}(t_1-t_1^{-1})=(t_1-t_1^{-1})\,\Omega_{\tilde{L}}(t_1)=\xi^{\alpha}(t_1-t_1^{-1})\,\Omega_L(t_1)\prod_{j=1}^{p-1}\Omega_{B\cup L}(\xi^j, t_1).$$

Putting $t_1=1$, by Lemmas 3 and 4 we have:

$$\begin{aligned}1&=\xi^{\alpha}\prod_{j=1}^{p-1}\Omega_{B\cup L}(\xi^j, 1)=\xi^{\alpha}\prod_{j=1}^{p-1}\frac{\sin(qj\pi/p)}{\sin(j\pi/p)}\\&=\xi^{\alpha}\,\xi^{p(p-1)(q-1)/2}=\xi^{\alpha+p(p-1)(q-1)/2}.\end{aligned}$$

Though α is equal to $p(p-1)(1-q)/2$ modulo $2p$, we may choose $\alpha=p(p-1)(1-q)/2$. Since q is the linking number of B and L, we have the desired form.

Step 2. The case where L and $\tilde{L}$ are knots.

Let $q=lk(B, L)$. We can obtain a (p, q)-torus knot from $\tilde{L}$ by changing some crossings of $\tilde{L}$. We induct on the number n of them. Since $\tilde{L}$ has period p, we may consider that n is a multiple of p. Suppose $\alpha=p(p-1)(1-q)/2$ for the case $n=(c-1)p$, $c\geq 1$. We consider the case that $n=cp$ and the signs of at least p crossings of them are positive. For three links L_+, L_- and L_0 which differ only in one place as shown in Fig. 2, from (1.5) and (1.10), we obtain:

$$\Omega_{L_+}(t)=\Omega_{L_-}(t)+(t-t^{-1})\,\Omega_{L_0}(t, t)\,. \tag{2.10}$$

And for three links $B\cup L_+$, $B\cup L_-$ and $B\cup L_0$ which differ only in one place, in which a string of B is not contained, as shown in Fig. 2, by (1.4), we have

$$\Omega_{B\cup L_+}(\xi^j, t)=\Omega_{B\cup L_-}(\xi^j, t)+(t-t^{-1})\,\Omega_{B\cup L_0}(\xi^j, t, t)\,. \tag{2.11}$$

If $L_+=L$, then by (2.10) and (2.11), (2.1) with $\mu=1$ becomes

$$\begin{aligned}\Omega_{\tilde{L}}(t) &= \xi^\alpha\{\Omega_{L_-}(t)+(t-t^{-1})\,\Omega_{L_0}(t, t)\}\\ &\times\prod_{j=1}^{p-1}\{\Omega_{B\cup L_-}(\xi^j, t)+(t-t^{-1})\,\Omega_{B\cup L_0}(\xi^j, t, t)\}\,.\end{aligned}$$

Since $\Omega_{L_0}(t, t)$, $\Omega_{B\cup L_0}(\xi^j, t, t)$ and $(t-t^{-1})\,\Omega_{L_-}(t)$ are elements of $\boldsymbol{C}[t^{\pm 1}]$, we see

$$\Omega_{\tilde{L}}(t)=\xi^\alpha\,\Omega_{L_-}(t)\prod_{j=1}^{p-1}\Omega_{B\cup L_-}(\xi^j, t)+g(t)$$

for some $g(t)\in\boldsymbol{C}[t^{\pm 1}]$. Thus

$$(t-t^{-1})\,\Omega_{\tilde{L}}(t)=\xi^\alpha(t-t^{-1})\,\Omega_{L_-}(t)\prod_{j=1}^{p-1}\Omega_{B\cup L_-}(\xi^j, t)+(t-t^{-1})\,g(t)\,.$$

Putting $t=1$, by (1.14) we have:

$$1=\xi^\alpha\prod_{j=1}^{p-1}\Omega_{B\cup L_-}(\xi^j, 1)\,. \tag{2.12}$$

Let $\tilde{L}_-$ be the p-fold covering link of L_- branched over B, then by assumption,

$$\Omega_{\tilde{L}_-}(t)=\xi^{p(p-1)(1-lk(B,L_-))/2}\,\Omega_{L_-}(t)\prod_{j=1}^{p-1}\Omega_{B\cup L_-}(\xi^j, t)\,.$$

Thus

$$(t-t^{-1})\,\Omega_{\tilde{L}_-}(t)=\xi^{p(p-1)(1-lk(B,L_-))/2}(t-t^{-1})\,\Omega_{L_-}(t)\prod_{j=1}^{p-1}\Omega_{B\cup L_-}(\xi^j, t)\,.$$

