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ASYMPTOTICS OF EIGENVALUES OF THE LAPLACIAN WITH SMALL SPHERICAL ROBIN BOUNDARY

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1. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^N$ with C^∞ boundary $\partial\Omega$. Let $\tilde{w}$ be a fixed point in Ω and $B(\varepsilon, \tilde{w})$ be the ball of radius ε with the center $\tilde{w}$. We put $\Omega_\varepsilon=\Omega\backslash\overline{B(\varepsilon, \tilde{w})}$. Consider the following eigenvalue problem

$$
\begin{aligned}
(1.1) \qquad & -\Delta u(x) = \lambda u(x) && x\in\Omega_\varepsilon \\
& u(x) = 0 && x\in\partial\Omega \\
& u(x)+k\varepsilon^\sigma \frac{\partial u}{\partial \nu_x}(x) = 0 && x\in\partial B(\varepsilon, \tilde{w}) .
\end{aligned}
$$

Here k denotes a positive constant. And σ is a real number. Here $\partial/\partial\nu_x$ denotes the derivative along the exterior normal direction with respect to Ω_ε.

Let $\mu_j(\varepsilon)>0$ be the j-th eigenvalue of (1.1). Let μ_j be the j-th eigenvalue of the problem

$$
\begin{aligned}
(1.2) \qquad & -\Delta u(x) = \lambda u(x) && x\in\Omega \\
& u(x) = 0 && x\in\partial\Omega .
\end{aligned}
$$

Let $G(x,y)$ (resp, $G_\varepsilon(x, y)$) be the Green function of the Laplacian in Ω (resp. Ω_ε) associated with the boundary condition (1.2) (resp. (1.1)).

Main aim of this paper is to show the following Theorems. Let $\varphi_i(x)$ be the L^2-normalized eigenfunction associated with μ_j. We have the following.

Theorem 1. *Assume $N=3$. We fix j and $\sigma\geqq 1$. Suppose that μ_j is simple. Then, for any fixed $s\in(0, 1)$,*

$$
\begin{aligned}
(1.3) \qquad & \mu_j(\varepsilon) = \mu_j+P_j\varepsilon+O(\varepsilon^{2-s}) && (\quad \sigma\geqq 2) \\
& \mu_j(\varepsilon) = \mu_j+P_j\varepsilon+O(\varepsilon^{\sigma}) && (1<\sigma<2) \\
& \mu_j(\varepsilon) = \mu_j+(1+k)^{-1}P_j\varepsilon+O(\varepsilon^{2-s}) && (\quad \sigma=1) ,
\end{aligned}
$$

where

$$P_j = 4\pi\varphi_j(\tilde{w})^2 .$$

Theorem 2. *Assume $N=3$. We fix j and $\sigma<1$. Suppose that μ_j is simple. Then,*

$$\begin{aligned} (1.4)\qquad \mu_j(\varepsilon) &= \mu_j+Q_j\varepsilon^{2-\sigma}+O(\varepsilon^{3-2\sigma}) && (0\leqq\sigma<1) \\ \mu_j(\varepsilon) &= \mu_j+Q_j\varepsilon^{2-\sigma}+R_j\varepsilon^3+O(\varepsilon^{3-2\sigma}) && (-1/2<\sigma<0) \\ \mu_j(\varepsilon) &= \mu_j+Q_j\varepsilon^{2-\sigma}+R_j\varepsilon^3+O(\varepsilon^4) && (-2<\sigma\leqq-1/2) \\ \mu_j(\varepsilon) &= \mu_j+R_j\varepsilon^3+O(\varepsilon^4) && (\sigma\leqq-2)\,, \end{aligned}$$

where

$$\begin{aligned} Q_j &= (4\pi/k)\,\varphi_j(\tilde{w})^2 \\ R_j &= -\pi(2\,|\mathrm{grad}\,\varphi_j(\tilde{w})|^2-(4/3)\mu_j\varphi_j(\tilde{w})^2)\,. \end{aligned}$$

Remark. The case $N=2$ is treated in Ozawa [10] and [11]. The singularity of $G(x, y)$ near $x=y$ in the case $N=3$ is stronger than that of the case $N=2$. When we use the Sobolev embedding; $W^{2,p}(\Omega)\hookrightarrow C^{2-N/p}(\bar{\Omega})$, we must take p larger as N increases. Therefore we may need some change of the mehtod devloped in the above papers.

When $N\geqq4$, we do not know whether the method we have used can be applied or not.

For the related papers we have Besson [2], Chavel and Feldman [3], Ozawa [8], [9], Rauch and Taylor [12] and the references in the above papers.

For other related problems on singular variation of domains the readers may refer to Arrieta, Hale and Han [1], Jimbo [4], Jimbo and Morita [5]. The Poisson equation with many small Robin holes is discussed in Kaizu [6], [7].

2. Outline of proof of Theorem 1 and Theorem 2

Hereafter we assume $N=3$.

We introduce the following kernel $p_\varepsilon(x, y)$.

$$\begin{aligned} (2.1)\qquad p_\varepsilon(x, y) = G(x, y)&+g(\varepsilon)G(x, \tilde{w})G(\tilde{w}, y) \\ &+h(\varepsilon)\langle\nabla_w G(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle \\ &+i(\varepsilon)\langle H_w G(x, \tilde{w}),\ H_w G(\tilde{w}, y)\rangle\,, \end{aligned}$$

where

$$\langle\nabla_w u(\tilde{w}),\ \nabla_w v(\tilde{w})\rangle = \sum_{n=1}^{3}\frac{\partial u}{\partial w_n}\frac{\partial v}{\partial w_n}\Big|_{w=\tilde{w}}$$

$$\langle H_w u(\tilde{w}),\ H_w v(\tilde{w})\rangle = \sum_{m,n=1}^{3}\frac{\partial^2 u}{\partial w_m\partial w_n}\frac{\partial^2 v}{\partial w_m\partial w_n}\Big|_{w=\tilde{w}}$$

when $w=(w_1, w_2, w_3)$ is an orthonormal frame of $\boldsymbol{R}^3$. Here $g(\varepsilon)$, $h(\varepsilon)$, $i(\varepsilon)$ are determined so that

$$(2.2)\qquad p_\varepsilon(x, y)+k\varepsilon^\sigma\frac{\partial}{\partial\nu_x}p_\varepsilon(x, y) \qquad x\in\partial B(\varepsilon, \tilde{w})$$

is small in some sense.

If we put

$$g(\varepsilon) = -(\gamma + (4\pi\varepsilon)^{-1} + k(4\pi)^{-1}\varepsilon^{\sigma-2})^{-1} \tag{2.3}$$

$$h(\varepsilon) = (k\varepsilon^{\sigma} - \varepsilon)/((4\pi)^{-1}\varepsilon^{-2} + k(2\pi)^{-1}\varepsilon^{\sigma-3}) \quad (\sigma<1)$$
$$= 0 \quad (\sigma \geqq 1) \tag{2.4}$$

and

$$i(\varepsilon) = k\varepsilon^{\sigma+1}/(3(4\pi)^{-1}\varepsilon^{-3} + 9k(4\pi)^{-1}\varepsilon^{\sigma-4}) \quad (\sigma<1)$$
$$= 0 \quad (\sigma \leqq 1), \tag{2.5}$$

the above aim for (2.2) to be small is attained.
Here

$$\gamma = \lim_{x \to \tilde{w}} (G(x, \tilde{w}) - (4\pi)^{-1} |x - \tilde{w}|^{-1}).$$

We put

$$(\boldsymbol{G}f)(x) = \int_{\Omega} G(x, y) f(y) dy$$

$$(\boldsymbol{G}_{\varepsilon}f)(x) = \int_{\Omega_{\varepsilon}} G_{\varepsilon}(x, y) f(y) dy$$

and

$$(\boldsymbol{P}_{\varepsilon}f)(x) = \int_{\Omega_{\varepsilon}} p_{\varepsilon}(x, y) f(y) dy$$

Let T and T_{ε} be operators on Ω and Ω_{ε}, respectively. Then, $||T||_p$, $||T_{\varepsilon}||_{p,\varepsilon}$ denotes the operator norm on $L^p(\Omega)$, $L^p(\Omega_{\varepsilon})$, respectively. Let f and f_{ε} be functions on Ω and Ω_{ε}, respectively. Then, $||f||_p$, $||f_{\varepsilon}||_{p,\varepsilon}$ denotes the norm on $L^p(\Omega)$, $L^p(\Omega_{\varepsilon})$, respectively.

At first we outline the proof of Theorem 1. A crucial part of our proof of Theorem 1 is the following.

Theorem 3. *Fix $\sigma \geqq 1$ and $s \in (0, 1)$. Then there exists a constant C_s independent of ε such that*

$$||(\boldsymbol{P}_{\varepsilon} - \boldsymbol{G}_{\varepsilon})f||_{2,\varepsilon} \leqq C_s \varepsilon^{2-s} ||f||_{p,\varepsilon} \tag{2.6}$$

holds for any $f \in L^p(\Omega_{\varepsilon})$ ($p>3$).

We put

$$\tilde{p}_{\varepsilon}(x, y) = G(x, y) + g(\varepsilon) G(x, \tilde{w}) G(\tilde{w}, y) \chi_{\varepsilon}(x) \chi_{\varepsilon}(y)$$
$$+ h(\varepsilon) \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle \chi_{\varepsilon}(x) \chi_{\varepsilon}(y)$$
$$+ i(\varepsilon) \langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y) \rangle \chi_{\varepsilon}(x) \chi_{\varepsilon}(y) \tag{2.7}$$

for the characteristic function $\chi_{\varepsilon}(x)$ of $\bar{\Omega}_{\varepsilon}$.

And we put

$$(\tilde{\boldsymbol{P}}_\varepsilon f)(x)=\int_\Omega \tilde{p}_\varepsilon(x,y)f(y)dy\,.$$

Since $\boldsymbol{G}_\varepsilon$ is approximated by $\boldsymbol{P}_\varepsilon$ and the difference between $\boldsymbol{P}_\varepsilon$ and $\tilde{\boldsymbol{P}}_\varepsilon$ is small in some sense, we know that everything reduces to our investigation of the perturbative analysis of $\boldsymbol{G}\rightarrow\tilde{\boldsymbol{P}}_\varepsilon$.

Next we outline the proof of Theorem 2. One important part of our proof of Theorem 2 is the following.

Theorem 4. *Fix $\sigma<1$. Then, there exists a constant C such that*

$$(2.8)\qquad \|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\varphi_j)\|_{2,\varepsilon}\leqq C\varepsilon^{4-\sigma}\qquad(0\leqq\sigma<1)$$
$$\leqq C\varepsilon^{4}\qquad(\quad\sigma<0)$$

hold.

We fix j and put

$$(2.9)\qquad \bar{p}_\varepsilon(x,y)=G(x,y)-(4\pi/3)\mu_j\varepsilon^3G(x,\tilde{w})G(\tilde{w},y)\xi_\varepsilon(x)\xi_\varepsilon(y)$$
$$+g(\xi)G(x,\tilde{w})G(\tilde{w},y)\xi_\varepsilon(x)\xi_\varepsilon(y)$$
$$+h(\varepsilon)\langle\nabla_wG(x,\tilde{w}),\nabla_wG(\tilde{w},y)\rangle\xi_\varepsilon(x)\xi_\varepsilon(y)$$
$$+i(\varepsilon)\langle H_wG(x,\tilde{w}),H_wG(\tilde{w},y)\rangle\xi_\varepsilon(x)\xi_\varepsilon(y)\,,$$

where $\xi_\varepsilon(x)\in C^\infty(\boldsymbol{R}^3)$ satisfies $|\xi_\varepsilon(x)|\leqq1$, $\xi_\varepsilon(x)=1$ for $x\in\boldsymbol{R}^3\backslash\overline{B(\varepsilon,\tilde{w})}$, $\xi_\varepsilon(x)=0$ for $x\in B(\varepsilon/2,\tilde{w})$ and $\xi_\varepsilon(x-\tilde{w})$ is rotationary invariant.
Furthermore we put

$$(\bar{\boldsymbol{P}}_\varepsilon f)(x)=\int_\Omega \bar{p}_\varepsilon(x,y)f(y)dy\,.$$

The other important part of our proof of Theorem 2 is the following.

