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AN ISOPERIMETRIC INEQUALITY FOR ORBIFOLDS

Dedicated to Professor Masaru Takeuchi on his sixtieth birthday

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Introduction

The purpose of this note is to prove, by a method of Anderson ([**1**]), an isoperimetric inequality of Gallot's type (see [**5**]) for compact Riemannian orbifolds. An *orbifold* is locally a quotient $U=\tilde{U}/G$ of uniformizing open set $\tilde{U}(\subseteqq \boldsymbol{R}^m)$ by a finite subgroup G of the group of the orientation presreving diffeomorphisms of $\tilde{U}$, where each element of G is assumed to act on $\tilde{U}$ with fixed point set of codimension $\geqq 2$ (see [**11**] for more details). As a result, we can generalize Gallot's theorem on a Sobolev constant ([**6**]) to orbifolds. This together with [**2**] implies the uniquencess of Einstein-Kähler metrics for Fano orbifolds, which can also be obtained directly from [**2**] by Li-Yau's results ([**8**], [**9**]) (see Remark 2.1).

In this note, we fix once for all a compact connected m-dimensional Riemannian orbifold (M, g). A subset $H\subset M$ is called a *smooth hypersurface* M, if $\pi^{-1}(U\cap H)$ ia a smooth hypersurface in $\tilde{U}$ for any local uniformizing system $\{\tilde{U}, G, \pi\colon \tilde{U}\to U\}$. By D_g, V_g and Ric_g, we denote the diameter, total volume and Ricci tensor of the orbifold (M, g), respectively. Moreover, by $\mu_g^{(i)}$ we mean the i-dimensional measure on M associated to g. We now set $\mathfrak{J}_g:=\inf\{\mu_g^{(m-1)}(\partial\Omega)/\mu_g^{(m)}(\Omega)^{(m-1)/m};\ \Omega\in\mathcal{J}\}$ where $\mathcal{J}$ denotes the set of all domains Ω in M such that $2\mu_g^{(m)}(\Omega)\leqq V_g$ and that its boundary $\partial\Omega$ is a smooth hypersurface in M. Then our main result states the following:

Theorem A. *If* $\mathrm{Ric}_g\geqq -(m-1)\varepsilon^2 g$, *for some constant* $\varepsilon\geqq 0$, *then there exists a constant* $K_1>0$, *depending only on m and* $A_g:=\varepsilon D_g$, *such that*

$$\mathfrak{J}_g\geqq K_1 V_g^{1/m} D_g^{-1}.$$

Finally, the usefulness of Anderson's method was pointed out to me by Professor S. Bando to whom I wish to give my hearty thanks. I also wish to thank Professor A. Kasue who taught me Li-Yau's results. Special thanks are due also to Professor T. Mabuchi for introducing me to this topic and also for constant encouragements.

1. The proof of Theorem A

Let p be an arbitary point in M. Take a local uniformizing system $\{\tilde{U}, G, \pi: \tilde{U}\to U\}$ over a neighborhood U of p. By aubse of terminology, let (T_pM, g_p) denote the inner product tangent space $(T_{\tilde{p}}\tilde{U}, (\pi^*g)_{\tilde{p}})$ at some point $\tilde{p}$ in $\pi^{-1}(p)$. For each $v\in T_pM$, consider the geodesic $\gamma_v: [0, \infty)\to(M, g)$ such that $\gamma_v(0)=p$ and $\dot{\gamma}_v(0)=v$. Note that γ_v is a V-manifold map in the sense of [**11**].We now put

$$\Sigma_p: = \{v\in T_pM; \gamma_v(1)\in\mathfrak{S}(M)\},$$
$$U_pM := \{v\in T_pM; g_p(v, v) = 1\} \cong S^{m-1},$$

