ALMOST QF RINGS AND ALMOST QF# RINGS

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In this paper we assume that every ring R is an associative ring with identity and R is two-sided artinian. The author has defined almost projective modules and almost injective modules in [8], and by making use of the concept of almost projectives he has defined almost hereditary rings in [7], whose class contains that of hereditary rings and serial rings. Similarly to [7] we shall define an almost QF ring, which is a generalization of QF rings.

It is well known that an artinian ring R is QF if and only if R is self injective. Following this fact, if R is almost injective as a right R-module, we call R a right almost QF ring. Analogously we call R a right almost QF# ring if every injective is right almost projective. On the other hand, the author studied rings with (*) (resp. (*)*) (see §1 for definitions) in [4]. K. Oshiro called such a ring a right H- (resp. co-H) ring in [10]. In this note we shall show that a right almost QF (resp. almost QF#) ring coincides with a right co-H (resp. H-) ring. In the final section we shall give a characterization of serial rings in terms of almost projectives and almost injectives.

In the forthcoming paper [9] we shall study certain conditions under which right almost QF rings are QF or serial.

1. Almost QF rings

In this paper we always assume that R is a two-sided artinian ring with identity and that every module is a unitary right R-module. We use the same notations in [7]. We have studied almost hereditary rings in [7], i.e. J, the Jacobson radical of R, is right almost projective. We shall study, in this paper, some kind of the dual concept to almost hereditary rings (see Theorem 1 below). We call R *a right almost* QF *ring* if R is right almost injective as a right R-module [8]. We can define similarly a *left almost* QF *ring*. It is clear that R is right almost QF if and only if every finitely generated projective R-module is right almost injective. Hence the concept of almost QF rings is presevred under Morita equivalence and we may assume that R is basic.

On the other hand, the author studied the two conditions in [3] and [4]. Let M be an R-module. If $M\mathrm{Soc}^l(R)\neq 0$ (resp. $M\,\mathrm{Soc}^r(R)\neq 0$) then we call M non-

small (resp. non-cosmall) [4], where $\mathrm{Soc}^l(R)$ (resp. $\mathrm{Soc}^r(R)$) is the left (resp. right) socle of R.

(∗)   *Every non-small module contains a non-zero injective module.*

(∗)∗   *Every non-cosmall module contains a projective direct summand.*

K. Oshiro called rings with (∗) (resp. (∗)∗) right H (resp. right co-H) rings in [10]. Relating with those two concepts we have

Theorem 1.   *Let R be an artinian ring.   Then the following are equivalent:*
1)   *R is right almost QF.*
2)   *The Jacobson radical J of R is almost injective as a right R-module.*
3)   *R is a right co-H ring.*

Proof. 1) ↔ 2).   This is clear from [8], Corollary 1* and Proposition 3.

1) ↔ 3).   We know, from the above implication, [8], Proposition 3, [4], Theorem 3.6 and [11], Theorem 4.1, that the structure of right almost QF rings coincides with that of right co-H rings.

Corollary.   *R is right almost QF if and only if R is right QF-2, QF-3 and every submodule containing a projective submodule of eR is local for any primitive idempotent e.*

Proof.   If R is right almost QF, R satisfies the conditions in the corollary by [8], Corollary 1* and Proposition 3.   Conversely assume that R satisfies the conditions.   Let A be a submodule of eR such that $eR \supset A \supset fR$, where e, f are primitive idempotents.   Then $A \approx gR/B$ by assumption, where g is a primitive idempotent.   Hence we obtain a natural epimorphism $\theta: gR \to A$.   Put $K = \theta^{-1}(fR)$.   Since fR is projective, $K = B \oplus K'$.   Further gR is uniform, and hence $B = 0$.   Therefore A is projective, and R is right almost by Theorem 1 and [8], Proposition 3.

2.   Almost QF* rings

In this section we shall study the dual concept to almost QF.   If every indecomposable injective module is almost projective, we call R *a right almost QF* ring*.   If every indecomposable injective module is loeal, we call R *right QF-2**.   If a projective cover of every (indecomposable) injective module is injective, we call R *right QF-3**.

As the dual to Theorem 1, the following theorem is clear from [1], Theorem 2, [8], Theorem 1 and [10], Theorem 3.18.

Theorem 2.   *Let R be artinian.   Then the following are equivalent:*
1)   *R is right almost QF*.*
2)   *R is a right H-ring.*

K. Oshiro [11] showed that R is right almost QF* if and only if R is left al-

most QF.

The following is dual to Corollary to Theorem 1.