Theorem 5. *Fix $\sigma<1$. Then, there exists a constant C such that*

$$(2.10)\qquad \|(\chi_\varepsilon\bar{\boldsymbol{P}}_\varepsilon-\boldsymbol{P}_\varepsilon\chi_\varepsilon)\varphi_j\|_{2,\varepsilon}\leqq C\varepsilon^{4-\sigma}\qquad(0<\sigma<1)$$
$$\leqq C\varepsilon^{4}\qquad(\quad\sigma\leqq0)$$

hold.

Since (2.8) and (2.10) are both $o(\varepsilon^3+\varepsilon^{2-\sigma})$, we know that everything reduces to our investigation of the perturbative analysis of $\boldsymbol{G}\rightarrow\bar{\boldsymbol{P}}_\varepsilon$.

3. Estimation of L^p-norm

We write $B(\varepsilon,\tilde{w})=B_\varepsilon$. In this section we show the following propositions.

Proposition 3.1. *Fix $\sigma \geqq 1$. Assume that $u_\varepsilon(x) \in C^\infty(\bar{\Omega}_\varepsilon)$ satisfies*

$$
\begin{aligned}
(3.1) \qquad & \Delta u_\varepsilon(x) = 0 \qquad x \in \Omega_\varepsilon \\
& u_\varepsilon(x) = 0 \qquad x \in \partial\Omega \\
& u_\varepsilon(x) + k\varepsilon^\sigma \frac{\partial u_\varepsilon}{\partial \nu_x}(x) = M(\omega) \quad x = \tilde{w} + \varepsilon\omega \in \partial B_\varepsilon \, (\omega \in S^2) \, .
\end{aligned}
$$

We fix $s \in (0, 1)$. Then,

$$
(3.2) \qquad ||u_\varepsilon||_{2,\varepsilon} \leqq C_s \varepsilon^{1-s} \operatorname*{Max}_\omega |M(\omega)|
$$

holds for a constant C_s independent of ε.

Proposition 3.2. *Fix $\sigma < 2$. Under the same assumptions of u_ε in Proposition* 3.1,

$$
(3.3) \qquad ||u_\varepsilon||_{2,\varepsilon} \leqq C \varepsilon^{2-\sigma} \operatorname*{Max}_\omega |M(\omega)|
$$

holds for a constant C independent of ε.

We take the same procedure as in Ozawa [9, section 1, pp. 260–262] to prove the above Propositions. But we need some change of the method developed in the above paper, since we put the Robin condition on ∂B_ε and we assume that $N=3$.

At first we prepare two Lemmas.

Lemma 3.3. *Fix $\alpha \in C^\infty(S^2)$ and $q > 1$. Then there exists at least one solution of*

$$
(3.4) \qquad \Delta v_\varepsilon(x) = 0 \qquad x \in \boldsymbol{R}^3 \backslash \bar{B}_\varepsilon
$$

$$
(3.5) \qquad v_\varepsilon(x) + k\varepsilon^\sigma \frac{\partial v_\varepsilon}{\partial \nu_x}(x) = \alpha(\omega) \quad x = \tilde{w} + \varepsilon\omega \in \partial B_\varepsilon (\omega \in S^2)
$$

satisfying

$$
(3.6) \qquad |v_\varepsilon(x)| \leqq C\varepsilon^{2-\sigma} \operatorname*{Max}_\omega |\alpha(\omega)| \, r^{-1}(\log(r/(r-\varepsilon)))^{1/2}
$$

$$
(3.7) \qquad |v_\varepsilon(x)| \leqq C_q \varepsilon^{1-\sigma/q} \operatorname*{Max}_\omega |\alpha(\omega)| \, (r-\varepsilon)^{-1/q'}
$$

for $r = |x - \tilde{w}| > \varepsilon$ and

$$
(3.8) \qquad ||v_\varepsilon||_{2,\varepsilon} \leqq C'_q \varepsilon^{1-(\sigma-1)/(2q)} \operatorname*{Max}_\omega |\alpha(\omega)| \, ,
$$

where q' satisfies $(1/q) + (1/q') = 1$.

Proof. We put $x = \tilde{w} + r\omega$ ($\omega \in S^2$) and

$$
\omega = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta) \quad (0 \leqq \theta < \pi, \, 0 \leqq \varphi < 2\pi).
$$

Let $P_n(z)$ be the Legendre polynomial and $P_n^m(z)$ be the associated Legendre function, that is,

$$P_n^m(z) = (1-z^2)^{m/2} \cdot d^m P_n(z)/dz^m \qquad (|z|<1,\ m \in \boldsymbol{Z}_+).$$

It is well-known that $\{P_n^m(\cos\theta)\cos m\varphi,\ P_n^m(\cos\theta)\sin m\varphi;\ 0\leqq m\leqq n\}_{n=0}^{\infty}$ is a complete orthogonal system of $L^2(S^2)$ consisting of eigenfunction of the Laplace-Beltrami operator Δ_{S^2} whose eigenvalue is $-n(n+1)$.

Therefore we have the Fourier expansion

$$\alpha(\omega) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi), \tag{3.9}$$

where

$$Y_n(\theta, \varphi) = \sum_{m=0}^{n} (a_{n,m}\cos m\varphi + b_{n,m}\sin m\varphi)\, P_n^m(\cos\theta). \tag{3.10}$$

By the Parseval relation, we see

$$\begin{aligned} &\sum_{n=0}^{\infty} (2n+1)^{-1}(a_{n,0}^2 + \sum_{m=1}^{n} ((n+m)!/2\cdot(n-m)!)(a_{n,m}^2+b_{n,m}^2)) \\ &= C\|\alpha\|_{L^2(S^2)}^2 \leqq C'(\operatorname*{Max}_{\omega}|\alpha(\omega)|)^2. \end{aligned} \tag{3.11}$$

We put

$$v_\varepsilon(x) = \sum_{n=0}^{\infty} (\sum_{m=0}^{n} (s_{n,m}\cos m\varphi + t_{n,m}\sin m\varphi) P_n^m(\cos\theta)) r^{-(n+1)}.$$

Then, it satisfies $\Delta v_\varepsilon(x)=0$ for $x\in \boldsymbol{R}^3\backslash \bar{B}_\varepsilon$.
We see that

$$\begin{aligned} v_\varepsilon(x) + k\varepsilon^\sigma \frac{\partial v_\varepsilon}{\partial \nu_x}(x)_{|x\in\partial B_\varepsilon} &= \alpha(\omega) \\ &= \sum_{n=0}^{\infty} (\sum_{m=0}^{n} (a_{n,m}\cos m\varphi + b_{n,m}\sin m\varphi) P_n^m(\cos\theta)) \end{aligned}$$

implies

$$\begin{aligned} a_{n,m} &= \varepsilon^{-(n+1)}(1+(n+1)k\varepsilon^{\sigma-1})s_{n,m} \\ b_{n,m} &= \varepsilon^{-(n+1)}(1+(n+1)k\varepsilon^{\sigma-1})t_{n,m} \end{aligned}$$

for $0\leqq m\leqq n$, $n\geqq 0$.
Thus we have

$$v_\varepsilon(x) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi)\,(\varepsilon/r)^{n+1}(1+(n+1)k\varepsilon^{\sigma-1})^{-1}, \tag{3.12}$$

and

$$|v_\varepsilon(x)|^2 \leq (\sum_{n=0}^{\infty} Y_n(\theta, \varphi)^2) \sum_{n=0}^{\infty} (\varepsilon/r)^{2n+2}(1+(n+1)k\varepsilon^{\sigma-1})^{-2}. \tag{3.13}$$

Since (3.9) holds in $L^2(S^2)$, we see that

$$\int_{\omega\in S^2} |v_\varepsilon(x)|^2\, d\omega$$
$$\leqq \|\alpha\|^2_{L^2(S^2)} \sum_{n=0}^{\infty} (\varepsilon/r)^{2n+2}(1+(n+1)k\varepsilon^{\sigma-1})^{-2}$$
$$\leqq C(\operatorname{Max}_{\omega} |\alpha(\omega)|)^2 (\sum_{n=0}^{\infty} (\varepsilon/r)^{2(n+1)q'})^{1/q'}$$
$$\times (\sum_{n=0}^{\infty} (1+(n+1)k\varepsilon^{\sigma-1})^{-2q})^{1/q}$$
$$\leqq C(\operatorname{Max}_{\omega} |\alpha(\omega)|)^2 (\varepsilon/r)^2 (\sum_{n=0}^{\infty} (\varepsilon/r)^n)^{1/q'}$$
$$\times (\int_0^\infty (1+k\varepsilon^{\sigma-1}t)^{-2q} dt)^{1/q}$$
$$= C(\operatorname{Max}_{\omega} |\alpha(\omega)|)^2 (\varepsilon/r)^2 (r/(r-\varepsilon))^{1/q'}((2q-1)k\varepsilon^{\sigma-1})^{-1/q} .$$

Therefore, we have

$$\|v_\varepsilon\|^2_{2,\varepsilon} \leqq \int_\varepsilon^R (\int_{\omega\in S^2} |v_\varepsilon(x)|^2 d\omega) r^2 dr$$
$$\leqq C_q(\operatorname{Max}_{\omega} |\alpha(\omega)|)^2\, \varepsilon^{2+(1-\sigma)/q} .$$

Thus we get (3.8).

Using the Schwarz inequality and the relation

$$P_n(\cos\theta)^2 + \sum_{m=1}^{n} (2\cdot(n-m)!/(n+m)!) P_n^m(\cos\theta)^2 = 1 ,$$

we see

$$(3.13) \qquad |Y_n(\theta, \varphi)|^2 \leqq a^2_{n,0} + \sum_{m=1}^{n} ((n+m)/!2\cdot(n-m)!)\,(a^2_{n,m}+b^2_{n,m}) .$$

From (3.11), (3.12) and (3.13), we have

$$|v_\varepsilon(x)| \leqq C \operatorname{Max}_{\omega} |\alpha(\omega)| R(\varepsilon, \sigma, r)^{1/2}(\varepsilon/r) ,$$

where

$$R(\varepsilon, \sigma, r) = \sum_{n=0}^{\infty} (\varepsilon/r)^{2n}(n+1)\,(1+(n+1)k\varepsilon^{\sigma-1})^{-2} .$$

Since

$$R(\varepsilon, \sigma, r) \leqq C\, \varepsilon^{2(1-\sigma)} \sum_{n=0}^{\infty} (n+1)^{-1}(\varepsilon/r)^{n+1}$$
$$\leqq C\, \varepsilon^{2(1-\sigma)} \log(r/(r-\varepsilon))$$

and

$$R(\varepsilon, \sigma, r) \leq (\sum_{n=0}^{\infty} (n+1)\,(\varepsilon/r)^{2nq'})^{1/q'}$$
$$\times (\sum_{n=0}^{\infty} (n+1)\,(1+(n+1)k\varepsilon^{\sigma-1})^{-2q})^{1/q}$$

$$\leqq (\sum_{n=0}^{\infty} (n+1)(\varepsilon/r)^n)^{1/q'}$$

$$\times (k^{-1}\varepsilon^{1-\sigma} \sum_{n=0}^{\infty} (1+(n+1)k\varepsilon^{\sigma-1})^{1-2q})^{1/q}$$

$$\leqq (r/(r-\varepsilon))^{2/q'}(k^{-1}\varepsilon^{1-\sigma} \int_0^\infty (1+k\varepsilon^{\sigma-1}t)^{1-2q}dt)^{1/q}$$

$$= (r/(r-\varepsilon))^{2/q'}(2(q-1)k^2\varepsilon^{2(\sigma-1)})^{-1/q}$$

hold, we get (3.6) and (3.7). q.e.d.

