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ON THE ADDITIVITY OF h-GENUS OF KNOTS

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Introduction

We say that $(V_1, V_2; F)$ is a Heegaard splitting of the 3-sphere S^3, if both V_1 and V_2 are handlebodies, $S^3=V_1\cup V_2$ and $V_1\cap V_2=\partial V_1=\partial V_2=F$. Then F is called a Heegaard surface of S^3.

Let K be a knot in S^3. Then it is well known that there exists a Heeagard surface of S^3 which contains K. Thus we define $h(K)$ as the minimum genus among all Heegaard surfaces of S^3 containing K, and we call it the h-genus of K. We note here that any two Heegaard surfaces of S^3 with the same genus are mutually ambient isotopic ([11]).

By the definition, it follows that $h(K)=0$ if and only if K is a trivial knot and that $h(K)=1$ if and only if K is a torus knot. Hence if $h(K)=1$ then K is prime. In this paper we show:

Theorem. *Let K_1 and K_2 be non-trivial knots in S^3. If $h(K_1\# K_2)=2$, then $h(K_1)=h(K_2)=1$.*

On the other hand, we show the following two propositions.

Proposition 1. *Let K_1 and K_2 be non-trivial knots in S^3 with $(1, 1)$-decompositions. Suppose neither K_1 nor K_2 are torus knots. Then $h(K_1)=h(K_2)=2$ and $h(K_1\# K_2)=3$.*

Here, we say that a knot K admits a (g, b)-decomposition, if there is a genus g Heegaard splitting $(V_1, V_2; F)$ of S^3 such that $V_i\cap K$ is a b-string trivial arc system in V_i $(i=1, 2)$ (cf. [2] and [6]).

REMARK. Since every 2-bridge knot admits a $(1, 1)$-decomposition, there are infinitely many knots satisfying the hypthesis of Proposition 1.

Proposition 2. *Let n be an integer greater than 1 and K_n the knot illustrated in* Figure 1. *Then $h(K_n)=3$ and $h(K_n\# K)=3$ for any 2-bridge knot K.*

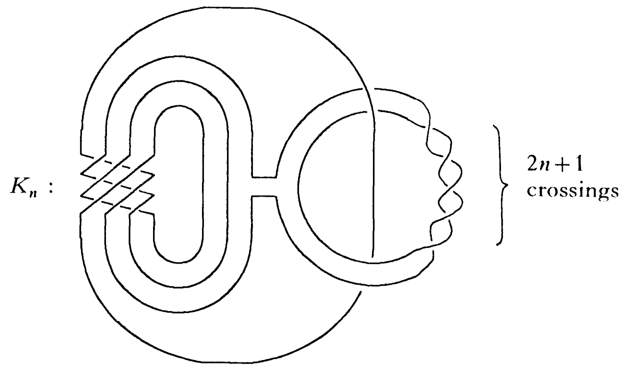

Figure 1

By Propositions 1 and 2, concerning h-genus we have the following "equalities": $2+2=3$, $3+1=3$ and $3+2=3$. Hence it seems difficult to determine $h(K_1)$ and $h(K_2)$ when $h(K_1 \# K_2)=3$.

Next, let $t(K)$ be the tunnel number of a knot K in S^3. Here the tunnel number of K is the minimum number of mutually disjoint arcs properly embedded in the exterior of K in S^3 whose complementary space is a handlebody. We call the family of such arcs an unknotting tunnel system for K. Concerning the relation between $t(K)$ and $h(K)$, C. Morin and M. Saito pointed out the following fact.

Fact. $t(K) \leq h(K) \leq t(K)+1$.

By Fact, we have the Venn diagram illustrated in Figure 2. For behavior of tunnel number of knots under connected sum, see [4], [5], [6], [7] and [9].

