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A DECOMPOSITION OF $BP\langle 2\rangle$ AND v_1-TORSION

DOMINIQUE ARLETTAZ AND JOHN KLIPPENSTEIN

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Introduction

For any p-local connective spectrum F (with p a prime number), the first author discovered in [2] integers ρ_j and maps $F\to\Sigma^{j+1}H(\pi_jF, 0)$ between F and Eilenberg-MacLane spectra such that the compositions

$$F \longrightarrow \Sigma^{j+1} H(\pi_jF, 0) \xrightarrow{p^{\rho_j}} \Sigma^{j+1} H(\pi_jF, 0)$$

are trivial. This enabled him to prove that in the Atiyah-Hirzebruch-Dold spectral sequence for the F-homology of any bounded below spectrum, $p^{\rho_j}d^{j+1}_{s,t}=0$ for all $j\geq 1$, s and t. Now, let us consider the Brown-Peterson spectrum BP with $BP_*=\mathbf{Z}_{(p)}[v_1, v_2, \cdots]$, where the degree of v_k is $|v_k|=2(p^k-1)$ for $k\geq 1$, and denote as usual by $BP\langle m\rangle$ the spectrum such that $BP\langle m\rangle_*\cong\mathbf{Z}_{(p)}[v_1\ v_2, \cdots, v_m]$ for any $m\geq 1$. This paper exploits a similar composite

$$BP\langle 2\rangle/(v_2^j) \longrightarrow \Sigma^{j|v_2|+1}BP\langle 1\rangle \xrightarrow{v_1^{(p+1)j+1}} \Sigma^{-|v_1|+1}BP\langle 1\rangle$$

which, as a consequence of calculations by the second author in [6] can be seen to be trivial. As a result, we can construct maps

$$f_j\colon BP\langle 2\rangle \to \Sigma^{-|v_1|}BP\langle 1\rangle\,,$$

for all $j\geq 1$, which we control on the homotopy level (see Theorem 2.1). These maps induce maps between the Atiyah-Hirzebruch-Dold spectral sequences for $BP\langle 2\rangle$ and $BP\langle 1\rangle$-homology respectively which provide information about the differentials in the Atiyah-Hirzebruch-Dold spectral sequence for $BP\langle 2\rangle$ (see Theorem 3.3). On the other hand, the triviality of the above composition implies torsion results on the differentials in a modified Bockstein spectral sequence for $BP\langle 2\rangle$ analogous to the BP Bockstein spectral sequence of Johnson and Wilson [5] (see Theorem 4.5).

In order to illustrate how this new information might be used in calculation,

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we have considered the problem of constructing $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$. $M(p^i)$ is the Moore spectrum having all integral homology groups trivial except for $\mathbf{Z}/p^i$ in dimension 0. Inductively, $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$ is the cofiber of a map $\Sigma^{j_n|v_n|} M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_{n-1}^{j_{n-1}}) \to M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_{n-1}^{j_{n-1}})$ which induces multiplication by $v_n^{j_n}$ in BP-homology. This means that $BP_* M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n}) \cong BP_*/(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$. The information about differentials in the spectral sequences mentioned in the previous paragraph can be used to obtain lower bounds on j_n in terms of i, and j_2 in terms of j_1, if $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$ exists. It should be noted that better bounds than ours can be obtained from ideas in Ravenel's book [8] (see Remark 3.6; see also the conditions on j_1 and j_2 established by Lin in [7]). Thus the main theorems of the present paper are Theorems 3.3 and 4.1 giving information about the differentials in the different spectral sequences. The results about the constructibility of $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$ are given simply to illustrate the use of these theorems. We hope that someone else will be able to use this information in a more novel way.

The paper is organized as follows. In the first section we show how to use the first author's results in [2] to derive the condition that if $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$ is constructible then $j_n \geq i$. In the second section we introduce a general construction of maps from $BP\langle m\rangle$ to $BP\langle m-1\rangle$. We apply it in Section 3 to the case $m=2$ in order to get the interesting maps between the Atiyah-Hirzebruch-Dold spectral sequences for $BP\langle 2\rangle$ and $BP\langle 1\rangle$ respectively; we illustrate the use of these maps by proving that if $M(p^i, v_1^{j_1}, v_2^{j_2})$ is constructible, then $j_2 > \frac{j_1}{p+1}$. Finally, we discuss in the fourth section the set up and general use of the modified Bockstein spectral sequence for $BP\langle 2\rangle$ and obtain torsion results on its differentials.