Lemma 3.4. *Fix $\beta \in C^\infty(\partial B_\varepsilon)$. Assume that $g \in H^2(\Omega_\varepsilon)$ satisfies*

$$\begin{aligned} \Delta g(x) &= 0 \qquad x \in \Omega_\varepsilon \\ g(x) &= 0 \qquad x \in \partial\Omega \\ g(x) + k\varepsilon^\sigma \frac{\partial g}{\partial \nu_x}(x) &= \beta(x) \qquad x \in \partial B_\varepsilon . \end{aligned} \tag{3.14}$$

Then,

$$\int_{\Omega_\varepsilon} |\nabla g(x)|^2 \, dx \leqq 4\pi k^{-1}\varepsilon^{2-\sigma}(\operatorname*{Max}_{\partial B_\varepsilon} |\beta(x)|)^2 .$$

Proof. Since $g \in H^2(\Omega_\varepsilon)$, we have Green's formula:

$$\int_{\Omega_\varepsilon} (g \cdot \Delta g + |\nabla g|^2) \, dx = \int_{\partial\Omega_\varepsilon} g \frac{\partial g}{\partial \nu_x} \, d\sigma_x .$$

By (3.14), we can see that

$$\int_{\Omega_\varepsilon} |\nabla g|^2 \, dx = \int_{\partial B_\varepsilon} g \frac{\partial g}{\partial \nu_x} \, d\sigma_x = \int_{\partial B_\varepsilon} (\beta(x) - k\varepsilon^\sigma \frac{\partial g}{\partial \nu_x}(x)) \frac{\partial g}{\partial \nu_x}(x) d\sigma_x .$$

Therefore, we have

$$\int_{\Omega_\varepsilon} |\nabla g|^2 \, dx + k\varepsilon^\sigma \int_{\partial B_\varepsilon} |\frac{\partial g}{\partial \nu_x}|^2 \, d\sigma_x = \int_{\partial B_\varepsilon} \beta(x) \frac{\partial g}{\partial \nu_x}(x) d\sigma_x . \tag{3.15}$$

Using the Schwarz inequality, we have

$$k\varepsilon^\sigma \int_{\partial B_\varepsilon} |\frac{\partial g}{\partial \nu_x}|^2 \, d\sigma_x \leqq \int_{\partial B_\varepsilon} \beta(x) \frac{\partial g}{\partial \nu_x}(x) d\sigma_x$$

$$\leqq (\int_{\partial B_\varepsilon} \beta(x)^2 \, d\sigma_x)^{1/2} (\int_{\partial B_\varepsilon} |\frac{\partial g}{\partial \nu_x}|^2 \, d\sigma_x)^{1/2} .$$

Thus,

$$(\int_{\partial B_\varepsilon} |\frac{\partial g}{\partial \nu_x}|^2 \, d\sigma_x)^{1/2} \leqq k^{-1}\varepsilon^{-\sigma} (\int_{\partial B_\varepsilon} \beta(x)^2 d\sigma_x)^{1/2} . \tag{3.16}$$

From (3.15) and (3.16), we get

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla g|^2 dx &\leqq \int_{\partial B_\varepsilon} \beta(x) \frac{\partial g}{\partial \nu_x}(x)\, d\sigma_x \\ &\leqq \Big(\int_{\partial B_\varepsilon} \beta(x)^2 d\sigma_x\Big)^{1/2} \Big(\int_{\partial B_\varepsilon} \Big|\frac{\partial g}{\partial \nu_x}\Big|^2 d\sigma_x\Big)^{1/2} \\ &\leqq k^{-1}\varepsilon^{-\sigma} \int_{\partial B_\varepsilon} \beta(x)^2 d\sigma_x \\ &\leqq 4\pi k^{-1}\varepsilon^{2-\sigma}(\operatorname*{Max}_{\partial B_\varepsilon} |\beta(x)|)^2 .\end{aligned}$$

q.e.d.

Now we are in a position to prove Propositions 3.1 and 3.2.

Proof of Proposition 3.1. Let $u_\varepsilon(x)$ be as in Proposition 3.1. We take an arbitrary $q>\sigma$. Firstly we put $\alpha(\omega)=M(\omega)$ and we take $v_\varepsilon^{(0)}$ so that it satisfies (3.4), (3.5), (3.7) and (3.8). Then $v_\varepsilon^{(0)}$ may not satisfy $v_\varepsilon^{(0)}(x)=0$ for $x\in\partial\Omega$. Let $v_\varepsilon^{(1)}$ be the harmonic function in Ω satisfying $v_\varepsilon^{(1)}(x)=v_\varepsilon^{(0)}(x)$ for $x\in\partial\Omega$. Put

$$M_\varepsilon = \operatorname*{Max}_{\omega} |M(\omega)|.$$

Then from (3.7) we see that $\operatorname{Max}\{|v_\varepsilon^{(1)}(x)|\,;\, x\in\bar\Omega\} \leqq \hat{C}_q\varepsilon^{1-\sigma/q} M_\varepsilon$ and $\operatorname{Max}\{|v_\varepsilon^{(1)}(x) + k\varepsilon^\sigma\partial v_\varepsilon^{(1)}(x)/\partial\nu_x|\,;\, x\in\partial B_\varepsilon\} \leqq \hat{C}_q\varepsilon^{1-\sigma/q} M_\varepsilon$, where $\hat{C}_q$ is a constant independent of ε. Secondly, we put $\alpha(\omega)=v_\varepsilon^{(1)}(x)+k\varepsilon^\sigma\partial v_\varepsilon^{(1)}(x)/\partial\nu_x$ for $x=\tilde{w}+\varepsilon\omega\in\partial B_\varepsilon$ and we take $v_\varepsilon^{(2)}$ so that it satisfies (3.4), (3.5), (3.7) and (3.8). Let $v_\varepsilon^{(3)}$ be the harmonic function in Ω satisfying $v_\varepsilon^{(3)}(x)=v_\varepsilon^{(2)}(x)$ for $x\in\partial\Omega$. Then, $\operatorname{Max}\{|v_\varepsilon^{(3)}(x)|\,;\, x\in\bar\Omega\} \leqq (\hat{C}_q\varepsilon^{1-\sigma/q})^2 M_\varepsilon$ and $\operatorname{Max}\{|v_\varepsilon^{(3)}(x)+k\varepsilon^\sigma\partial v_\varepsilon^{(3)}(x)/\partial\nu_x|\,;\, x\in\partial B_\varepsilon\} \leqq (\hat{C}_q\varepsilon^{1-\sigma/q})^2 M_\varepsilon$.

By repeating this procedure we have

$$\begin{aligned}\Delta v_\varepsilon^{(2n+1)}(x) &= 0 \qquad x\in\Omega \\ v_\varepsilon^{(2n+1)}(x) &= v_\varepsilon^{(2n)}(x) \qquad x\in\partial\Omega\end{aligned}$$

and

$$\Delta v_\varepsilon^{(2n+2)}(x) = 0 \qquad x\in \boldsymbol{R}^3\backslash\bar{B}_\varepsilon$$

$$v_\varepsilon^{(2n+2)}(x)+k\varepsilon^\sigma\frac{\partial v_\varepsilon^{(2n+2)}}{\partial\nu_x}(x) = v_\varepsilon^{(2n+1)}(x)+k\varepsilon^\sigma\frac{\partial v_\varepsilon^{(2n+1)}}{\partial\nu_x}(x) \qquad x\in\partial B_\varepsilon$$

for $n=0, 1, 2, \cdots$.

Then, by induction,

$$\operatorname*{Max}_{\bar\Omega} |v_\varepsilon^{(2n+1)}(x)| \leqq (\hat{C}_q\varepsilon^{1-\sigma/q})^{n+1} M_\varepsilon \tag{3.17}$$

$$\operatorname*{Max}_{\partial B_\varepsilon} \Big|v_\varepsilon^{(2n+1)}(x)+k\varepsilon^\sigma\frac{\partial v_\varepsilon^{(2n+1)}}{\partial\nu_x}(x)\Big| \leqq (\hat{C}_q\varepsilon^{1-\sigma/q})^{n+1} M_\varepsilon \tag{3.18}$$

$$(3.19)\qquad |v_\varepsilon^{(2n)}(x)|\leqq(\hat{C}_q\varepsilon^{1-\sigma/q})^{n+1}M_\varepsilon(r-\varepsilon)^{-1/q'}\qquad (r>\varepsilon)$$

$$(3.20)\qquad \|v_\varepsilon^{(2n)}\|_{2,\varepsilon}\leqq C_q'\varepsilon^{1-(\sigma-1)/(2q)}(\hat{C}_q\varepsilon^{1-\sigma/q})^n M_\varepsilon$$

hold for $n\geqq 0$.

Since $q>\sigma$, we can take ε so that $\hat{C}_q\varepsilon^{1-\sigma/q}<1/2$. We put

$$(3.21)\qquad w_\varepsilon(x)=\sum_{n=0}^\infty(-1)^n v_\varepsilon^{(n)}(x)\,.$$

From (3.17) and (3.19), we can see that the right hand side of (3.21) is uniformly convergent on $\bar{\Omega}\backslash B_\eta$ for any $\eta>\varepsilon$. Since $v_\varepsilon^{(n)}$ is harmonic in Ω_ε, we see that $w_\varepsilon(x)$ is harmonic in Ω_ε, $w_\varepsilon(x)=0$ for $x\in\partial\Omega$ and

$$\frac{\partial w_\varepsilon}{\partial x_j}(x)=\sum_{n=0}^\infty(-1)^n\frac{\partial v_\varepsilon^{(n)}}{\partial x_j}(x)\qquad x\in\Omega_\varepsilon\,,\qquad j=1,2,3\,.$$

We put

$$g_\varepsilon^{(n)}(x)=u_\varepsilon(x)-\sum_{i=0}^{2n+1}(-1)^i v_\varepsilon^{(i)}(x)\,.$$

Then,

$$(3.22)\qquad \nabla g_\varepsilon^{(n)}(x)\to\nabla(u_\varepsilon-w_\varepsilon)(x)\qquad (n\to\infty)\qquad \text{for}\quad x\in\Omega_\varepsilon\,.$$

It is easy to see that $g_\varepsilon^{(n)}$ is harmonic in Ω_ε, $g_\varepsilon^{(n)}(x)=0$ for $x\in\partial\Omega$ and

$$g_\varepsilon^{(n)}(x)+k\varepsilon^\sigma\frac{\partial g_\varepsilon^{(n)}}{\partial\nu_x}(x)=v_\varepsilon^{(2n+1)}(x)+k\varepsilon^\sigma\frac{\partial v_\varepsilon^{(2n+1)}}{\partial\nu_x}(x)\qquad x\in\partial B_\varepsilon\,.$$

Therefore, by Lemma 3.4 and (3.18), we have

$$\begin{aligned}\int_{\Omega_\varepsilon}|\nabla g_\varepsilon^{(n)}|^2\,dx&\leqq 4\pi k^{-1}\varepsilon^{2-\sigma}\operatorname*{Max}_{\partial B_\varepsilon}\Big|v_\varepsilon^{(2n+1)}(x)+k\varepsilon^\sigma\frac{\partial v_\varepsilon^{(2n+1)}}{\partial\nu_x}(x)\Big|\\&\leqq 4\pi k^{-1}\varepsilon^{2-\sigma}(\hat{C}_q\varepsilon^{1-\sigma/q})^n M_\varepsilon\,.\end{aligned}$$