Putting $t=1$, we have:

$$1=\xi^{p(p-1)(1-lk(B,L_-))/2}\prod_{j=1}^{p-1}\Omega_{B\cup L_-}(\xi^j, 1)\,. \tag{2.13}$$

From (2.12) and (2.13) we obtain:

$$\xi^{\alpha}=\xi^{p(p-1)(1-lk(B,L_-))/2}.$$

Since $lk(B, L_-)=lk(B, L_+)$, we may choose

$$\alpha=p(p-1)(1-lk(B, L))/2.$$

Similarly we can prove in case the signs of the crossings are negative.

Step 3. The case where $L=K_1\cup K_2$ is a 2-component link such that $lk(K_1, K_2)\neq 0$ and $\tilde{K}_2$ is a knot.

Suppose $\tilde{K}_1$ has ν_1 components. By Proposition 1,

$$\Omega_{\tilde{L}}(t_1, \cdots, t_1, t_2)=\xi^{\alpha}t_1^{k_1}t_2^{k_2}\Omega_L(t_1, t_2)\prod_{j=1}^{p-1}\Omega_{B\cup L}(\xi^j, t_1, t_2),$$

and so putting $t_1=1$, we have:

$$\Omega_{\tilde{L}}(1, \cdots, 1, t_2)=\xi^{\alpha}t_2^{k_2}\Omega_L(1, t_2)\prod_{j=1}^{p-1}\Omega_{B\cup L}(\xi^j, 1, t_2). \tag{2.14}$$

By (1.2), we have the following.

$$\Omega_{\tilde{L}}(1, \cdots, 1, t_2)=(t_2^{pl/\nu_1}-t_2^{-pl/\nu_1})^{\nu_1}\Omega_{\tilde{K}_2}(t_2), \tag{2.15}$$

$$\Omega_L(1, t_2)=(t_2^l-t_2^{-l})\Omega_{K_2}(t_2); \tag{2.16}$$

$$\Omega_{B\cup L}(\xi^j, 1, t_2)=(\xi^{jy}t_2^l-\xi^{-jy}t_2^{-l})\Omega_{B\cup K_2}(\xi^j, t_2), \tag{2.17}$$

where $l=lk(K_1, K_2)$ and $y=lk(B, K_1)$. Since $\tilde{K}_2$ is a knot, by Step 2,

$$\Omega_{\tilde{K}_2}(t_2)=\xi^{p(p-1)(1-x)/2}\Omega_{K_2}(t_2)\prod_{j=1}^{p-1}\Omega_{B\cup K_2}(\xi^j, t_2), \tag{2.18}$$

where $x=lk(B, K_2)$. Using (2.15) and (2.18), we have:

$$\begin{aligned}&\Omega_{\tilde{L}}(1, \cdots, 1, t_2)\\&\quad=\xi^{p(p-1)(1-x)/2}(t_2^{pl/\nu_1}-t_2^{-pl/\nu_1})^{\nu_1}\Omega_{K_2}(t_2)\prod_{j=1}^{p-1}\Omega_{B\cup K_2}(\xi^j, t_2).\end{aligned}\tag{2.19}$$

And substituting (2.16) and (2.17) to (2.14), we have:

$$\begin{aligned}&\Omega_{\tilde{L}}(1, \cdots, 1, t_2)\\&\quad=\xi^{\alpha}t_2^{k_2}(t_2^l-t_2^{-l})\Omega_{K_2}(t_2)\prod_{j=1}^{p-1}\{(\xi^{jy}t_2^l-\xi^{-jy}t_2^{-l})\Omega_{B\cup K_2}(\xi^j, t_2)\}\\&\quad=\xi^{\alpha}t_2^{k_2}\{\prod_{j=0}^{p-1}(\xi^{jy}t_2^l-\xi^{-jy}t_2^{-l})\}\Omega_{K_2}(t_2)\prod_{j=1}^{p-1}\Omega_{B\cup K_2}(\xi^j, t_2)\\&\quad=\xi^{\alpha+p(p-1)y/2}t_2^{k_2}(t_2^{pl/\nu_1}-t_2^{-pl/\nu_1})^{\nu_1}\\&\qquad\times\Omega_{K_2}(t_2)\prod_{j=1}^{p-1}\Omega_{B\cup K_2}(\xi^j, t_2).\quad\text{(by Lemma 2)}\end{aligned}\tag{2.20}$$

From (2.19) and (2.20) we obtain:

$$\xi^{-\alpha+p(p-1)(p-x-y)/2}=t_2^{k_2}\,.$$

Since $x+y=lk(B, L)$, we may choose that $\alpha=p(p-1)(1-lk(B, L))$ and $k_2=0$. Hence we obtain $k_1=0$ by (2.2).