where $\mathfrak{S}(M)$ is the set of all singular points in M. The mapping $\exp_p: T_pM\backslash\Sigma_p\to M\backslash\mathfrak{S}(M)$, which sends $v\in T_pM\backslash\Sigma_p$ to $\gamma_v(1)$, is a C^∞ map between ordinary manifolds. Let $d\sigma_p$ denote the standard volume form on $U_pM\cong S^{m-1}$. We now define the function $\theta_p: T_pM\backslash\Sigma_p\to[0, \infty)$ by $(\exp_p^* d\mu_g^{(m)})(v)=\theta_p(v)dr\wedge d\sigma_p$, where $r(v):=g_p(v, v)^{1/2}$ with $v\in T_pM$. Let $U_p^0M(\neq\emptyset)$ be the set of all $v\in U_pM$ such that $tv\notin\Sigma_p$ for some $t>0$. Then for $v\in U_p^0M$, we can define $c(v)$ as the supremum of all $t>0$ such that $\gamma_v|_{[0,t]}$ is a minimizing geodesic. Define the associated cut locus Cut(p) by $\mathrm{Cut}(p):=\cup_{v\in U_p^0M}\exp_p(c(v)v)$. For each $\delta\geqq 0$, let $b_\delta(r)$ and $s_\delta(r)$ be the total volume of a geodesic r-ball and r-sphere, respectively, in a simply connected m-dimensional space form of constant curvature $-\delta^2$. Throughout this note, we assume $\mathrm{Ric}_g\geqq -(m-1)\varepsilon^2 g$ for some constant $\varepsilon\geqq 0$. Then Bishop's inequality for ordinary manifolds is valid also for orbifolds, i.e.,

$$\theta_p(tv)(\varepsilon^{-1}\sinh(\varepsilon t))^{1-m}\geqq\theta_p(t'v)({}^{-1}\varepsilon\sinh(\varepsilon t'))^{1-m},$$

whenever $v\in U_p^0M$ and $0<t\leqq t'\leqq c(v)$.

By Bishop's inequality, Gromov's result ([**7**; (B)]) for ordinary manifolds is valid also for orbifolds. Therefore, we obtain

Lemma 1.1. *Let W be a Borel set in M and H a smooth hypersurface in M. Let q be a point in M such that, for all $p\in W$, any minimizing geodesic γ between p and q always meets H. Moreover, let h be the point in $H\cap\gamma$ nearest to p. Now, for $v\in U_q^*M$, we put*

$$d_1(\gamma) := \mathrm{dist}_g(q, h),$$
$$d_2(v) := \min(\sup\{t\geqq 0; \gamma_v(t)\in W\}, c(v)),$$

where $U_q^*M(\neq\emptyset)$ is the set of all $v\in U_q^0M$ such that $\gamma_v(t)\in W$ and that $\gamma_v|_{[0,t]}$ is a minizing geodesic for some $t>0$. Then

$$\mu_g^{(m)}(W)/\mu_g^{(m-1)}(H)\leqq\sup_{v\in U_q^*M}(\{b_\varepsilon(d_2(v))-b_\varepsilon(d_1(\gamma_v))\}/s_\varepsilon(d_1(\gamma_v))).$$

By this lemma and Bishop's inequality, Anderson's result ([**1**; Lemmas

4.2, 4.3]) for ordinary manifolds is true also for orbifolds as follows. (To obtain a correct statement, the condition (1.2.1) below seems to be necessary, though Anderson does not state it explicitly.)