Using Fatou's Lemma and (3.22), we see that

$$\int_{\Omega_\varepsilon}|\nabla(u_\varepsilon-w_\varepsilon)|^2\,dx\leqq\liminf_{n\to\infty}\int_{\Omega_\varepsilon}|\nabla g^{(n)}|^2\,dx\leqq 0\,.$$

Thus, $u_\varepsilon-w_\varepsilon=\text{constant}$ a.e. Ω_ε. Since $u_\varepsilon(x)=w_\varepsilon(x)=0$ for $x\in\partial\Omega$, $u_\varepsilon=w_\varepsilon$ a.e. Ω_ε. Therefore,

$$(3.23)\qquad u_\varepsilon(x)=\sum_{n=0}^\infty(-1)^n v_\varepsilon^{(n)}(x)\qquad x\in\Omega_\varepsilon\,.$$

From (3.17) and (3.20), we have

$$\|\sum_{n=0}^{2n'+1}(-1)^n v_\varepsilon^{(n)}\|_{2,\varepsilon} \leqq \sum_{n=0}^{n'}(\|v_\varepsilon^{(2n)}\|_{2,\varepsilon}+\|v_\varepsilon^{(2n+1)}\|_{2,\varepsilon})$$
$$\leqq \sum_{n=0}^{n'}(C'_q\varepsilon^{1-(\sigma-1)/(2q)}+\hat{C}_q\varepsilon^{1-\sigma/q})(1/2)^n M_\varepsilon$$
$$\leqq C_q\varepsilon^{1-\sigma/q}M_\varepsilon .$$

Using Fatou's Lemma and (3.23), we see that

$$\int_{\Omega_\varepsilon}|u_\varepsilon(x)|^2\,dx \leqq \liminf_{u'\to\infty}\int_{\Omega_\varepsilon}|\sum_{n=0}^{2n'+1}(-1)^n v_\varepsilon^{(u)}(x)|^2\,dx$$
$$\leqq (C_q\varepsilon^{1-\sigma/q}M_\varepsilon)^2 .$$

Thus we get (3.2). q.e.d.

Proof of Proposition 3.2. Let $\{v_\varepsilon^{(n)}(x)\}_{n=0}^\infty$ be the sequence of functions as in the proof of Proposition 3.1. Then, by using (3.6), we can get

$$\text{(3.24)}\qquad \operatorname*{Max}_{\bar{\Omega}}|v_\varepsilon^{(2n+1)}(x)| \leqq (\hat{C}\varepsilon^{2-\sigma})^{n+1}M_\varepsilon$$
$$\operatorname*{Max}_{\partial B_\varepsilon}|v_\varepsilon^{(2n+1)}(x)+k\varepsilon^\sigma\frac{\partial v_\varepsilon^{(2n+1)}}{\partial\nu_x}(x)| \leqq (\hat{C}\varepsilon^{2-\sigma})^{n+1}M_\varepsilon$$

$$\text{(3.25)}\qquad |v_\varepsilon^{(2n)}(x)| \leqq (\hat{C}\varepsilon^{2-\sigma})^{n+1}M_\varepsilon r^{-1}(\log(r/(r-\varepsilon)))^{1/2}$$

for $n\geqq 0$. Here $\hat{C}$ is a constant independent of ε and $M_\varepsilon=\operatorname*{Max}_\omega|M(\omega)|$.

Since $\sigma<2$, we can take ε so that $\hat{C}\varepsilon^{2-\sigma}<1/2$. Then, by the same argument as in the proof of Proposition 3.1, we can see that

$$\text{(3.26)}\qquad u_\varepsilon(x)=\sum_{n=0}^\infty(-1)^n v_\varepsilon^{(n)}(x)\qquad x\in\Omega_\varepsilon .$$

From (3.24), (3.25) and (3.26), we have

$$\text{(3.27)}\qquad |u_\varepsilon(x)| \leqq \sum_{n=0}^\infty(|v_\varepsilon^{(2n)}(x)|+|v_\varepsilon^{(2n+1)}(x)|)$$
$$\leqq C\varepsilon^{2-\sigma}M_\varepsilon r^{-1}(\log(r/(r-\varepsilon)))^{1/2}\qquad (r>\varepsilon) .$$

Now (3.3) easily follows from (3.27). q.e.d.

4. Proof of Theorem 3

From this section to section 7, we assume $\sigma\geqq 1$. By (2.3) we see that

$$\text{(4.1)}\qquad g(\varepsilon)=-4\pi\varepsilon+O(\varepsilon^2+\varepsilon^\sigma)\qquad (\sigma>1)$$
$$=-4\pi(1+k)^{-1}\varepsilon+O(\varepsilon^2)\qquad (\sigma=1) .$$

We take an arbitrary fixed point $x\in\partial B_\varepsilon$. Without loss of generality we may assume $\tilde{w}=0$ and $x=\varepsilon e_1$. Here we put $e_1=(1, 0, 0)$. We put $p_\varepsilon(x, y)$ as before and

$$S(x, y) = G(x, y)-(4\pi)^{-1}|x-y|^{-1}.$$

Then, $S(x, y)\in C^\infty(\Omega\times\Omega)$ and

$$\begin{aligned} &p_\varepsilon(x, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}p_\varepsilon(x, y)_{|x=\varepsilon e_1} \\ &= G(x, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}G(x, y)-g(\varepsilon)k\varepsilon^\sigma\frac{\partial}{\partial x_1}S(x, \tilde{w})G(\tilde{w}, y) \\ &\quad +g(\varepsilon)\left((4\pi)^{-1}\varepsilon^{-1}+S(x, \tilde{w})+k(4\pi)^{-1}\varepsilon^{\sigma-2}\right)G(\tilde{w}, y) \end{aligned}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.
Since $\gamma=S(\tilde{w}, \tilde{w})$, $S(x, \tilde{w})=\gamma+O(\varepsilon)$ as $\varepsilon\to 0$. By (4.1),

$$(4.2)\qquad \begin{aligned} &p_\varepsilon(x, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}p_\varepsilon(x, y)_{|x=\varepsilon e_1} \\ &= G(x, y)-G(\tilde{w}, y)+O(\varepsilon^2)G(\tilde{w}, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}G(x, y) \end{aligned}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.

We take an arbitrary $f\in L^p(\Omega_\varepsilon)$ and let $\tilde{f}$ be the extension of f to Ω defined by 0 on B_ε. Then we have

$$(4.3)\qquad \begin{aligned} &(\boldsymbol{P}_\varepsilon f)(x)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon f)(x)_{|x=\varepsilon e_1} \\ &= (\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})+O(\varepsilon^2)(\boldsymbol{G}\tilde{f})(\tilde{w})-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x) \end{aligned}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.

By the Sobolev embedding theorem and a priori estimate

$$(4.4)\qquad ||\boldsymbol{G}\tilde{f}||_{C^{1+\tau}(\bar{\Omega})}\leqq C||\tilde{f}||_p\leqq C||f||_{p,\varepsilon}$$

hold for $\tau=1-3/p$ $(p>3)$. Therefore we have

$$|(\boldsymbol{P}_\varepsilon f)(x)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon f)(x)|_{x=\varepsilon e_1}\leqq C\varepsilon||f||_{p,\varepsilon}.$$

We put $u_\varepsilon=(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)f$. Then u_ε satisfies (3.1) because $\boldsymbol{G}_\varepsilon f$ satisfies the given Robin condition on ∂B_ε. By Proposition 3.1, we have (2.6).

5. Convergence of eigenvalues for $\sigma\geqq 1$

We put $\tilde{p}_\varepsilon(x, y)$, $\tilde{\boldsymbol{P}}_\varepsilon$ as before. Then,

$$\tilde{\boldsymbol{P}}_\varepsilon = A_0 + g(\varepsilon) A_1 , \tag{5.1}$$

where $A_0 = \boldsymbol{G}$ and

$$(A_1 f)(x) = G(x, \tilde{w}) \chi_\varepsilon(x) (\boldsymbol{G} \chi_\varepsilon f)(\tilde{w}) . \tag{5.2}$$

Since

$$|(A_1 f)(x)| \leqq C |x - \tilde{w}|^{-1} \chi_\varepsilon(x) ||f||_p \qquad (p > 3/2) ,$$

we have

$$\begin{aligned} ||A_1 f||_p &\leqq C ||f||_p \qquad (3/2 < p < 3) \\ &\leqq C \varepsilon^{3/p-1} ||f||_p \qquad (p > 3) . \end{aligned} \tag{5.3}$$

From (4.1), (5.1) and (5.3) we have

$$||(\tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{G}) f||_2 \leqq |g(\varepsilon)| \; ||A_1 f||_2 \leqq C \varepsilon ||f||_2$$

for any $f \in L^2(\Omega)$. Therefore we get the following.

Lemma 5.1. *There exists a constant C independent of ε such that*

$$||\tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{G}||_2 \leqq C \varepsilon \tag{5.4}$$

holds.

Next we want to estimate $||\chi_\varepsilon \tilde{\boldsymbol{P}}_\varepsilon \chi_\varepsilon - \tilde{\boldsymbol{P}}_\varepsilon||_2$. It does not exceed

$$||(1 - \chi_\varepsilon) \tilde{\boldsymbol{P}}_\varepsilon \chi_\varepsilon||_2 + ||\tilde{\boldsymbol{P}}_\varepsilon (1 - \chi_\varepsilon)||_2 . \tag{5.5}$$

Notice that $(1 - \chi_\varepsilon) \chi_\varepsilon = 0$ in $g(\varepsilon)$ term. By (5.1),

$$||(1 - \chi_\varepsilon) \tilde{\boldsymbol{P}}_\varepsilon v||_2 \leqq C |B_\varepsilon|^{1/2} ||\boldsymbol{G} v||_2 \leqq C \varepsilon^{3/2} ||v||_2$$

hold for any $v \in L^2(\Omega)$. Therefore we get

$$\begin{aligned} &||(1 - \chi_\varepsilon) \tilde{\boldsymbol{P}}_\varepsilon||_2 \leqq C \varepsilon^{3/2} \\ &||(1 - \chi_\varepsilon) \tilde{\boldsymbol{P}}_\varepsilon \chi_\varepsilon||_2 \leqq C \varepsilon^{3/2} . \end{aligned} \tag{5.6}$$

Since we have the duality

$$((1 - \chi_\varepsilon) \tilde{\boldsymbol{P}}_\varepsilon)^* = \tilde{\boldsymbol{P}}_\varepsilon (1 - \chi_\varepsilon) ,$$

we get

$$||\tilde{\boldsymbol{P}}_\varepsilon (1 - \chi_\varepsilon)||_2 \leqq C \varepsilon^{3/2} . \tag{5.7}$$

Summing up these facts, we get the following.

Lemma 5.2. *There exists a constant C independent of ε such that*

$$\|\chi_\varepsilon \tilde{P}_\varepsilon \chi_\varepsilon - \tilde{P}_\varepsilon\|_2 \leqq C\varepsilon^{3/2}$$

holds.

Notice that the j-the eigenvalue of $\boldsymbol{P}_\varepsilon$ is equal to the j-th eigenvalue of $\chi_\varepsilon \tilde{\boldsymbol{P}}_\varepsilon \chi_\varepsilon$. By virtue of Theorem 3, Lemmas 5.1 and 5.2, we see that there exists a constant C independent of ε such that

$$(5.8) \qquad |\mu_j(\varepsilon)^{-1} - \mu_j^{-1}| \leqq C(\varepsilon^{2-s} + \varepsilon + \varepsilon^{3/2}) \leqq C \cdot \varepsilon$$

hold.