Step 4. The case where L is a knot and $\tilde{L}$ has more than 1 component.

We take a knot K such that $lk(L, K)\neq 0$ and $\gcd(lk(B, K), p)=1$. Let $M=L\cup K$. By Proposition 1 and Step 3, we have:

$$\Omega_{\tilde{M}}(t_1, \cdots, t_1, t_2)=\xi^{p(p-1)(1-lk(B,M))/2}\,\Omega_M(t_1, t_2)\prod_{j=1}^{p-1}\Omega_{B\cup M}(\xi^j, t_1, t_2)\,.$$

putting $t_2=1$, we have:

$$\Omega_{\tilde{M}}(t_1, \cdots, t_1, 1)=\xi^{p(p-1)(1-lk(B,M))/2}\,\Omega_M(t_1, 1)\prod_{j=1}^{p-1}\Omega_{B\cup M}(\xi^j, t_1, 1)\,. \tag{2.21}$$

By (1.2), we have the following.

$$\Omega_{\tilde{M}}(t_1, \cdots, t_1, 1)=(t_1^{pl}-t_1^{-pl})\,\Omega_{\tilde{L}}(t_1, \cdots, t_1)\,; \tag{2.22}$$

$$\Omega_M(t_1, 1)=(t_1^{l}-t_1^{-l})\,\Omega_L(t_1)\,; \tag{2.23}$$

$$\Omega_{B\cup M}(\xi^j, t_1, 1)=(\xi^{jx}t_1^{l}-\xi^{-jx}t_1^{-l})\,\Omega_{B\cup L}(\xi^j, t_1)\,, \tag{2.24}$$

where $l=lk(L, K)$ and $x=lk(B, K)$. Since L is a knot, by Proposition 1 we have:

$$\Omega_{\tilde{L}}(t_1, \cdots, t_1)=\xi^{\alpha}\,\Omega_L(t_1)\prod_{j=1}^{p-1}\Omega_{B\cup L}(\xi^j, t_1) \tag{2.25}$$

Combining (2.21)-(2.25) and Lemma 2 as in Step 3, we obtain:

$$\xi^{\alpha}=\xi^{p(p-1)(1-lk(B,M)+x)/2}\,.$$

Since $lk(B, M)=lk(B, L)+lk(B, K)$, we may choose $\alpha=p(p-1)(1-lk(B, L))/2$.

Step 5. The case where $L=K_1\cup K_2$ is a 2-component link with $lk(K_1,K_2)\neq 0$.

Since we have the desired result for K_2 by Steps 2 and 4, we can prove $\alpha=p(p-1)(1-lk(B, L))/2$ and $k_2=0$ as in Step 3. Since $k_1+k_2=0$, we obtain $k_1=0$.

Step 6. The case where $L=K_1\cup\cdots\cup K_\mu$, $\mu\geq 2$ is a μ-component link such that there exists a component K_i satisfying $lk(K_i, K_j)\neq 0$ for any $j\neq i$.

We induct on the number μ of the components of L. In case $\mu=2$, we have the desired result by Step 5. We assume that the Theorem holds for the μ-component links. Suppose $L=K_1\cup\cdots\cup K_{\mu+1}$ is a $(\mu+1)$-component link such that K_i, $i\neq\mu+1$, is a component with $lk(K_i, K_j)\neq 0$ for any $j\neq i$. Since the Theorem holds for the link $L'=K_1\cup\cdots\cup K_\mu$ by assumption, as in Step 3, we have:

$$\xi^{-\alpha+p(p-1)(1-lk(B,L')-y)/2}=t_1^{k_1}\cdots t_\mu^{k_\mu},$$

where $y=lk(B, K_{\mu+1})$. We may choose that $\alpha=p(p-1)(1-lk(B,L')-y)/2=p(p-1)(1-lk(B,L))/2$ and $k_i=0, 1\leq i\leq\mu$. Since $\sum_{i=1}^{\mu+1}k_i=0$, we obtain $k_{\mu+1}=0$.