Proposition 1. *Let R be an artinian ring. Then R is right almost QF^* if and only if* 1): *R is right QF-2^*,* 2): *R is right QF-3^* and* 3): *if eR/A is injective for $A \neq 0$, then eR/B is uniform for any $B \subset A$.* 1) *together with* 2) *is equivalent to* 4): *every indecomposable injective is a factor module of some local projective and injective module, where e is a primitive idempotent.*

Proof. If R is right almost QF^*, we obtain the conditions 1), 2) and 3) by [8], Corollary 1^*. Conversely we assume 1), 2) and 3). Let eR/A be injective and $B \subset A$. Then eR/B is uniform by 3). Take a diagram

$$0 \to eR/B \xrightarrow{\ i\ } E(eR/B)$$

$$\downarrow \nu \quad \nearrow \mu$$

$$eR/A$$

where i is the inclusion and ν is the natural epimorphism. Since eR/A is injective, we have μ with $\mu i = \nu$. ν being an epimorphism, $E(eR/B) = eR/B + \mu^{-1}(0)$. Further $E(eR/B)$ is local by 1) and 3), and hence $E(eR/B) = eR/B$. Therefore eR/A is almost projective by [8], Theorem 1^*. Hence R is right almost QF^*.

If R is hereditary and QF, then R is semisimple. Concerning with this fact, we have

Proposition 2. *Let R be an artinian ring. Then the following are equivalent:*
   1) *R is serial.*
   2) *R is right almost QF^* and right co-serial. (cf.* [11], *Theorem 6.1.)*
   3) *R is right almost QF and right almost hereditary.*
   4) *R is right almost QF^* and right almost hereditary.*
   5) *R is left almost QF and right almost hereditary.*

Proof. 1) $\to$ 2), 3), 4) and 5). This is clear from [7] and [10].

2) $\to$ 1). Let f be a primitive idempotent and $E = E(fR)$. Then we may assume $E = e_1 R \oplus \cdots \oplus e_p R \oplus g_1 R/A_1 \oplus \cdots \oplus g_q R/A_q$, where the $e_i R$ and the $g_j R$ are injective and $A_j \neq 0$ for all j by Proposition 1. Let θ be the natural epimorphism of $(\sum_{i=1}^{p} \oplus e_i R \oplus \sum_{j=1}^{q} \oplus g_j R)$ onto E with $\theta^{-1}(0) = A_1 \oplus \cdots \oplus A_q$. Then since fR is projective, $\theta^{-1}(fR) = P \oplus \theta^{-1}(0)$; $P \approx fR$. Set $E_1 = \Sigma \oplus e_i R$ and $E_2 = \Sigma \oplus g_j R$ and $\pi_1 : E \to E_1$ the projection. Since $\theta^{-1}(0)$ is essential in E_2, $P \cap E_2 = 0$. Hence $fR \approx P \approx \pi_1(P) \subset E_1$. Next we shall show that every submodule of E_1 is standard. Take submodules $K_i \subset L_i \subset e_i R$ for $i = 1, 2$ such that

$\mu: L_1/K_1 \approx L_2/K_2$. We may assume $|e_1R/K_1| \leq |e_2R/K_2|$. Since e_iR is uniserial, we can suppose that e_1R/K_1 and e_2R/K_2 are contained in $F=E(L_1/K_1)$, which is also uniserial. We can extend μ to an automorphism μ^* of F. Since $|e_1R/K_1| \leq |e_2R/K_2|$, $\mu^*(e_1R/K_1) \subset e_2R/K_2$. On the other hand, e_1R being projective, μ^* is liftable to a homomorphism of e_1R into e_2R. Hence fR is a standard submodule of E_1 by [6], Lemma 5. Accordingly $fR \subset$ some e_iR (isomorphically) for fR is local. Therefore fR is uniserial for any f, and hence R is serial by [2], Theorem 5.4.

3) $\rightarrow$ 1). If R is hereditary, R is a serial ring in the first category by Corollary to Theorem 1 and [7], Corollary 3. Assume that R is of the form (9) in [7], Theorem 2. Now we follow [7] for the notations. Then h_aR is not injective provided R is not serial by [7], Corollary 3. However h_aR must be contained in an injective and projective fR by Corollary to Theorem 1, which is impossible from the construction of R in [7].

4) $\rightarrow$ 1). Let R be right almost hereditary and right almost QF[*]. We use the same notations as in [7]. If R is not serial, $T_1 \neq 0$ in [7], the figure (9). Then $E(h_1R/h_1J)$ is a factor module of an injective and projective fR by Proposition 1, which is impossible by the structure of R. Hence R is serial.

4) $\rightarrow$ 5). This is clear from the remark after Theorem 2 [11].

3. Serial rings

We have studied generalizations of QF rings in §§1 and 2. We shall consider, in this section, the remaining generalizations following previous sections.

Theorem 3. *Let R be artinian. Then the following are equivalent:*
1) *Every almost projective is almost injective.*
2) *Every almost injective is almost projective.*
3) *R is serial.*

Proof. 3) $\rightarrow$ 1) and 2). This is clear from [7], Figure (2) and [8], Corollary 1 and Corollary 1[*].