For the later convenience we estimate $\|(\chi_\varepsilon \tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{P}_\varepsilon \chi_\varepsilon) f\|_{2,\varepsilon}$. We put $v_\varepsilon = (\chi_\varepsilon \tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{P}_\varepsilon \chi_\varepsilon) f$. Then, $v_\varepsilon = (\boldsymbol{G}\hat{\chi}_\varepsilon f)$ on Ω_ε. Here $\hat{\chi}_\varepsilon$ is the characteristic function on $\boldsymbol{B}_\varepsilon$. Thus v_ε satisfies $\Delta v_\varepsilon(x) = 0$ for $x \in \Omega_\varepsilon$, $v_\varepsilon(x) = 0$ for $x \in \partial\Omega$ and

$$(5.9) \qquad |v_\varepsilon(x)| \leqq C\left(\int_{B_\varepsilon} |x-y|^{-p'} dy\right)^{1/p'} \|f\|_p \leqq \begin{cases} C\varepsilon^{2-3/p}\|f\|_p & (3/2 < p < \infty) \\ C\varepsilon^2 \|f\|_\infty & (p = \infty) \end{cases}$$

for $x \in \partial \boldsymbol{B}_\varepsilon$.

By the maximum principle, we get the following.

Lemma 5.3. *There exists a constant C independent of ε such that*

$$(5.10) \qquad \begin{aligned} \|(\chi_\varepsilon \tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{P}_\varepsilon \chi_\varepsilon) f\|_{2,\varepsilon} &\leqq C\varepsilon^{2-3/p}\|f\|_p \qquad (3/2 < p < \infty) \\ &\leqq C\varepsilon^2 \|f\|_\infty \qquad (p = \infty) \end{aligned}$$

hold for any $f \in L^p(\Omega)$.

6. Perturbational calculus for $\tilde{\boldsymbol{P}}_\varepsilon$

In this section we consider the behaviour of eigenvalues of $\tilde{\boldsymbol{P}}_\varepsilon$ as ε tends to 0. We put

$$\lambda(\varepsilon) = \lambda_0 + g(\varepsilon)\lambda_1$$
$$\psi(\varepsilon) = \psi_0 + g(\varepsilon)\psi_1$$

so that $\lambda(\varepsilon)$ and $\psi(\varepsilon)$ is an approximate eigenvalue of $\tilde{\boldsymbol{P}}_\varepsilon$ and an approximate eigenfunction of $\tilde{\boldsymbol{P}}_\varepsilon$, respectively.

Let λ_0 be a simple eigenvalue of A_0 and ψ_0 be a solution of

$$(6.1) \qquad (A_0 - \lambda_0)\psi_0 = 0\,, \qquad \|\psi_0\|_2 = 1\,.$$

Next we solve the following equations:

$$(6.2) \qquad (A_0 - \lambda_0)\psi_1 = (\lambda_1 - A_1)\psi_0$$

$$(6.3) \qquad (\psi_0, \psi_1)_2 = 0 ,$$

where $(\ ,\ )_2$ denotes the inner product on $L^2(\Omega)$.

By the Fredholm alternative theory, we see that

$$(6.4) \qquad \lambda_1 = (A_1\psi_0, \psi_0)_2$$

is the condition such that the unique solution of ψ_1 of (6.2) and (6.3) exists.

Hereafter we put $\lambda_0 = \mu_j^{-1}$. Then $\psi_0 = \varphi_j$. It is easy to see

$$(6.5) \qquad \lambda_1 = |(G\chi_\varepsilon\varphi_j)(\tilde{w})|^2 = \mu_j^{-2}\varphi_j(\tilde{w})^2 + O(\varepsilon^2)$$

$$(6.6) \qquad (\tilde{\boldsymbol{P}}_\varepsilon - \lambda(\varepsilon))\psi(\varepsilon) = g(\varepsilon)^2(A_1 - \lambda_1)\psi_1 .$$

From (5.3), (6.2), (6.4) and (6.6), we have the following.

Lemma 6.1. *There exists a constant C independent of ε such that*

$$(6.7) \qquad \|(\tilde{\boldsymbol{P}}_\varepsilon - \lambda(\varepsilon))\psi(\varepsilon)\|_{2,\varepsilon} \leqq Cg(\varepsilon)^2 \leqq C\varepsilon^2$$

hold.

By (5.3), (6.2) and (6.4), we have

$$(6.8) \qquad \|\psi_1\|_p,\ \|A_1\|_p \leqq C\varepsilon^{3/p-1} \qquad (p>3) .$$

Now we have the following.

Lemma 6.2. *Fix $s \in (0, 1)$. Then, there exist constants C, C_s independent of ε such that*

$$(6.9) \qquad \|(\boldsymbol{P}_\varepsilon - \boldsymbol{G}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon} \leqq C_s\varepsilon^{2-s}$$

$$(6.10) \qquad \|(\chi_\varepsilon\tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{P}_\varepsilon\chi_\varepsilon)\psi(\varepsilon)\|_{2,\varepsilon} \leqq C\varepsilon^2$$

hold.

Proof. By (6.8), Theorem 3 and Lemma 5.3, we have

$$\begin{aligned}&\|(\boldsymbol{P}_\varepsilon - \boldsymbol{G}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon} \\ &\leqq C_s\varepsilon^{2-s}(1 + |g(\varepsilon)|\varepsilon^{3/p-1}) \leqq C_s\varepsilon^{2-s} \qquad (p>3)\end{aligned}$$

and

$$\begin{aligned}&\|(\chi_\varepsilon\tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{P}_\varepsilon\chi_\varepsilon)\psi(\varepsilon)\|_{2,\varepsilon} \\ &\leqq C(\varepsilon^2 + |g(\varepsilon)|\varepsilon^{2-3/p}\varepsilon^{3/p-1}) \leqq C\varepsilon^2 \qquad (p>3) .\end{aligned}$$

q.e.d.

7. Proof of Theorem 1

Now we are in a position to prove Theorem 1. We fix $s \in (0, 1)$. Then, by Lemmas 6.1 and 6.2, we have

$$\begin{aligned}
&\|(\boldsymbol{G}_\varepsilon-\lambda(\varepsilon))(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}\\
\leqq&\|(\boldsymbol{G}_\varepsilon-\boldsymbol{P}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}+\|(\boldsymbol{P}_\varepsilon\chi_\varepsilon-\chi_\varepsilon\tilde{\boldsymbol{P}}_\varepsilon)\psi(\varepsilon)\|_{2,\varepsilon}\\
&\qquad+\|\chi_\varepsilon(\tilde{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\|_{2,\varepsilon}\\
\leqq&C_s\varepsilon^{2-s}.
\end{aligned}$$

Since $\|\psi(\varepsilon)\|_{2,\varepsilon}\in(1/2, 2)$ for small ε, there exists at least one eigenvalue $\lambda^*(\varepsilon)$ of $\boldsymbol{G}_\varepsilon$ satisfying

$$|\lambda^*(\varepsilon)-\lambda(\varepsilon)|\leqq C_s\varepsilon^{2-s}. \tag{7.1}$$

We here represent $\lambda(\varepsilon)$ explicitly as follows:

$$\begin{aligned}
\lambda(\varepsilon) &= \mu_j^{-1}+g(\varepsilon)(\mu_j^{-2}\varphi_j(\tilde{w})^2+0(\varepsilon^2))\\
&=\begin{cases}\mu_j^{-1}-4\pi\mu_j^{-2}\varphi_j(\tilde{w})^2\varepsilon+0(\varepsilon^2+\varepsilon^\sigma) & (\sigma>1)\\ \mu_j^{-1}-4\pi(1+k)^{-1}\mu_j^{-2}\varphi_j(\tilde{w})^2\varepsilon+0(\varepsilon^2) & (\sigma=1)\end{cases}
\end{aligned}\tag{7.2}$$

By (7.1), (7.2) and the fact (5.8), we see that $\lambda^*(\varepsilon)$ must be $\mu_j(\varepsilon)^{-1}$. Then, (1.3) easily follows from (7.1) and (7.2). Therefore we get the desired Theorem 1.

8. Proof of Theorem 4

From this section we assume $\sigma<1$. By (2.3), (2.4) and (2.5), we see that

$$\begin{aligned}
g(\varepsilon) &= -(4\pi/k)\varepsilon^{2-\sigma}+O(\varepsilon^{3-2\sigma})\\
h(\varepsilon) &= 2\pi\varepsilon^3+O(\varepsilon^{4-\sigma})\\
i(\varepsilon) &= (4\pi/9)\varepsilon^5+O(\varepsilon^{6-\sigma}).
\end{aligned}\tag{8.1}$$

We take an arbitrary $x\in\partial B_\varepsilon$. Without loss of generality we may assume that $\tilde{w}=0$ and $x=\varepsilon e_1$. We put $S(x, y)$ as before. Then, the same calculation as in p. 263 of Ozawa [9] yields

$$\begin{aligned}
&\langle\nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\\
&= (4\pi)^{-1}\varepsilon^{-2}\frac{\partial}{\partial w_1}G(\tilde{w}, y)+\langle\nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle
\end{aligned}\tag{8.2}$$

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\langle\nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\\
&= -(2\pi)^{-1}\varepsilon^{-3}\frac{\partial}{\partial w_1}G(\tilde{w}, y)+\frac{\partial}{\partial x_1}\langle\nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle
\end{aligned}\tag{8.3}$$

$$\begin{aligned}
&\langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle-\langle H_w S(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle\\
&= 3(4\pi)^{-1}\varepsilon^{-3}\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y)-(4\pi)^{-1}\varepsilon^{-3}\Delta_w G(\tilde{w}, y)
\end{aligned}\tag{8.4}$$

(8.5)
$$\frac{\partial}{\partial x_1}\langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle - \frac{\partial}{\partial x_1}\langle H_w S(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle$$
$$= -9(4\pi)^{-1}\varepsilon^{-4}\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y) + 3(4\pi)^{-1}\varepsilon^{-4}\Delta_w G(\tilde{w}, y)$$

for $x=\varepsilon e_1$, $\tilde{w}=0$. We recall that

(8.6)
$$\Delta_w G(\tilde{w}, y) = 0 \qquad \text{for} \quad y\in\Omega_\varepsilon .$$

We put $p_\varepsilon(x, y)$ as before. By (8.2), (8.3), (8.4), (8.5) and (8.6), we have

(8.7)
$$p_\varepsilon(x, y) - k\varepsilon^\sigma \frac{\partial}{\partial x_1} p_\varepsilon(x, y)_{|x=\varepsilon e_1} = \sum_{j=1}^{7} L_j ,$$

where

$$L_1 = G(x, y)$$
$$L_2 = g(\varepsilon)\,((4\pi\varepsilon)^{-1} + \gamma + (4\pi)^{-1}k\varepsilon^{\sigma-2})G(\tilde{w}, y)$$
$$L_3 = g(\varepsilon)O(\varepsilon^\sigma)G(\tilde{w}, y)$$
$$L_4 = (4\pi)^{-1}(\varepsilon^{-2} + 2k\varepsilon^{\sigma-3})h(\varepsilon)\frac{\partial}{\partial w_1}G(\tilde{w}, y) - k\varepsilon^\sigma\frac{\partial}{\partial x_1}G(x, y)$$
$$L_5 = 3(4\pi)^{-1}(\varepsilon^{-3} + 3k\varepsilon^{\sigma-4})i(\varepsilon)\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y)$$
$$L_6 = h(\varepsilon)\,\langle\nabla_w S(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle - k\varepsilon^\sigma h(\varepsilon)\frac{\partial}{\partial x_1}\langle\nabla_w S(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle$$
$$L_7 = i(\varepsilon)\,\langle H_w S(x, \tilde{w}),\ H_w G(\tilde{w}, y)\rangle - k\varepsilon^\sigma i(\varepsilon)\frac{\partial}{\partial x_1}\langle H_w S(x, \tilde{w}),\ H_w G(\tilde{w}, y)\rangle$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.
Here we used the fact that

$$S(x, \tilde{w}) = \gamma + O(\varepsilon) \qquad \text{as} \quad \varepsilon\to 0 .$$

By (2.3), (2.4), (2.5) and (8.6), we get the following.