Lemma 1.2. *Let H be a smooth hypersurface in M which disconnects $M\backslash H$ into two open sets Ω_0 and Ω_1. Let $B_\rho(p)$ be a geodesic ρ-ball in M, centered at $p\in M$, such that*

$$(1.2.1) \qquad \mu_g^{(m-1)}(\partial B_\rho(p)\cap \mathrm{Cut}(p))=0\,.$$

For q in $\partial B_\rho(p)\backslash\Omega_1$, let $\eta(q)$ denote the distance from p to H along a minimizing geodesic γ between p and q, where $\eta(q)=0$ if γ and H do not intersect. Then by setting $\Pi(H,p):=\min\{\mu_g^{(m)}(\Omega_0\cap B_\rho(p)),\ \mu_g^{(m)}(\Omega_1\cap B_\rho(p))\}$, we have the following three inequalities:

$$(1.2.2) \qquad \Pi(H,p)\leqq 2b_\varepsilon(4\rho)s_\varepsilon(2\rho)^{-1}\mu_g^{(m-1)}(H\cap B_{2\rho}(p))\,,$$

$$(1.2.3) \qquad \int_{q\in\partial B_\rho(p)\backslash\Omega_1}\frac{s_\varepsilon(\eta(q))}{s_\varepsilon(\rho)}d\mu_g^{(m-1)}\leqq \mu_g^{(m-1)}(H\cap B_q(p))\,,$$

$$(1.2.4) \qquad \int_{q\in\partial B_\rho(p)\backslash\Omega_1}\frac{b_\varepsilon(\rho)-b_\varepsilon(\eta(q))}{s_\varepsilon(\rho)}d\mu_g^{(m-1)}\leqq \mu_g^{(m)}(\Omega_0\cap B_\beta(p))\,.$$

Proof. Following Anderson's arguments in [**1**; Lemmas 4.2, 4.3], we explain the outline of the proof. By Gromov's result ([7; (C)]), we may assume that

$$\Omega_0\cap B_\rho(p)\ni q_0 \quad\text{and}\quad \Omega_1\cap B_\rho(p)\supseteqq W$$

for some point q_0 and some subset W such that, for any point q in W, each minimizing geodesic γ between q_0 and q intersects H at a point h satisfying

$$\mathrm{dist}_g(q_0,h)\geqq \mathrm{dist}_g(q,h) \quad\text{and}\quad 2\mu_g^{(m)}(W)\geqq \mu_g^{(m)}(\Omega_1\cap B_\rho(p))\,.$$

Then by Lemma 1.1, we obtain

$$\Pi(H,p)\leqq 2\mu_g^{(m-1)}(H\cap B_{2\rho}(p))\sup_{v\in U_{q_0}^*M} b_\varepsilon(d_2(v))s_\varepsilon(d_1(\gamma_v))^{-1}\,,$$

which immediately implies (1.2.2). Next, we define a map $E\colon U_p'M\to M$ by $E(v):=\exp_p(\rho v)$, where $U_p'M:=\{v\in U_p^0M \mid \rho v\notin \Sigma_p\}$. By the condition (1.2.1), there exists an open subset U of $\partial B_\rho(p)\backslash\mathfrak{S}(M)$ with $\mu_g^{(m-1)}(\partial B_\rho(p))=\mu_g^{(m-1)}(U)$ such that every $q\in U$ and p are joined by a unique minimizing geodesic γ_q from p to q. By setting $U_1:=U\cap(\partial B_\rho(p)\backslash\Omega_1)$, we define a map $G\colon E^{-1}(U_1)\to H$ by $G(v):=\gamma_{E(v)}(\eta(E(v)))$. Then Bishop's inequality shows that

$$\begin{aligned}\mu_g^{(m-1)}(H\cap B_p(p))&\geqq \mu_g^{(m-1)}(G(E^{-1}(U_1)))\\ &\geqq \int_{q\in U_1}\theta_p(\eta(q)\dot\gamma_q(0))\theta_p(\rho\dot\gamma_q(0))^{-1}d\mu_g^{(m-1)}\,,\end{aligned}$$

from which, we obtain (1.2.3). Now, we can similarly obtain (1.2.4). ■

By using these lemmas, we can now prove our main result as follows.