Step 7. The case where $L=K_1\cup\cdots\cup K_\mu$, $\mu\geq 2$, is a μ-component link. We take a knot K_0 such that $lk(K_0, K_i)\neq 0$ for any K_i, $1\leq i\leq\mu$. For the link $L'=K_0\cup L$, by Step 6, we obtain;

$$\begin{aligned}&\Omega_{\tilde{L}'}(t_0,\cdots,t_0,t_1,\cdots,t_1,\cdots,t_\mu,\cdots,t_\mu)\\&=\xi^{p(p-1)(1-lk(B,L'))/2}\,\Omega_{L'}(t_0,t_1,\cdots,t_\mu)\prod_{j=1}^{p-1}\Omega_{B\cup L'}(\xi^j,t_0,t_1,\cdots,t_\mu).\end{aligned}$$

As in Step 4 we have:

$$\xi^{-\alpha+p(p-1)(1-lk(B,L))/2}=t_1^{k_1}\cdots t_\mu^{k_\mu}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. From Theorem 1 we obtain the following relation:

$$\begin{aligned}&(t-t^{-1})\,\Omega_{\tilde{L}}(t,\cdots,t)\\&=\sqrt{-1}^{(p-1)(1-lk(B,L))}(t-t^{-1})\,\Omega_L(t,\cdots,t)\prod_{j=1}^{p-1}\Omega_{B\cup L}(\xi^j,t,\cdots,t)\end{aligned}$$

By (1.10) and (1.11), we have the desired formula.

3. Application

Theorem 3. *Let $L=K_1\cup\cdots\cup K_\mu$, $\mu\geq 1$, be an oriented μ-component link such that the Conway polynomial of L is not zero and let B be a trivial knot with $B\cap L=\emptyset$. Let $\tilde{L}$ be the p-fold covering link of L branched over B, where $p\geq 2$. Let $a_n(L)$ and $a_n(\tilde{L})$ be the coefficients of degree n of the Conway polynomials of L and $\tilde{L}$, respectively, and let $m=\min\{n\,|\,a_n(L)\neq 0\}$ and $\tilde{m}=\min\{n\,|\,a_n(\tilde{L})\neq 0\}$. If $\tilde{L}$ has μ components, then $\tilde{m}=m$ and $a_{\tilde{m}}(\tilde{L})=p^{\mu-1}a_m(L)$.*

We need two lemmas to prove Theorem 3.

Lemma 5 ([6, p.24]). *For any integer $p>1$,*

$$\prod_{j=1}^{p-1}\sin(j\pi/p)=\frac{p}{2^{p-1}}.$$

Proof. By Lemma 2, we have:

$$\prod_{j=1}^{p-1}(\xi^jX-\xi^{-j}X^{-1})=\sqrt{-1}^{(p-1)}\frac{X^p-X^{-p}}{X-X^{-1}}.$$

Putting $X=1$, we obtain:

$$\prod_{j=1}^{p-1}(\xi^j-\xi^{-j})=\sqrt{-1}^{(p-1)}p\,.$$

Since $2\sqrt{-1}\sin(j\pi/p)=\xi^j-\xi^{-j}$, we have:

$$\begin{aligned}2^{p-1}\prod_{j=1}^{p-1}\sin(j\pi/p)&=\prod_{j=1}^{p-1}\{2\sin(j\pi/p)\}\\&=\prod_{j=1}^{p-1}\{\sqrt{-1}^{(-1)}(\xi^j-\xi^{-j})\}\\&=\sqrt{-1}^{(1-p)}\prod_{j=1}^{p-1}(\xi^j-\xi^{-j})\\&=\mathrm{p}\,.\end{aligned}$$

This completes the proof.

Lemma 6. *Let $L=K_0\cup K_1\cup\cdots\cup K_\mu$, $\mu\geq 1$, be a $(\mu+1)$-component link such that K_0 is a trivial knot. Let $\Omega_L(t_0, t_1, \cdots, t_\mu)$ be the Conway potential function of L. Then for any integer $p>1$,*

$$\Omega_L(\xi^j, 1, \cdots, 1)=\frac{\prod_{\nu=1}^{\mu}\{2\sqrt{-1}\sin(\mathrm{r}_\nu j\pi/p)\}}{2\sqrt{-1}\sin(j\pi/p)}\,,$$

and

$$\prod_{j=1}^{p-1}\Omega_L(\xi^j, 1, \cdots, 1)=\sqrt{-1}^{(p-1)(r-1)}p^{\mu-1}\,,$$

where $r_\nu=lk(K_0, K_\nu)$, $1\leq\nu\leq\mu$, and $r=\sum_{\nu=1}^{\mu}r_\nu$.