(8.8)
$$p_\varepsilon(x, y) - k\varepsilon^\sigma\frac{\partial}{\partial x_1}p_\varepsilon(x, y)_{|x=\varepsilon e_1}$$
$$= G(x, y) - G(\tilde{w}, y) - \varepsilon\frac{\partial}{\partial w_1}G(\tilde{w}, y) + L_3 + L_6 + L_7$$
$$-k\varepsilon^\sigma\Big(\frac{\partial}{\partial x_1}G(x, y) - \frac{\partial}{\partial w_1}G(\tilde{w}, y) - \varepsilon\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y)\Big)$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.

We take an arbitrary $f\in L^p(\Omega_\varepsilon)$ and let $\tilde{f}$ be the extension of f to Ω defined by 0 on B_ε. By (8.8),

$$(8.9)\qquad (\boldsymbol{P}_\varepsilon f)(x)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon f)(x)_{|x=\varepsilon e_1}$$
$$=(\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})-\varepsilon\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})+I_0(\varepsilon,\tilde{f})$$
$$-k\varepsilon^\sigma\Big(\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x)-\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})-\varepsilon\frac{\partial^2}{\partial w_1^2}(\boldsymbol{G}\tilde{f})(\tilde{w})\Big),$$

where

$$I_0(\varepsilon,\tilde{f})=g(\varepsilon)O(\varepsilon^\sigma)(\boldsymbol{G}\tilde{f})(\tilde{w})$$
$$+h(\varepsilon)\langle\nabla_w S(x,\tilde{w}),\nabla_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle$$
$$-k\varepsilon^\sigma h(\varepsilon)\frac{\partial}{\partial x_1}\langle\nabla_w S(x,\tilde{w}),\nabla_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle$$
$$+i(\varepsilon)\langle H_w S(x,\tilde{w}),H_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle$$
$$-k\varepsilon^\sigma i(\varepsilon)\frac{\partial}{\partial x_1}\langle H_w S(x,\tilde{w}),H_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.

By (4.4), we have

$$(8.10)\qquad |(\boldsymbol{G}\tilde{f})(\tilde{w})|\leqq C\|f\|_{p,\varepsilon}$$
$$|(\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})-\varepsilon\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})|\leqq C\varepsilon^{2-3/p}\|f\|_{p,\varepsilon}$$
$$\left|\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x)-\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})\right|\leqq C\varepsilon^{1-3/p}\|f\|_{p,\varepsilon}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$, $p>3$.

Furthermore,

$$(8.11)\qquad \left|\frac{\partial}{\partial w_n}(\boldsymbol{G}\tilde{f})(\tilde{w})\right|\leqq C\Big(\int_{\Omega_\varepsilon}|y-\tilde{w}|^{-2p'}dy\Big)^{1/p'}\|\tilde{f}\|_p$$
$$\leqq\begin{cases}C\varepsilon^{1-3/p}\|f\|_{p,\varepsilon} & (1<p<3)\\ C\|f\|_{p,\varepsilon} & (p>3)\end{cases}$$

for $1\leqq n\leqq 3$, where p' satisfies $(1/p)+(1/p')=1$. Also,

$$(8.12)\qquad \left|\frac{\partial^2}{\partial w_m\partial w_n}(\boldsymbol{G}\tilde{f})(\tilde{w})\right|\leqq C\Big(\int_{\Omega_\varepsilon}|y-\tilde{w}|^{-3p'}dy\Big)^{1/p'}\|\tilde{f}\|_p$$
$$\leqq C\varepsilon^{-3/p}\|f\|_{p,\varepsilon}\qquad (p>1)$$

for $1 \leqq m, n \leqq 3$.

Summing up these facts, we get

$$|(\boldsymbol{P}_\varepsilon f)(x) - k\varepsilon^\sigma \frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon f)(x)|_{x=\varepsilon e_1} \leqq C\varepsilon^{1+\sigma-3/p}||f||_{p,\varepsilon}$$

for $p>3$.

Therefore we have the following by Proposition 3.2.

Lemma 8.1. *For a constant C independent of* ε,

$$||(\boldsymbol{P}_\varepsilon - \boldsymbol{G}_\varepsilon)f||_{2,\varepsilon} \leqq C\varepsilon^{3-3/p}||f||_{p,\varepsilon} \tag{8.13}$$

holds for any $f \in L^p(\Omega_\varepsilon)$ $(p>3)$.

The right hand side of (8.13) is not $O(\varepsilon^3)$. On the other hand, the right hand side of (2.8) is $o(\varepsilon^3)$. Therefore we need some sharper estimate to get Theorem 4.

We put $v_\varepsilon(x) = ((\boldsymbol{P}_\varepsilon - \boldsymbol{G}_\varepsilon)(\chi_\varepsilon \varphi_j))(x)$. As we get (8.9),

$$\begin{aligned} &v_\varepsilon(x) - k\varepsilon^\sigma \frac{\partial}{\partial x_1} v_\varepsilon(x)|_{x=\varepsilon e_1} \\ &= I_1(\varepsilon) - I_2(\varepsilon) - k\varepsilon^\sigma(I_3(\varepsilon) - I_4(\varepsilon)) + I_5(\varepsilon), \end{aligned} \tag{8.14}$$

where

$$\begin{aligned} I_1(\varepsilon) &= (\boldsymbol{G}\varphi_j)(x) - (\boldsymbol{G}\varphi_j)(\tilde{w}) - \varepsilon \frac{\partial}{\partial w_1}(\boldsymbol{G}\varphi_j)(\tilde{w}) \\ I_2(\varepsilon) &= (\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(x) - (\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(\tilde{w}) - \varepsilon \frac{\partial}{\partial w_1}(\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(\tilde{w}) \\ I_3(\varepsilon) &= \frac{\partial}{\partial x_1}(\boldsymbol{G}\varphi_j)(x) - \left(\frac{\partial}{\partial w_1} + \varepsilon \frac{\partial^2}{\partial w_1^2}\right)(\boldsymbol{G}\varphi_j)(\tilde{w}) \\ I_4(\varepsilon) &= \frac{\partial}{\partial x_1}(\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(x) - \left(\frac{\partial}{\partial w_1} + \varepsilon \frac{\partial^2}{\partial w_1^2}\right)(\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(\tilde{w}) \end{aligned}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$, and $I_5(\varepsilon)$ is given by replacing f with $\chi_\varepsilon \varphi_j$ in the term $I_0(\varepsilon, \tilde{f})$ of (8.9).

Since $\boldsymbol{G}\varphi_j = \mu_j^{-1}\varphi_j$,

$$|I_1(\varepsilon)| \leqq C\varepsilon^2, \ |I_3(\varepsilon)| \leqq C\varepsilon^2. \tag{8.15}$$

Using (8.11), (8.12) with $f = \chi_\varepsilon \varphi_j$, we have

$$|I_5(\varepsilon)| \leqq C(\varepsilon^2 + \varepsilon^{3+\sigma}). \tag{8.16}$$

Furthermore,

$$(8.17) \qquad |I_2(\varepsilon)| \leqq C\varepsilon^{2-3/p} \|\hat{\chi}_\varepsilon \varphi_j\|_p \qquad (p>3)$$
$$\leqq C\varepsilon^{2-3/p} |B_\varepsilon|^{1/p} \leqq C\varepsilon^2 .$$

Now we want to estimate $I_4(\varepsilon)$. We put $L(x, y)=(4\pi)^{-1}|x-y|^{-1}$. Then, we have

$$(8.18) \qquad I_4(\varepsilon) = I_6(\varepsilon)+I_7(\varepsilon)+I_8(\varepsilon) ,$$

where

$$I_6(\varepsilon) = \frac{\partial}{\partial x_1}\Big(\int_{B_\varepsilon} L(x, y)(\varphi_j(y)-\varphi_j(x))dy\Big)_{|x=\varepsilon e_1}$$
$$-\frac{\partial}{\partial w_1}\int_{B_\varepsilon} L(w, y)(\varphi_j(y)-\varphi_j(w))dy$$
$$-\varepsilon\frac{\partial^2}{\partial w_1^2}\int_{B_\varepsilon} L(w, y)\Big(\varphi_j(y)-\varphi_j(w)-\sum_{n=1}^{3}(y_n-w_n)\frac{\partial\varphi_j}{\partial w_n}(w)\Big)dy$$
$$I_7(\varepsilon) = \frac{\partial}{\partial x_1}(\varphi_j(x)F(x))_{|x=\varepsilon e_1}-\Big(\frac{\partial}{\partial w_1}+\varepsilon\frac{\partial^2}{\partial w_1^2}\Big)(\varphi_j(w)F(w))$$
$$-\varepsilon\sum_{n=1}^{3}\frac{\partial^2}{\partial w_1^2}\Big(\frac{\partial\varphi_j}{\partial w_n}(w)K_n(w)\Big)$$
$$I_8(\varepsilon) = \frac{\partial}{\partial x_1}(\boldsymbol{S}\hat{\chi}_\varepsilon\varphi_j)(x)_{|x=\varepsilon e_1}-\Big(\frac{\partial}{\partial w_1}+\varepsilon\frac{\partial^2}{\partial w_1^2}\Big)(\boldsymbol{S}\hat{\chi}_\varepsilon\varphi_j)(w)$$

for $w=0$. Here we put operator $\boldsymbol{S}$ and functions F, K_n as follows:

$$(\boldsymbol{S}f)(x) = \int_\Omega S(x, y)f(y)dy$$
$$F(x) = \int_{B_\varepsilon} L(x, y)dy$$

and

$$K_n(w) = \int_{B_\varepsilon} L(w, y)(y_n-w_n)dy \qquad (n=1, 2, 3) .$$

It is easy to see that

$$(8.19) \qquad |I_8(\varepsilon)| \leqq C\varepsilon^2$$

$$(8.20) \qquad |I_6(\varepsilon)| \leqq C\int_{B_\varepsilon} |x-y|^{-1}dy_{|x=\varepsilon e_1}+C\int_{B_\varepsilon} |\tilde{w}-y|^{-1}dy$$
$$+C\varepsilon\int_{B_\varepsilon} |\tilde{w}-y|^{-1}dy$$
$$\leqq C\varepsilon^2 .$$

The simple calculation yields

$$\begin{aligned}(8.21)\qquad F(x) &= (\varepsilon^3/3)|x|^{-1} \qquad \text{for} \quad x\in \boldsymbol{R}^3\backslash\bar{\boldsymbol{B}}_\varepsilon \\ &= \varepsilon^2/2-|x|^2/6 \qquad \text{for} \quad x\in \boldsymbol{B}_\varepsilon\,,\end{aligned}$$

$$(8.22)\qquad K_n(w) = w_n(|w|^2/5-\varepsilon^2)/3 \qquad \text{for} \quad w\in \boldsymbol{B}_\varepsilon\,.$$

Therefore, we see that

$$(8.23)\qquad F(x) = \varepsilon^2/3\,, \quad \partial F(x)/\partial x_1 = -\varepsilon/3$$

for $x=\varepsilon e_1$, and

$$(8.24)\qquad F(w) = \varepsilon^2/2\,, \quad \partial F(w)/\partial w_1 = 0\,, \quad \partial^2 F(w)/\partial w_1^2 = -1/3$$

$$(8.25)\qquad K_n(w) = \partial^2 K_n(w)/\partial w_1^2 = 0\,, \quad \partial K_n(w)/\partial w_1 = -\delta_{1,n}(\varepsilon^2/3)$$

for $w=0$, where $\delta_{1,n}$ is the Kronecker delta. Summing up these facts, we have

$$(8.26)\qquad I_7(\varepsilon) = -\varepsilon(\varphi_j(\varepsilon e_1)-\varphi_j(0))/3+O(\varepsilon^2) = O(\varepsilon^2)\,.$$

From (8.14), (8.15), (8.16), (8.17), (8.18), (8.19), (8.20) and (8.26), we see that

$$|v_\varepsilon(x)-k\varepsilon^\sigma \frac{\partial}{\partial x_1} v_\varepsilon(x)|_{x=\varepsilon e_1} \leqq C(\varepsilon^2+\varepsilon^{2+\sigma})\,.$$

By Proposition 3.2, we have

$$||v_\varepsilon||_{2,\varepsilon} \leqq C\varepsilon^4(1+\varepsilon^{-\sigma})\,.$$

Therefore, we get the desired Theorem 4.