Proof of Theorem A. Though we basically follow the arguments in [**1**; Theorem 4.1] for ordinary manifolds, the difference between his case and our case will be emphasized down below. Fix a real number $k\gg 2$. Let $\Omega\in\mathcal{J}$ and put $\Omega^c:=M\backslash(\Omega\cup\partial\Omega)$. For each $p\in\Omega$, choose a $\rho>0$ such that

$$\mu_g^{(m-1)}(\partial B_\rho(p)\cap \mathrm{Cut}(p))=0\,, \tag{1.3}$$

$$k\mu_g^{(m)}(B_\rho(p)\cap\Omega^c)\leqq 2\mu_g^{(m)}(B_\rho(p)\cap\Omega)\leqq 2k\mu_g^{(m)}(B_\rho(p)\cap\Omega^c)\,, \tag{1.4}$$

where the existence of such a ρ is guaranteed by $\mu_g^{(m)}(\mathrm{Cut}(p))=0$ and $2\mu_g^{(m)}(\Omega)\leqq V_g$. From (1.4), we have $2k\mu_g^{(m)}(B_\rho(p))\leqq 2(k+2)\mu_g^{(m)}(\Omega)\leqq(k+2)V_g$. By applying (1.2.3) and (1.2.4) to $\partial\Omega$ and Ω^c, we have

$$\begin{aligned}&\mu_g^{(m-1)}(\partial B_\rho(p)\backslash\Omega)\\ &\leqq s_\varepsilon(\rho)b_\varepsilon(\rho)^{-1}\mu_g^{(m)}(B_\rho(p)\cap\Omega^c)+\int_{y\in\partial B_\rho(p)\backslash\Omega} b_\varepsilon(\eta(y))b_\varepsilon(\rho)^{-1}d\mu_g^{(m-1)}\\ &\leqq s_\varepsilon(\rho)b_\varepsilon(\rho)^{-1}\mu_g^{(m)}(B_\rho(p)\cap\Omega^c)+\mu_g^{(m-1)}(\partial\Omega\cap B_\rho(p))\,.\end{aligned}$$

Then from (1.2.2), we obtain

$$\mu_g^{(m-1)}(\partial B_\rho(p)\backslash\Omega)\leqq\alpha(m,\varepsilon,\rho,1)\mu_g^{(m-1)}(\partial\Omega\cap B_{2\rho}(p))\,,$$

where $\alpha(m,\varepsilon,\rho,j):=2j\,b_\varepsilon(4\rho)s_\varepsilon(\rho)s_\varepsilon(2\rho)^{-1}b_\varepsilon(\rho)^{-1}+1$. By Bishop's inequality, we have

$$b_\varepsilon(ar)\leqq b_\varepsilon(r)a^m\Big(\int_0^1\Big(\frac{\sinh(aA_gt)}{aA_g}\Big)^{m-1}dt\Big)\Big(\int_0^1\Big(\frac{\sinh(A_gt)}{A_g}\Big)^{m-1}dt\Big)^{-1},$$

for all $a\geqq 1$ and $0\leqq r\leqq D_g$. From this inequality, we can show $\alpha(m,\varepsilon,\rho,j)\leqq c_0(m,A_g,j)$ for some constant $c_0(m,A_g,j)>0$ depending only on m, A_g, j (and independent of ε,ρ). In this note, $c_i(\alpha,\cdots,\beta)$ means a positive constant depending only on $\alpha,\cdots,\beta$. Therefore, we obatin

$$\mu_g^{(m-1)}(\partial B_\rho(p)\backslash\Omega)\leqq c_0(m,A_g,1)\mu_g^{(m-1)}(\partial\Omega\cap B_{2\rho}(p))\,. \tag{1.5}$$

In view of (1.4), applying the same arguments as above to $\partial B_\rho(p)\backslash\Omega^c$, we also obtain

$$\mu_g^{(m-1)}(\partial B_\rho(p))\backslash\Omega^c)\leqq c_0(m,A_g,k)\mu_g^{(m-1)}(\partial\Omega\cap B_{2\rho}(p))\,. \tag{1.6}$$