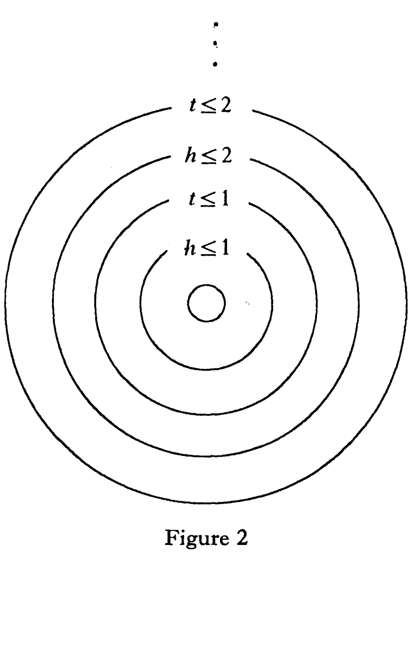

Figure 2

1. Proof of Fact and Propositions 1 and 2

Proof of Fact. Let $\{\gamma_1, \gamma_2, \cdots, \gamma_{t(K)}\}$ be an unknotting tunnel system for K. Put $V_1 = N(K) \cup N(\gamma_2 \cup \gamma_2 \cup \cdots \cup \gamma_{t(K)})$ and $V_2 = cl(S^3 - V_1)$, where $N(K)$ is a regular neighborhood of K in S^3 and $N(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{t(K)})$ a regular neighborhood of $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{t(K)}$ in $E(K) = cl(S^3 - N(K))$. Then by the definition of the tunnel number $t(K)$, (V_1, V_2) is a genus $t(K)+1$ Heegaard splitting of S^3. Since K is a core of a handle of V_1, K is ambient isotopic to a loop in ∂V_1. Hence we have $h(K) \leq t(K)+1$.

Conversely, let $(V_1, V_2; F)$ be a genus $h(K)$ Heegaard splitting of S^3 such that K is contained in F. Let Γ be a core graph of V_1, i.e. $cl(V_1 - N(\Gamma))$ is homeomorphic to $F \times I$, where I is a unit interval. Let α be a "trivial" arc connecting a point in K and a point in Γ. Then, since $cl(V_1 - N(\Gamma))$ is homeomorphic to $F \times I$, $cl(S^3 - N(\Gamma \cup \alpha \cup K))$ is a genus $h(K)+1$ handlebody. This shows that K has an unknotting tunnel system consisting of $h(K)$ arcs. Hence we have $t(K) \leq h(K)$. This completes the proof of the fact. ■

To prove Propostion 1, we prepare a lemma.

Lemma 1. *A knot K admits a $(1, 1)$-decomposition if and only if there is a genus two Heegaard splitting $(V_1, V_2; F)$ of S^3 satisfying the following conditions: K is contained in F, and there is a cancelling disk pair (D_1, D_2) of (V_1, V_2) such that $D_1 \cap K$ is a single point.*

Here, we say that (D_1, D_2) is a cancelling disk pair of (V_1, V_2) if D_i is a non-separating disk properly embedded in V_i $(i=1, 2)$ and $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$ is a single point.

Proof of Lemma 1. Suppose K admits a $(1, 1)$-decomposition. Then there is a genus one Heegaard splitting (W_1, W_2) of S^3 such that $W_i \cap K$ is a trivial arc properly embedded in W_i, say α_i, $(i=1, 2)$. Let $N(\alpha_1)$ be a regular neighborhood of α_1 in W_1. Let C_1 and C_1' be the components of $N(\alpha_1) \cap \partial W_1$. Then $C_1 \cup C_1'$ is two disks which is a regular neighborhood of $\partial \alpha_1$ in ∂W_1. Since α_1 is a trivail arc in W_1, there is a disk in W_1, say E, such that ∂E is a union of α_1 and an arc in ∂W_1, say γ_1. We may assume that $\gamma_1 \cap C_1$ ($\gamma_1 \cap C_1'$ resp.) is an arc, say β_1 (β_1' resp.). Put $\Delta_1 = E \cap N(\alpha_1)$ and $D_1 = cl(E - \Delta_1)$.