9. Convergence of eigenvalues for $\sigma<1$

We put A_0, A_1 as before. Then,

$$(9.1)\qquad \tilde{\boldsymbol{P}}_\varepsilon = A_0+g(\varepsilon)A_1+h(\varepsilon)A_2+i(\varepsilon)A_3\,,$$

where

$$(9.2)\qquad (A_2 f)(x) = \langle \nabla_w G(x, \tilde{w}), \nabla_w(G\chi_\varepsilon f)(\tilde{w})\rangle \chi_\varepsilon(x)$$

$$(9.3)\qquad (A_3 f)(x) = \langle H_w G(x, \tilde{w}), H_w(G\chi_\varepsilon f)(\tilde{w})\rangle \chi_\varepsilon(x)\,.$$

Using (8.11) and (8.12), we have

$$\begin{aligned}(9.4)\qquad ||A_2 f||_p &\leqq C\Big(\int_{\Omega_\varepsilon} |x-\tilde{w}|^{-2p}dx\Big)^{1/p}||\nabla_w(G\chi_\varepsilon f)||_\infty \\ &\leqq \begin{cases} C\varepsilon^{-1}||f||_p & (3/2<p<3) \\ C\varepsilon^{3/p-2}||f|_p & (p>3)\,,\end{cases}\end{aligned}$$

and

$$(9.5) \qquad \|A_3 f\|_p \leqq C\left(\int_{\Omega_\varepsilon} |x-\tilde{w}|^{-3p} dx\right)^{1/p} \|H_w(G\chi_\varepsilon f)\|_\infty$$
$$\leqq C\varepsilon^{-3}\|f\|_p \qquad (p>1).$$

Here we put

$$(H_w v)(w) = \sum_{m,n=1}^{3} \frac{\partial^2 v}{\partial w_m \partial w_n}(w).$$

From (5.3), (8.1), (9.1), (9.4) and (9.5),

$$\|(\tilde{P}_\varepsilon - G)f\|_2 \leqq C(|g(\varepsilon)| + |h(\varepsilon)|\varepsilon^{-1} + |i(\varepsilon)|\varepsilon^{-3})\|f\|_2$$
$$\leqq C(\varepsilon^2 + \varepsilon^{2-\sigma})\|f\|_2$$

hold for any $f \in L^2(\Omega)$.

Therefore we get the following.

Lemma 9.1. *There exists a constant C independent of ε such that*

$$(9.6) \qquad \|\tilde{P}_\varepsilon - G\|_2 \leqq C(\varepsilon^2 + \varepsilon^{2-\sigma})$$

holds.

Notice that Lemma 5.2 is valid for $\sigma<1$ because $(1-\chi_\varepsilon)\chi_\varepsilon=0$. As we get (5.8),

$$(9.7) \qquad |\mu_j(\varepsilon)^{-1} - \mu_j^{-1}| \leqq C(\varepsilon^{3-3/p} + \varepsilon^2 + \varepsilon^{2-\sigma} + \varepsilon^{3/2}) \leqq C(\varepsilon^{3/2} + \varepsilon^{2-\sigma})$$

hold for a constant C independent of ε.

10. Perturbational calculus for $\bar{P}_\varepsilon$

We recall (2.9). Then,

$$(10.1) \qquad \bar{P}_\varepsilon = A_0 + \bar{g}(\varepsilon)\bar{A}_1 + h(\varepsilon)\bar{A}_2 + i(\varepsilon)\bar{A}_3,$$

where

$$(10.2) \qquad \bar{g}(\varepsilon) = g(\varepsilon) - (4\pi/3)\mu_j \varepsilon^3$$

and $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ is given by replacing χ_ε with ξ_ε in (5.2), (9.2), (9.3), respectively.

Furthermore we put $\lambda_0 = \mu_j^{-1}$, $\psi_0 = \varphi_j$ and

$$\lambda(\varepsilon) = \lambda_0 + \bar{g}(\varepsilon)\lambda_1 + h(\varepsilon)\lambda_2 + i(\varepsilon)\lambda_3$$
$$\psi(\varepsilon) = \psi_0 + \bar{g}(\varepsilon)\psi_1 + h(\varepsilon)\psi_2 + i(\varepsilon)\psi_3.$$

Then,

(10.3) $$(A_0-\lambda_0)\psi_0 = 0\,, \quad ||\psi_0||_2 = 1\,.$$

Next we consider the following equations:

(10.4) $$(A_0-\lambda_0)\psi_n = (\lambda_n-\bar{A}_n)\psi_0, \quad (\psi_0, \psi_n)_2 = 0 \qquad (n=1, 2, 3)\,.$$

By the Fredholm alternative theory, we see that

(10.5) $$\lambda_n = (\bar{A}_n\psi_0, \psi_0)_2 \qquad (n=1, 2, 3)$$

is the condition such that the unique solution ψ_n of (10.4) exists.

Since $\xi_\varepsilon=0$ on $B_{\varepsilon/2}$, $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ satisfies the same inequality as in (5.3), (9.4), (9.5), respectively. Then, by the Fredholm theory and the estimate of the $L^p(\Omega)$ norm of the right hand side of (10.4), we get the following.

Lemma 10.1. *For a constant C independent of* ε,

$$\begin{aligned} ||\psi_1||_p, ||\bar{A}_1||_p &\leqq C \qquad (3/2<p<3)\\ &\leqq C\varepsilon^{3/p-1} \qquad (p>3)\\ ||\psi_2||_p, ||\bar{A}_2||_p &\leqq C\varepsilon^{-1} \quad (3/2<p<3)\\ &\leqq C\varepsilon^{3/p-2} \qquad (p>3)\\ ||\psi_3||_p, ||\bar{A}_3||_p &\leqq C\varepsilon^{-3} \qquad (p>1) \end{aligned}$$

hold.

In view of (10.1), (10.3) and (10.4), we have

(10.6) $$\begin{aligned} &(\bar{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\\ &= g(\varepsilon)^2(\bar{A}_1-\lambda_1)\psi_1+h(\varepsilon)^2(\bar{A}_2-\lambda_2)\psi_2+i(\varepsilon)^2(\bar{A}_3-\lambda_3)\psi_3\\ &\quad +g(\varepsilon)h(\varepsilon)\left((\bar{A}_1-\lambda_1)\psi_2+(\bar{A}_2-\lambda_2)\psi_1\right)\\ &\quad +h(\varepsilon)i(\varepsilon)\left((\bar{A}_2-\lambda_2)\psi_3+(\bar{A}_3-\lambda_3)\psi_2\right)\\ &\quad +i(\varepsilon)g(\varepsilon)\left((\bar{A}_3-\lambda_3)\psi_1+(\bar{A}_1-\lambda_1)\psi_3\right)\,. \end{aligned}$$

By (10.5), (10.6) and Lemma 10.1, we see that

$$||(\bar{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)||_2 \leqq C(g(\varepsilon)^2+\varepsilon^4) \leqq C\varepsilon^4(1+\varepsilon^{-2\sigma})\,.$$

Therefore we get the following.

Lemma 10.2. *For a constant C independent of* ε,

(10.7) $$||(\bar{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)||_2 \leqq C\varepsilon^4(1+\varepsilon^{-2\sigma})$$

holds.

On the other hand, by Lemmas 8.1, 10.1 and Theorem 4, we see that

$$\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}$$
$$\leqq C(\varepsilon^4(1+\varepsilon^{-\sigma})+|g(\varepsilon)|\varepsilon^2+|h(\varepsilon)|\varepsilon+|i(\varepsilon)|\varepsilon^{-3/p}) \qquad (p>3)$$
$$\leqq C\varepsilon^4(1+\varepsilon^{-\sigma}).$$

Therefore, we get the following.

Lemma 10.3. *For a constant C independent of* ε,

$$\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}\leqq C\varepsilon^4(1+\varepsilon^{-\sigma}) \tag{10.8}$$

holds.

11. Proof of Theorem 5

We put

$$J_\varepsilon(x;v)=(\chi_\varepsilon\bar{\boldsymbol{P}}_\varepsilon v-\boldsymbol{P}_\varepsilon\chi_\varepsilon v)(x) \tag{11.1}$$

for $v\in L^p(\Omega)$.
Then, we see that

$$\begin{aligned}\Delta J_\varepsilon(x;v)&=0 \qquad x\in\Omega_\varepsilon\\ J_\varepsilon(x;v)&=0 \qquad x\in\partial\Omega.\end{aligned} \tag{11.2}$$

As we get (8.9), we have

$$\begin{aligned}&J_\varepsilon(x;v)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}J_\varepsilon(x;v)_{|x=\varepsilon e_1}\\ &=\sum_{n=9}^{11}I_n(\varepsilon;v)-k\varepsilon^\sigma(I_{12}(\varepsilon;v)+I_{13}(\varepsilon;v)),\end{aligned} \tag{11.3}$$

where $I_9(\varepsilon;v)$ is given by $I_0(\varepsilon;\tilde{f})$ in (8.9) with $f=\xi_\varepsilon\hat{\chi}_\varepsilon v=(\xi_\varepsilon-\chi_\varepsilon)v$ and

$$\begin{aligned}I_{10}(\varepsilon;v)&=(\boldsymbol{G}\hat{\chi}_\varepsilon v)(x)-(\boldsymbol{G}\xi_\varepsilon\hat{\chi}_\varepsilon v)(\tilde{w})-\varepsilon\frac{\partial}{\partial w_1}(\boldsymbol{G}\xi_\varepsilon\hat{\chi}_\varepsilon v)(\tilde{w})\\ I_{11}(\varepsilon;v)&=-(4\pi/3)\mu_j\varepsilon^3G(x,\tilde{w})(\boldsymbol{G}\xi_\varepsilon v)(\tilde{w})\\ I_{12}(\varepsilon;v)&=\frac{\partial}{\partial x_1}(\boldsymbol{G}\hat{\chi}_\varepsilon v)(x)-(\frac{\partial}{\partial w_1}+\varepsilon\frac{\partial^2}{\partial w_1^2})(\boldsymbol{G}\xi_\varepsilon\hat{\chi}_\varepsilon v)(\tilde{w})\\ I_{13}(\varepsilon;v)&=-(4\pi/3)\mu_j\varepsilon^3\frac{\partial}{\partial x_1}G(x,\tilde{w})(\boldsymbol{G}\xi_\varepsilon v)(\tilde{w})\end{aligned}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.