Therefore, by adding (1.5) and (1.6), we have

$$\mu_g^{(m-1)}(\partial B_\rho(p))\leqq c_1(m,A_g,k)\mu_g^{(m-1)}(\partial\Omega\cap B_{2\rho}(p))\,. \tag{1.7}$$

For each $q\in\Omega$, we define $\rho(q)$ to be the smallest positive real number satisfying

(1.3) and (1.4). Then there exists a finite points $p_1, p_2, \cdots, p_N$ in Ω such that $B_{2\rho(p_i)}(p_i)$, $i=1, 2, \cdots, N$ are mutually disjoint and that $\cup_{i=1}^N B_{6\rho(p_i)}(p_i) \geqq \Omega$. Then by (1.7),

$$\min\{\mu_g^{(m-1)}(\partial B_{\rho(p_i)}(p_i))/\mu_g^{(m)}(B_{\rho(p_i)}(p_i))^{(m-1)/m};\ i=1, 2, \cdots, N\}$$
$$\leqq c_2(m, A_g, k)\mu_g^{(m-1)}(\partial\Omega)/\mu_g^{(m)}(\Omega)^{(m-1)/m}.$$

Therefore, we obtain

$$(1.8)\quad \mathfrak{J}_g \geqq c_3(m, A_g, k) \inf\{\mu_g^{(m-1)}(\partial B_\rho(p))/\mu_g^{(m)}(B_\rho(p))^{(m-1)/m};\ B_\rho(p)\in\mathcal{B}\},$$

where $\mathcal{B}$ is the set of all geodesic balls $B_\rho(p)$ in M such that $2k\mu_g^{(m)}(B_\rho(p)) \leqq (k+2)V_g$ and that $\mu_g^{(m-1)}(\partial B_\rho(p)\cap \mathrm{Cut}(p))=0$. Take a $B_\rho(p_0)\in\mathcal{B}$ for arbitary $p_0\in M$. If there exists a point $p_1\in M$ such that $\mathrm{dist}_g(p_0, p_1)=3\rho$, then by Lemma 1.1 applied to $H=\partial B_\rho(p_0)\backslash\{\mathfrak{S}(M)\cup \mathrm{Cut}(p_0)\}$, $q=p_1$ and $W=B_\rho(p_0)$, we have

$$\mu_g^{(m)}(B_\rho(p_0)) \leqq b_\varepsilon(8\rho)s_\varepsilon(\rho)^{-1}\mu_g^{(m-1)}(\partial B_\rho(p_0)).$$

Otherwise, by Lemma 1.1 applied to $H=\partial B_\rho(p_0)\backslash\{\mathfrak{S}(M)\cup \mathrm{Cut}(p_0)\}$, $q=p_0$ and $W=M\backslash(B_\rho(p_0)\cup\partial B_\rho(p_0))$, we obtain

$$\mu_g^{(m)}(B_\rho(p_0)) \leqq 3^m(k+2)(k-2)^{-1}b_\varepsilon(3\rho)s_\varepsilon(\rho)^{-1}\mu_g^{(m-1)}(\partial B_\rho(p_0)).$$

In both cases, by $b_\varepsilon(\rho)s_\varepsilon(\rho)^{-1} \leqq b_\varepsilon(\rho)^{1/m}s_0(1)^{-1/m}$, it now follows that

$$(1.9)\quad \mu_g^{(m)}(B_\rho(p_0)) \leqq c_4(m, A_g, k)s_0(1)^{-1/m}b_\varepsilon(\rho)^{1/m}\mu_g^{(m-1)}(\partial B_\rho(p_0)).$$

Finally, Theorem A is straightforward from (1.8), (1.9) and Bishop's inequality. ■

2. Applications

In this section, we shall give applications of Theorem A. First, we immediately obtain the following Sobolev inequality of Gallot's type, by the methods in [6].