Put $V_1 = cl(W_1 - N(\alpha_1))$. Then V_1 is a genus two handlebody and D_1 is a non-separating disk properly embedded in V_1. Put $V_2 = W_2 \cup N(\alpha_1)$. Then (V_1, V_2) is a genus two Heegaard splitting of S^3. Let $C_2 \cup C_2'$ be the image of $C_1 \cup C_1'$ in ∂W_2. Since α_2 is a trivial arc in W_2, there is disk in W_2, say Δ_2, such that $\partial \Delta_2$ is a union of α_2 and an arc in ∂W_2, say γ_2. We may assume that $\gamma_2 \cap C_2$ ($\gamma_2 \cap C_2'$ resp.) is an arc, say β_2 (β_2' resp.). Moreover we may assume that β_1 (β_1' resp.) is identified with β_2 (β_2' resp).

Put $A=\Delta_1 \cup \Delta_2$ in V_2. Then by the above observation, A is an annlus in V_2 such that ∂A is a union of K and a loop in ∂V_2, say K'. Then we can regard K' as K. Let D_2 be a disk properly embedded in $N(\alpha_1)$ parallel to C_1. Then D_2 is a non-separating disk properly embedded in V_2 intersecting K' in a single point. Moreover by the definition of D_1 and D_2, we see that (D_1, D_2) is a cancelling disk pair of the Heegaard splitting (V_1, V_2). This completes the proof of "if" part of the lemma.

Conversely by tracing back the above argument, we complete the proof of the lemma. ■

Proof of Propostion 1. By Lemma 1, for $i=1, 2$, we have a genus two Heegaard splitting $(V_1^i, V_2^i; F^i)$ of S^3 satisfying the following conditions: K_i is contained in F_i, $V_1^1 \cap V_1^2=\emptyset$ and there is a cancelling disk pair (D_1^i, D_2^i) of (V_i^1, V_2^i) such that $D_1^i \cap K_i$ is a single point. Hence $h(K_1)\leq 2$ and $h(K_2)\leq 2$. Let $N(D_1^i)$ be a regular neighborhood of D_1^i in V_1^i $(i=1, 2)$, and put $U_1^i=cl(V_1^i-N(D_1^i))$. Let W_1 be a genus three handlebody in S^3 obtained from U_1^1 and U_1^2 by identifying $cl(\partial U_1^1-\partial V_1^1)$ with $cl(\partial U_1^2-\partial V_1^2)$, and put $W_2=cl(S^3-W_1)$. Then since (D_1^i, D_2^i) is a cancelling disk pair of (V_1^i, V_2^i), (W_1, W_2) is a genus three Heegaard splitting of S^3. Moreover since $D_1^i \cap K$ is a single point, $K_1 \# K_2$ is contained in ∂W_1 (see Figure 3). Hence we have $h(K_1 \# K_2)\leq 3$. On the other hand, since K_i is not a torus knot $(i=1, 2)$, we have $h(K_1)\geq 2$ and $h(K_2)\geq 2$. And by Theorem we have $h(K_1 \# K_2)\geq 3$. This completes the proof of the proposition. ■