It is easy to see that

$$|I_9(\varepsilon;v)|\leqq C(\varepsilon^2+\varepsilon^{3+\sigma})\|v\|_p \qquad (p>3) \tag{11.4}$$

$$|I_{11}(\varepsilon;v)|\leqq C\varepsilon^2\|v\|_p \qquad (p>3/2) \tag{11.5}$$

(11.6) $$|I_{13}(\varepsilon; v)| \leqq C\varepsilon\|v\|_p \qquad (p>3/2).$$

We have

(11.7) $$\begin{aligned}|I_{10}(\varepsilon; v)| &\leqq C\Big(\int_{B_\varepsilon} |x-y|^{-p'}dy\Big)^{1/p'}_{|x=\varepsilon e_1} \|v\|_p \\ &+C\Big(\int_{B_\varepsilon} |\tilde{w}-y|^{-p'}dy\Big)^{1/p'} \|v\|_p \\ &+C\varepsilon\Big(\int_{B_\varepsilon} |\tilde{w}-y|^{-2p'}dy\Big)^{1/p'} \|v\|_p \\ &\leqq \begin{cases} C\varepsilon^{2-3/p}\|v\|_p & (3/2<p<\infty) \\ C\varepsilon^2\|v\|_\infty & (p=\infty), \end{cases}\end{aligned}$$

and

(11.8) $$\begin{aligned}|I_{12}(\varepsilon; v)| &\leqq C\Big(\int_{B_\varepsilon} |x-y|^{-2p'}dy\Big)^{1/p'}_{|x=\varepsilon e_1} \|v\|_p \\ &+C\Big(\int_{B_\varepsilon} |\tilde{w}-y|^{-2p'}dy\Big)^{1/p'} \|v\|_p \\ &+C\varepsilon\Big(\int_{B_\varepsilon} |\tilde{w}-y|^{-3p'}dy\Big)^{1/p'} \|v\|_p \\ &\leqq C\varepsilon^{1-3/p} \|v\|_p \qquad (3<p<\infty).\end{aligned}$$

Summing up these facts, we have

$$|J_\varepsilon(x; v)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}J_\varepsilon(x; v)|_{x=\varepsilon e_1}\leqq C\varepsilon^{1+\sigma-3/p}\|v\|_p$$

for $p>3$. By Proposition 3.2, we get the following.

Lemma 11.1. *There exists a constant C independent of ε such that*

(11.9) $$\|J_\varepsilon(\cdot\,; v)\|_{2,\varepsilon}\leqq C\varepsilon^{3-3/p}\|v\|_p$$

holds for any $v\in L^p(\Omega)$ $(p>3)$.

Next we estimate $\|J_\varepsilon(\cdot\,; \varphi_j)\|_{2,\varepsilon}$. We see that

(11.10) $$I_{12}(\varepsilon; \varphi_j) = I_4(\varepsilon)+\sum_{n=14}^{16} I_n(\varepsilon),$$

where

$$\begin{aligned}I_{14}(\varepsilon) &= \Big(\frac{\partial}{\partial w_1}+\varepsilon\frac{\partial^2}{\partial w_1^2}\Big)(S(1-\xi_\varepsilon)\hat{\chi}_\varepsilon\varphi_j)(\tilde{w}) \\ I_{15}(\varepsilon) &= \frac{\partial}{\partial w_1}\int_{B_\varepsilon} L(w, y)(1-\xi_\varepsilon(y))\varphi_j(y)dy_{|w=0} \\ I_{16}(\varepsilon) &= \varepsilon\frac{\partial^2}{\partial w_1^2}\int_{B_\varepsilon} L(w, y)(1-\xi_\varepsilon(y))\varphi_j(y)dy_{|w=0}.\end{aligned}$$

Since $S(x, y) \in C^\infty(\Omega \times \Omega)$,

$$|I_{14}(\varepsilon)| \leqq C\varepsilon^2 . \tag{11.11}$$

In section 8, we have already showed the following.

$$|I_4(\varepsilon)| \leqq C\varepsilon^2 \tag{11.12}$$

$$\frac{\partial}{\partial w_1} \int_{B_\varepsilon} L(w, y)\varphi_j(y)dy = O(\varepsilon^2) \tag{11.13}$$

$$\frac{\partial^2}{\partial w_1^2} \int_{B_\varepsilon} L(w, y)\varphi_j(y)dy = -(1/3)\varphi_j(\tilde{w}) + O(\varepsilon^2) \tag{11.14}$$

for $w = \tilde{w} = 0$.

On the other hand, we see that

$$\Delta_w \int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)dy = -(\hat{\chi}_\varepsilon \xi_\varepsilon)(w) .$$

Since $\xi_\varepsilon(w) = 0$ for $w \in B_{\varepsilon/2}$ and $\xi_\varepsilon(w)$ is rotationary invariant, we have

$$\int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)dy = \text{Constant} = O(\varepsilon^2) \qquad \text{for} \quad w \in B_{\varepsilon/2} \tag{11.15}$$

and

$$|\xi_\varepsilon(y)| = |\xi_\varepsilon(y) - \xi_\varepsilon(w)| \leqq C\varepsilon^{-1}|y - w| \tag{11.16}$$

for $w \in B_{\varepsilon/2}$, $y \in B_\varepsilon$.

Therefore, we have the following.

$$\begin{aligned} &\frac{\partial}{\partial w_1} \int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)\varphi_j(y)dy_{|w=0} \\ &= \frac{\partial}{\partial w_1} \int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)(\varphi_j(y) - \varphi_j(w))dy_{|w=0} \\ &\qquad + \frac{\partial}{\partial w_1}(\varphi_j(w) \int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)dy)_{|w=0} \\ &= O(\varepsilon^2) \end{aligned} \tag{11.16}$$

$$\begin{aligned} &\frac{\partial^2}{\partial w_1^2} \int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)\varphi_j(y)dy_{|w=0} \\ &= \frac{\partial^2}{\partial w_1^2} \int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)(\varphi_j(y) - \varphi_j(w))dy_{|w=0} \\ &\qquad + \frac{\partial^2}{\partial w_1^2}(\varphi_j(w) \int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)dy)_{|w=0} \\ &= O(\varepsilon) \end{aligned} \tag{11.17}$$

Summing up these facts, we have

$$I_{12}(\varepsilon;\varphi_j)=-(\varepsilon/3)\varphi_j(\tilde{w})+O(\varepsilon^2)\,. \tag{11.18}$$

It is easy to see that

$$I_{13}(\varepsilon;\varphi_j)=(\varepsilon/3)\varphi_j(\tilde{w})+O(\varepsilon^2)\,. \tag{11.19}$$

Thus, by (11.3), (11.4), (11.5), (11.7), (11.18) and (11.19), we have

$$|J_\varepsilon(x;\varphi_j)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}J_\varepsilon(x;\varphi_j)|_{x=\varepsilon e_1}\leqq C(\varepsilon^2+\varepsilon^{2+\sigma})\,.$$

By Proposition 3.2, we get the desired Theorem 5.

Furthermore, we have the following.

Lemma 11.2. *There exists a constant C independent of ε such that*

$$||J_\varepsilon(\cdot\,;\psi(\varepsilon))||_{2,\varepsilon}\leqq C\varepsilon^4(1+\varepsilon^{-\sigma}) \tag{11.20}$$

holds.

Proof. We recall that $\psi(\varepsilon)=\varphi_j+g(\varepsilon)\psi_1+h(\varepsilon)\psi_2+i(\varepsilon)\psi_3$. We put $p>3$ in Lemma 10.1. Then, (11.20) easily follows from Lemmas 10.1, 11.1 and Theorem 5. q.e.d.

REMARK. By neglecting $I_{11}(\varepsilon;v)$ and $I_{13}(\varepsilon;v)$ in (11.13), we have

$$||(\chi_\varepsilon\hat{P}_\varepsilon-P_\varepsilon\chi_\varepsilon)\varphi_j||_{2,\varepsilon}\leqq C\varepsilon^3\,, \tag{11.21}$$

where

$$\hat{P}_\varepsilon=\bar{P}_\varepsilon+(4\pi/3)\mu_j\varepsilon^3\bar{A}_1\,.$$

Since the remainder term of an asymptotic formula (1.4) is $O(\varepsilon^4)$ for $\sigma\leqq-2$, the estimate (11.21) is weak in the sense that the right hand side is $O(\varepsilon^3)$. Therefore, the existence of the term $(4\pi/3)\mu_j\varepsilon^3\,G(x,\tilde{w})G(\tilde{w},y)\xi_\varepsilon(x)\xi_\varepsilon(y)$ in (2.9) is essential to get Theorem 2.

12. Proof of Theorem 2

Now we are in a position to prove Theorem 2. As in section 7, by Lemmas 10.2, 10.3 and 11.2, we have

$$||(G_\varepsilon-\lambda(\varepsilon))\,(\chi_\varepsilon\psi(\varepsilon))||_{2,\varepsilon}\leqq C\varepsilon^4(1+\varepsilon^{-2\sigma})\,.$$

Since $||\psi(\varepsilon)||_{2,\varepsilon}\in(1/2,2)$ for small ε, there exists at least one eigenvalue $\lambda^*(\varepsilon)$ of G_ε satisfying

$$|\lambda^*(\varepsilon)-\lambda(\varepsilon)|\leqq C\varepsilon^4(1+\varepsilon^{-2\sigma})\,. \tag{12.1}$$

We here represent $\lambda_1, \lambda_2, \lambda_3$ explicitly as follows.

$$\lambda_1 = (\int_\Omega G(w, y)\xi_\varepsilon(y)\varphi_j(y)dy)^2_{|w=\tilde{w}} \tag{12.2}$$
$$= \mu_j^{-2}\varphi_j(\tilde{w})^2+O(\varepsilon^2)$$

$$\lambda_2 = \sum_{n=1}^{3} (\frac{\partial}{\partial w_n} \int_\Omega G(w, y)\xi_\varepsilon(y)\varphi_j(y)dy)^2_{|w=\tilde{w}} \tag{12.3}$$

$$\lambda_3 = \sum_{m,n=1}^{3} (\frac{\partial^2}{\partial w_m \partial w_n} \int_\Omega G(w, y)\xi_\varepsilon(y)\varphi_j(y)dy)^2_{|w=\tilde{w}} \tag{12.4}$$

Since $\xi_\varepsilon(y)=0$ for $y\in B_{\varepsilon/2}$,

$$|\lambda_3| \leqq C(\int_{\Omega\setminus B_{\varepsilon/2}} |y-\tilde{w}|^{-3}dy)^2 \leqq C|\log \varepsilon|^2 . \tag{12.5}$$

On the other hand, by (8.25) and (11.15), we see that

$$\frac{\partial}{\partial w_n} \int_\Omega G(w, y)(1-\xi_\varepsilon(y))\varphi_j(y)dy$$
$$= \frac{\partial}{\partial w_n} \int_\Omega S(w, y)(1-\xi_\varepsilon(y))\varphi_j(y)dy$$
$$+ \frac{\partial}{\partial w_n} \int_\Omega L(w, y)(1-\xi_\varepsilon(y))(\varphi_j(y)-\varphi_j(w))dy$$
$$+ \frac{\partial}{\partial w_n}(\varphi_j(w) \int_\Omega L(w, y)(1-\xi_\varepsilon(y))dy)$$
$$= O(\varepsilon) \qquad (n=1, 2, 3)$$

for $w=\tilde{w}=0$.

Thus, we have

$$\lambda_2 = \mu_j^{-2} |\mathrm{grad}\, \varphi_j(\tilde{w})|^2+O(\varepsilon) . \tag{12.6}$$

From (12.2), (12.5) and (12.6),

$$\lambda(\varepsilon) = \mu_j^{-1}-\mu_j^{-2}(Q_j\varepsilon^{2-\sigma}+R_j\varepsilon^3)+O(\varepsilon^{3-2\sigma}+\varepsilon^{4-\sigma}+\varepsilon^4) , \tag{12.7}$$

where Q_j, R_j are as mentioned before.

By (12.1), (12.7) and the fact (9.7), we see that $\lambda^*(\varepsilon)$ must be $\mu_j(\varepsilon)^{-1}$. Then,

$$|\mu_j(\varepsilon)^{-1}-\mu_j^{-1}(1-\mu_j^{-1}(Q_j\varepsilon^{2-\sigma}+R_j\varepsilon^3))| \tag{12.8}$$
$$\leqq C(\varepsilon^{3-2\sigma}+\varepsilon^4)$$

holds.

Theorem 2 easily follows from (12.8).

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