Proof. By (1.2) we have:

$$\begin{aligned}\Omega_L(t_0, 1, \cdots, 1)&=\{\prod_{\nu=1}^{\mu}(t_0^{r_\nu}-t_0^{-r_\nu})\}\,\Omega_{K_0}(t_0)\\&=\{\prod_{\nu=1}^{\mu}(t_0^{r_\nu}-t_0^{-r_\nu})\}/(t_0-t_0^{-1})\,.\end{aligned}$$

Thus

$$\begin{aligned}\Omega_L(\xi^j, 1, \cdots, 1)&=\frac{\prod_{\nu=1}^{\mu}(\xi^{jr_\nu}-\xi^{-jr_\nu})}{\xi^j-\xi^{-j}}\\&=\frac{\prod_{\nu=1}^{\mu}\{2\sqrt{-1}\sin(r_\nu j\pi/p)\}}{2\sqrt{-1}\sin(j\pi/p)}\,.\end{aligned}$$

Hence

$$\begin{aligned}\prod_{j=1}^{p-1}\Omega_L(\xi^j, 1, \cdots, 1)&=\prod_{j=1}^{p-1}\{\frac{\prod_{\nu=1}^{\mu}\{2\sqrt{-1}\sin(r_\nu j\pi/p)\}}{2\sqrt{-1}\sin(j\pi/p)}\}\\&=\frac{\prod_{\nu=1}^{\mu}\{\prod_{j=1}^{p-1}\{2\sqrt{-1}\sin(r_\nu j\pi/p)\}\}}{\prod_{j=1}^{p-1}\{2\sqrt{-1}\sin(j\pi/p)\}}\end{aligned}$$

$$= (2\sqrt{-1})^{(p-1)(\mu-1)} \frac{\prod_{\nu=1}^{\mu} \{\prod_{j=1}^{p-1} \sin(r_\nu j\pi/p)\}}{\prod_{j=1}^{p-1} \sin(j\pi/p)}\,.$$

By Lemma 4, this equals to

$$(2\sqrt{-1})^{(p-1)(\mu-1)} \frac{\prod_{\nu=1}^{\mu} \{\sqrt{-1}^{(p-1)(r_\nu-1)} \prod_{j=1}^{p-1} \sin(j\pi/p)\}}{\prod_{j=1}^{p-1} \sin(j\pi/p)}$$

$$= (2\sqrt{-1})^{(p-1)(\mu-1)} \sqrt{-1}^{(p-1)} (\textstyle\sum_{\nu=1}^{\mu} r_\nu - \mu) \{\displaystyle\prod_{j=1}^{p-1} \sin(j\pi/p)\}^{\mu-1}\,.$$

By Lemma 5, this equals to

$$(2\sqrt{-1})^{(p-1)(\mu-1)} \sqrt{-1}^{(p-1)(r-\mu)} \left(\frac{p}{2^{p-1}}\right)^{\mu-1}$$

$$= \sqrt{-1}^{(p-1)(r-1)} p^{\mu-1}\,.$$

This completes the proof.

Proof of Theorem 3. By Theorem 2 we have:

$$\nabla_{\tilde{L}}(z) = \sqrt{-1}^{(p-1)(1-r)} \nabla_L(z) \prod_{j=1}^{p-1} \Psi_{B\cup L}(2\sqrt{-1} \sin(j\pi/p), z)\,, \tag{3.1}$$

where $r = lk(B, L)$. Since L and $\tilde{L}$ are μ-component links, the Conway polynomials of L and $\tilde{L}$ are of the form:

$$\nabla_L(z) = a_{\mu-1}(L) z^{\mu-1} + a_{\mu+1}(L) z^{\mu+1} + \cdots\,, \tag{3.2}$$

$$\nabla_{\tilde{L}}(z) = a_{\mu-1}(\tilde{L}) z^{\mu-1} + a_{\mu+1}(\tilde{L}) z^{\mu+1} + \cdots\,. \tag{3.3}$$