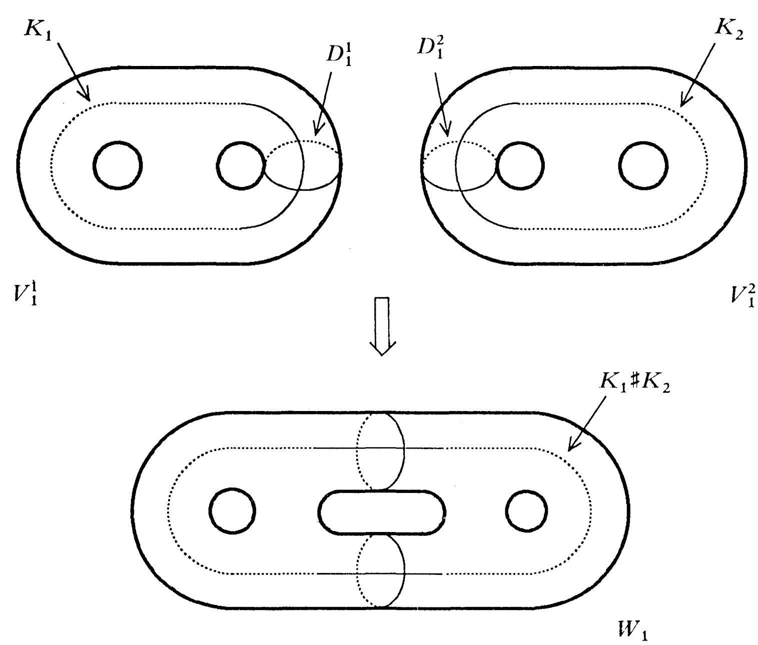

Figure 3

Proof of Propostion 2. By Theorem 3 of [5], we have $t(K_n)=2$ and $t(K_n \# K)=2$ for any 2-bridge knot K. Hence by Fact, $2 \leq h(K_n) \leq 3$ and $2 \leq h(K_n \# K) \leq 3$. If $h(K_n \# K)=2$, then by Theorem, we have $h(K_n)=1$, a contradiction. Hence we have $h(K_n \# K)=3$.

Suppose $h(K_n)=2$. Then there is a genus two Heegaard surface F of S^3 containing K_n. Then we have the following two cases.

Case 1 : K_n is a separating loop in F.

In this case, K_n bounds a punctured torus in S^3. Then by Ch. 8 of [8], the degree of the Alexander polynomial of K_n is at most two. However, the degree of the Alexander polynomial of K_n is $2n+10$. This is a contradiction, and hence Case 1 does not occur. Since the calculation of the Alexander polynomial is a routine matter, we leave it to the readers.

Case 2 : K_n is a non-separating loop in F.

Since, the orientation preserving mapping class group of F is generated by Dehn twists along the loops a_1, b_1, a_2, b_2 and a_3 indicated in Figure 4 ([3]), K_n is an image of the loop a_1 after a sequence of the Dehn twists. This shows that the orientation preserving involution h of S^3 indicated in Figure 4 fixes K_n setwise, and reverses the orientation of K_n (cf. [1] and [10]). Then by the proof of Theorem 3 of [5], we have a contradiction. This completes the proof of the proposition. ■

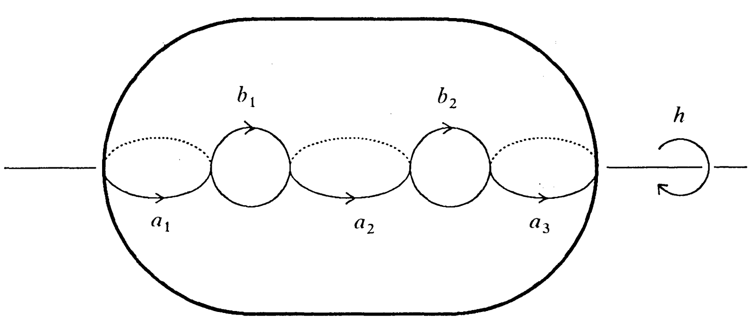

Figure 4

2. Proof of Theorem

Lemma 2. *Let V be a solid torus in S^3 and K a non-trivial knot in S^3 contained in ∂V. If K intersects a meridian of V more than once algebraically, then K is prime.*

Proof of Lemma 2. Put $\partial V=F$. Let S be a 2-sphere in S^3 intersecting K in two points. Then we may assume that each component of $S \cap F$ is a loop and that $\#(S \cap F)$ is minimum among all 2-spheres ambient isotopic rel. K to S, where $\#(\cdot)$ denotes the number of the components. Since S

intersects K in two points, we have the following two cases (see Figure 5).

Case I : $S \cap F = C_0^* \cup C_1 \cup \cdots \cup C_n$

Case II : $S \cap F = C_1^* \cup C_2^* \cup C_1 \cup \cdots \cup C_n$,

where C_i^* ($i=0, 1, 2$) is a loop intesecting K and C_i ($i=1, 2, \cdots, n$) is a loop not intersecting K.