1) or 2) $\rightarrow$ 3). Let $\{g_iR\}$ be the set of indecomposable, projective and injective modules, which is not empty by 1) or 2) and [8], and rename $\{g_iR\} = \{e_1R, \cdots, e_pR, f_1R, \cdots, f_qR\}$, where the e_iR are uniserial and the f_jR are not. Assume $q > 0$, i.e. R is not serial. We shall find the set of non-projective, non-injective, non-uniserial and indecomposable almost projectives. Since f_jR is not uniserial, there exists an integer k_j such that $f_jR/\mathrm{Soc}_r(f_jR)$ is injective for all $0 \leq r < k_j$ and $f_jR/\mathrm{Soc}_{k_j}(f_jR)$ is not injective. Then $f_jR/\mathrm{Soc}_{k_j}(f_jR)$ is non-projective, non-injective, non-uniserial and almost projective module by [8], Corollary 1. Further $f_jR/\mathrm{Soc}_{k_j}(f_jR) \not\approx f_iR/\mathrm{Soc}_{k_i}(f_iR)$ for $i \neq j$. Therefore since e_iR is uniserial, we obtain just q non-projective, non-injective, non-uniserial almost projective modules

$$\{f_j R/\mathrm{Soc}_{k_j}(f_j R)\}_{j \leq q}$$

by [8], Theorem 1. On the other hand, if $f_i J$ is projective, $f_i J \approx gR$ for some primitive idempotent g (note that $f_i R$ is uniform). Continuing this arument, we may suppose

$$f_i R \supset f_i J \supset \cdots \supset f_i J^{r_i - 1} (i \leq q), f_i J^{k-1} \approx f_{ik} R \text{ and } f_{ir_i} J \text{ is not projective, where}$$

the f_{is} are primitive idempotents.

Hence since $f_s R$ is not uniserial, non-projective, non-injective, non-uniseial almost injective is of $f_s J^{r_s} (=f_{sr_s} J)$ for $s \leq q$ by [8], Theorem $1^{\ast}$. Further $f_s J^{r_s} \not\approx f_t J^{r_t}$ for $s \neq t$, since $f_s R$ and $f_t R$ are indecomposable and injective. Hence we obtain just q non-projective, non-injective, non-uniserial and almost injective modules

$$\{f_j J^{r_j}\}_{j \leq q}.$$

From 1) we shall show that $f_{jr_j} J$ is local. By 1) $f_j R/\mathrm{Soc}_{k_j}(f_j R)$ is almost injective. Hence $\{f_i R/\mathrm{Soc}_{k_i}(f_i R)\}_{i \leq q} = \{f_s J^{r_s}\}_{s \leq q}$ up to isomorphism. Therefore the $f_s J^{r_s} = f_{sr_s} J$ are local. On the other hand, since every indecomposable projective gR is almost injective by assumption, $gR \approx e_i J^t$ for some t or $gR \approx f_i J^{t'}$ for some $t' \leq r_i - 1$ by [8], Corollary $1^{\ast}$. Hence gJ is local from the above. Therefore R is right serial by [5], Proposition 1, and R is serial by [10], Theorem 6.1.

Assume 2). Then $\{f_i R/\mathrm{Soc}_{k_i}(f_i R)\}_{i \leq q} = \{f_s J^{r_s}\}_{s \leq q}$ as above. Hence the $f_i R/\mathrm{Soc}_{k_i}(f_i R)$ are uniform and R is co-serial by [8], Corollary 1 and [5], Proposition 1'. Therefore R is serial by Proposition 2.

Proposition 3. *Let R be artinian. Then the following are equivalent:*
1) *Every almost projective is injective.*
2) *Every almost injective is projective.*
3) *R is semi-simple.*

Proof. Assume 1). Then R is QF. Since $eR/\mathrm{Soc}(eR)$ is almost projective by [8], Theorem 1, $eR/\mathrm{Soc}(eR)$ is injective by 1). Hence $eR/\mathrm{Soc}(eR)$ is projective. Therefore $\mathrm{Soc}(eR) = eR$. Remaining parts are also clear.

The final type is

Proposition 4. *Let R be as above. Then the following are equivalent:*
1) *Every clmost projective is projective.*
2) *Every almost injective is injective.*
3) *R is a direct sum of semi-simple rings and rings whose every projective is never injective.*

Proof. 1) $\rightarrow$ 3). Let R be basic and eR injective. Then $eR/\mathrm{Soc}(eR)$ is almost projecive by [8], Theorem 1, and hence $eR/\mathrm{Soc}(eR)$ is projective by 1). Therefore eR is simple and $R = eR \oplus (1-e)R$ as rings. Thus we obtain 3).

The remaining implications are also clear.

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