If we put

$$f_L(z) = \sqrt{-1}^{(p-1)(1-r)} \prod_{j=1}^{p-1} \Psi_{B\cup L}(2\sqrt{-1} \sin(j\pi/p), z)\,,$$

from (3.1), (3.2) and (3.3) we may assume that

$$f_L(z) = b_0 + b_2 z^2 + \cdots\,,$$

where $b_{2i} \in \mathbf{Z}$. To prove the theorem, we have only to check $b_0 = p^{\mu-1}$. Using (1.11) and Lemma 6, we have:

$$\begin{aligned} b_0 &= f_L(0) \\ &= \sqrt{-1}^{(p-1)(1-r)} \prod_{j=1}^{p-1} \Psi_{B\cup L}(2\sqrt{-1} \sin(j\pi/p), 0) \\ &= \sqrt{-1}^{(p-1)(1-r)} \prod_{j=1}^{p-1} \Omega_{B\cup L}(\xi^j, 1, 1, \cdots, 1) \\ &= \sqrt{-1}^{(p-1)(1-r)} \sqrt{-1}^{(p-1)(r-1)} p^{\mu-1} \\ &= p^{\mu-1} \end{aligned}$$

This completes the proof.

Let $L=K_1\cup\cdots\cup K_\mu$, $\mu\geq 1$, be a link with period n and $L^*=k_1\cup\cdots\cup k_\nu$, $\nu\geq 1$, be a factor link of L. For a covering map $f_n\colon L\to L^*$, we deonte the number of the components of $f_n^{-1}(k_i)$, $1\leq i\leq\nu$, by c_i. Then we say that L has period $n(\nu; c_1, \cdots, c_\nu)$.

Corollary 1. *Let L be an oriented μ-component link such that the Conway polynomial $\nabla_L(z)$ of L is not zero. If L has period $n(\mu; 1, \cdots, 1)$, then $a_m(L)\equiv 0 \pmod{n^{\mu-1}}$, where $a_m(L)$ is the coefficient of the lowest degree of the Conway polynomial of L.*

Proof. Since L has period $n(\mu; 1, \cdots, 1)$ and $\nabla_L(z)\neq 0$, a factor link L^* has μ components and the Conway polynomial of L^* is not zero by Theorem 2. By Theorem 3 we have $a_m(L)=n^{\mu-1}a_m(L^*)$. This completes the proof.

EXAMPLE. We calculate the Conway polynomial of a p-component link C_p, $p\geq 2$, as shown in Fig. 4. If we consider the Whitehead link $B\cup L$ as shown in Fig. 5, we can regard C_p as the p-fold covering link of L. Hence we can obtain the Conway polynomial of C_p using Theorem 2. Since $\Psi_{B\cup L}(z_1, z_2) = -z_1 z_2$, we have:

$$\begin{aligned}\nabla_{C_p}(z) &= \sqrt{-1}^{p-1}\prod_{j=1}^{p-1}(-2\sqrt{-1}\sin(j\pi/p)\times z)\\ &= 2^{p-1}z^{p-1}\prod_{j=1}^{p-1}\sin(j\pi/p)\\ &= 2^{p-1}z^{p-1}\frac{p}{2^{p-1}}\\ &= pz^{p-1}.\end{aligned}$$

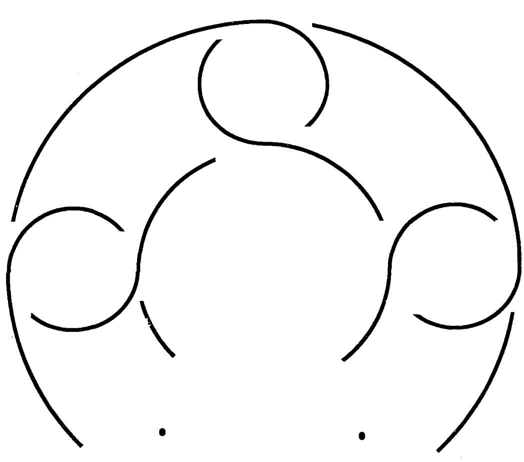

Fig. 4

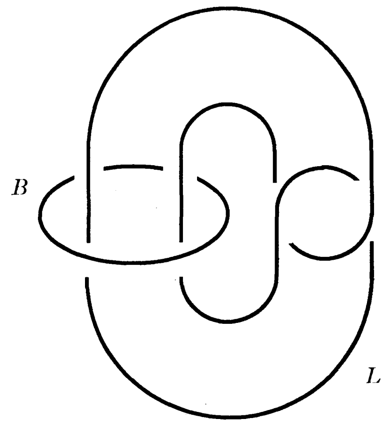

Fig. 5

If p is more than two, it is easy to see that $p \not\equiv 0 \pmod{n^{p-1}}$ for any integer $n>1$. By corollary 1, we find that $C_p, p \geq 3$, does not have period $n(p; 1, \cdots, 1)$ for $n \geq 2$.

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