Theorem B. *Let $L_1^p(M)$ be the Sobolev space of all functions f on a Riemannian orbifold (M, g) satisfying $\|f\|_{L^p}+\|df\|_{L^p}<\infty$, in terms of orbifold L^p-norm. Assume $\mathrm{Ric}_g \geqq -(m-1)\varepsilon^2 g$ for some constant $\varepsilon\geqq 0$. Then there exists a constant $K_2>0$, depending only on m and $A_g:=\varepsilon D_g$, such that*

$$\begin{cases} \|df\|_{L^2} \geqq K_2 V_g^{1/m} D_g^{-1}\|f\|_{L^{2m/(m-2)}}, & \text{if } m\geqq 3; \\ \|df\|_{L^2} \geqq K_2 V_g^{1/4} D_g^{-1}\|f\|_{L^4}, & \text{if } m=2, \end{cases}$$

for all $f\in L_1^2(M)$ with $\int_M f d\mu_g^{(m)}=0$.

As in [6], approximation of C^∞ functions by Morse functions gives us an effective tool in the proof of the Sobolev inequality for ordinary manifolds. For orbifolds, this approximation is also possible by the same methods as in [**10**; Theorem 2.7], though we have to use [**12**; Lemma 4.8] in place of [**10**; Lemma A]. Note that Theorem B implies the following theorem by the same arguments as in [**2**].

Theorem C. *Let $G_g(p, q)$ be the Green function (see* [**3**]) *for the Laplacian Δ_g on a Riemannian orbifold (M, g). If $\mathrm{Ric}_g \geqq -(m-1)\varepsilon^2 g$ for some constant $\varepsilon \geqq 0$, then we have a constant $K_3>0$, depending only on m and $A_g := \varepsilon D_g$, such that*

$$G_g(p, q) \geqq -K_3 D_g^2 V_g^{-1},$$

for all $(p, q) \in M \times M$ with $p \neq q$.

REMARK 2.1. For $m \geqq 3$, Theorem C can be proved without using Theorem B as follows. We denote by $H(p, q, t)$ the heat kernel for (M,g) (see [**3**]). A simple calculation shows $H_0(p, p, t) := H(p, p, t) - V_g^{-1} \leqq e^{-\lambda_1 t/2} H(p, p, t/2)$, where λ_1 is the first positive eigenvalues for $-\Delta_g$. This inequality and Li-Yau's results ([**8**; Theorem 7], [**9**; Corollary 3.1]) are valid also for orbifolds, and hence

$$H_0(p, p, t) \leqq c_5(m, A_g) V_g^{-1} (t D_g^{-2})^{-m/2},$$

for all $p \in M$ and $t \geqq c_6(m, A_g) D_g^2$. Thus, we can prove Theorem C by same arguments as in [**2**].

Now, we assume that X is a compact connected complex orbifold having positive first Chern class in the sense of orbifolds. Let $\mathcal{E}(X)$ denote the set of all Einstein-Kähler orbifold metrics g such that $\mathrm{Ric}_g = g$.

EXAMPLE 2.2. In [**4**], Galicki-Nitta constructed examples of compact self-dual Einstein orbifolds, of real dimension 4, by quaternionic Kähler reduction. Their twistor spaces are compact Einstein-Kähler orbifolds, of complex dimension 3, with positive first Chern class. .

Let $\mathrm{Aut}(X)$ be the group of all automorphisms of X and $\mathrm{Aut}^0(X)$ its identity component. Then $\mathrm{Aut}^0(X)$ naturally acts on $\mathcal{E}(X)$. In view of Theorem C, we can also prove the following theorem by the same arguements as in [**2**].

Theorem D. *If $\mathcal{E}(X) \neq \emptyset$, then $\mathcal{E}(X)$ consists of a single* $\mathrm{Aut}^0(X)$*-orbit.*

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