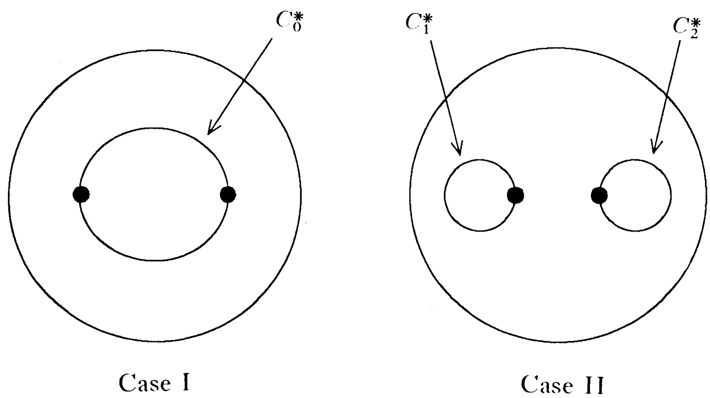

Figure 5

Claim 1 : There is no component of $\{C_i\}_{i=1}^n$ which is innermost in S.

Proof. Suppose there is an innermost component of $\{C_i\}_{i=1}^n$, say C_k, and let D be the disk in S bounded by C_k such that $D \cap (S \cap F - C_k) = \emptyset$. Then D is a disk properly embedded in V or in $cl(S^3 - V)$. By the minimality of $\#(S \cap F)$, D is essential in V or in $cl(S^3 - V)$. If D is in V, then D is a meridian disk of V. Then by the hypothesis of the lemma, D intersects K, a contradiction. If D is in $cl(S^3 - V)$, then ∂D is a prefered longitude of V. Then by the hypothesis of the lemma, D intersects K, a contradiction. This completes the proof of the claim.

Calim 2 : Case II does not occur.

Proof. Suppose we are in Case II. Then by Claim 1, C_1^* bounds a disk in S, say D, such that $D \cap (S \cap F - C_1^*) = \emptyset$. Since ∂D $(= C_1^*)$ intersects K in a single point, D is a non-separating disk properly embedded in V or in $cl(S^3 - V)$. If D is in V, then D is a meridian disk of V. This contradicts the hypothesis of the lemma. If D is in $cl(S^3 - V)$, then V is an unknotted solid torus. Then K is a $(n, 1)$-torus knot for some integer n. Hence K is a trivial knot, a contradiction. This completes the proof of the claim.

Suppose we are in Case I. By Claim 1, we have $S \cap F = C_0^*$. Let D_1 and D_2 be the two disks in S bounded by C_0^*. We may assume that D_1 is in V. If D_1 is a meridian disk of V, then since a core of V intersects D_1 in a single point, S is a non-separating 2-sphere in S^3, a contradiction. Hence D_1 is a separating disk in V. Then D_1 is isotopic rel. ∂D_1 to a disk in ∂V, say

D. Let B be the 3-ball in S^3 bounded by S containing D. Then $(B, B\cap K)$ is a trivial ball pair because $B\cap K$ is an arc properly embedded in $D\subset B$. This completes the proof of the lemma. ■

Lemma 3. *Let $(V_1, V_2; F)$ be a genus two Heegaard splitting of S^3 and K a non-trivial knot in S^3 contained in F. Suppose there is a non-separating disk properly embeded in V_1, say D, such that $D\cap K$ consists of at most one point. Then K is prime.*

Proof of Lemma 3. Let $N(D)$ be a regular neighborhood of D in V_1 such that $N(D)\cap K=\emptyset$ or an arc according as $D\cap K=\emptyset$ or a point.

Case I : $N(D)\cap K=\emptyset$.

Put $V=cl(V_1-N(D))$. Then V is a solid torus and K is a knot in ∂V. Since K is a non-trivial knot, K intersects a meridian of V algebraically. If K intersects a meridian of V more than once algebraically, then by Lemma 2, K is prime.

Suppose K intersects a meridian of V in a single point. Then K is ambient isotopic to a core of V, say K'. Since V_1 is obtained by attaching a 1-handle $N(D)$ to V, and $S^3-V_1=V_2$ is a handlebody, we see that K' is a tunnel number one knot. Then, since tunnel number one knots are prime ([7]), K' is prime. Hence K is prime. This completes the proof of Case I.

Crse II : $N(D)\cap K$ is an arc.

Put $\alpha=cl(K-N(D))$ and $cl(\partial N(D)-\partial V_1)=D_1\cup D_2$. Then α is an arc in ∂V_1 connecting the disks D_1 and D_2. Let $N(D_1\cup D_2\cup\alpha)$ be a regular neighborhood of $D_1\cup D_2\cup\alpha$ in V_1 and put $cl(\partial N(D_1\cup D_2\cup\alpha)-\partial V_1)=D_1^*\cup D_2^*\cup E$, where D_i^* is a disk parallel to D_i $(i=1, 2)$. Then E is a disk properly embedded in V_1 which splits V_1 into two solid tori $N(D\cup K)$ and W, where $N(D\cup K)$ is a regular neighborhood of $D\cup K$ in V_1 and $W=cl(V_1-N(D\cup K))$. Then since K is isotopic to a core of W, K is a tunnel number one knot. Hence K is prime, and this completes the proof of the lemma. ■

Proof of Theorem. Put $K=K_1\# K_2$. Let $(V_1, V_2; F)$ be a genus two Heegaard splitting of S^3 whose Heegarrd surface contains K, and let S be a 2-sphere which gives the non-trivial connected sum of K. We may assume that each component of $S\cap F$ is a loop and that $\#(S\cap F)$ is minimum among all 2-spheres ambient isotopic rel. K to S. Then similarly to the proof of Lemma 2, we have the following two cases (see Figure 5).

Case I : $S\cap F=C_0^*\cup C_1\cup\cdots\cup C_n$

Case II : $S\cap F=C_1^*\cup C_2^*\cup C_1\cup\cdots\cup C_n$,

where C_i^* $(i=0, 1, 2)$ is a loop interescting K and C_i $(i=1, 2, \cdots, n)$ is a loop not intersecting K.

Claim 1: There is no component of $\{C_i\}_{i=1}^n$ which is innermost in S.

Proof. Suppose there is an innermost component of $\{C_i\}_{i=1}^n$, say C_k, and let D be the disk in S bounded by C_k such that $D\cap(S\cap F-C_k)=\emptyset$. Then we may assume that D is a disk properly embedded in V_1. By the minimality of $\#(S\cap F)$, D is an essential disk in V_1. If D is a non-separating disk of V_1, then by Lemma 3 K is prime, a contradiction. If D splits V_1 into two solid tori, say W_1 and W_2. Then we may assume that K is contained in W_1. Let D' be a meridian disk of W_2 with $D'\cap D=\emptyset$. Then D' is a non-separating disk of V_1 such that $D'\cap K=\emptyset$. Then by Lemma 3, K is prime. This contradiction completes the proof of the claim.

Claim 2 : Case II does not occur.

Proof. Suppose we are in Case II. Then by Claim 1, C_1^* bounds a disk in S, say D, such that $D\cap(S\cap F-C_1^*)=\emptyset$. Then we may assume that D is properly embedded in V_1. Since $\partial D\,(=C_1^*)$ intersects K in a single point, D is a non-separating disk of V_1, and satisfies the hypothesis of Lemma 3. Hence K is prime. This contradiction completes the proof of the claim.

Now suppose we are in Case I. By Claim 1, we have $S\cap F=C_0^*$. Let D_1 and D_2 be the two disks in S bounded by C_0^*. We may assume that D_i is contained in V_i $(i=1,2)$. For $i=1$ or 2, if D_i is a non-separating disk in V_i, then since a core of a handle of V_i intersects D_i in a single point, S is a non-separating 2-sphere in S^3, a contradiction. Hence both D_1 and D_2 are separating disks in V_1 and in V_2 respectively. This shows that both K_1 and K_2 are contained in genus one Heegaard surfaces and completes the proof of Theorem. ■

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