1. The Atiyah-Hirzebruch-Dold spectral sequence for $C(m)$

The existence of universal bounds for the additive order of the differentials in the Atiyah-Hirzebruch-Dold spectral sequence was deduced in [2] from torsion results on the Postnikov k-invariants of spectra which produced maps between the Atiyah-Hirzebruch-Dold spectral sequence and a spectral sequence having only one-non-trivial line in its E^2-term.

The purpose of this first section is to show that these results have direct consequences as soon as we concentrate our attention on specific examples. But our main goal is to motivate and illustrate the basic ideas of the method we shall develop in the next sections.

If m is a positive integer, let $C(m)$ be the spectrum such that

$$\pi_* C(m) \cong BP_*/(v_1, v_2, \cdots, v_{m-1}, v_{m+1}, v_{m+2}, \cdots) \cong \mathbf{Z}_{(p)}[v_m]$$

(see Yagita's [9] for the existence of $C(m)$). The non-trivial Postnikov k-invariants of $C(m)$ are $k^{h|v_m|+1}(C(m))\in H^{h|v_m|+1}(C(m)[h|v_m|-1]; \mathbf{Z}_{(p)})$ for $h\geq 1$, where we write $C(m)[j]$ for the j-th Postnikov section of $C(m)$. If we apply the method of Theorem 1.4 of [2] to the spectrum $C(m)$, we get:

Lemma 1.1. *For any $h\geq 1$, the k-invariant $k^{h|v_m|+1}(C(m))$ has order dividing p^h.*

Proof. The spectrum $C(m)[h|v_m|-1]$ has non-trivial homotopy groups (which are isomorphic to $\mathbf{Z}_{(p)}$) only in dimensions 0, $|v_m|, 2|v_m|, \cdots, (h-1)|v_m|$; notice in particular that $C(m)[h|v_m|-1]=C(m)[(h-1)|v_m|]$. Therefore, we can consider the cofibrations of spectra (where $H(G, n)$ denotes the Eilenberg-MacLane spectrum having all homotopy groups trivial except for G in dimension n)

$$C(m)[d|v_m|] \to C(m)[(d-1)|v_m|] \to H(\mathbf{Z}_{(p)}, d|v_m|+1)$$

and the corresponding long exact homology sequences

$$\cdots \to H_{h|v_m|+2}H(\mathbf{Z}_{(p)},d|v_m|+1) \to H_{h|v_m|+1}C(m)[d|v_m|] \to H_{h|v_m|+1}C(m)[(d-1)|v_m|] \to \cdots$$

for $d=1, 2, \cdots, h-1$. According to Cartan's [4], $H_{h|v_m|+2}H(\mathbf{Z}_{(p)}, d|v_m|+1)$ is a direct sum of copies of $\mathbf{Z}/p$, as is $H_{h|v_m|+1}C(m)[0]$. By induction, it is then clear that $p^h H_{h|v_m|+1}C(m)[h|v_m|-1]=0$, and analogously that $p^h H_{h|v_m|}C(m)[h|v_m|-1]=0$. Finally, the universal coefficient theorem implies that the exponent of the cohomology group $H^{h|v_m|+1}(C(m)[h|v_m|-1]; \mathbf{Z}_{(p)})$ divides p^h and the proof is complete.

Now, consider the Atiyah-Hirzebruch-Dold spectral sequence for $C(m)$

$$E^2_{s,t} \cong H_s(X, \pi_t C(m)) \cong H_s X\otimes\pi_t C(m) \Rightarrow C(m)_{s+t}(X)\,,$$

where X is any bounded below spectrum. The non-trival differentials in this spectral sequence are $d^{h|v_m|+1}$ for $h\geq 1$.

Corollary 1.2. *In the Atiyah-Hirzebruch-Dold spectral sequence for $C(m)$, the differentials satisfy*

$$p^h\, d^{h|v_m|+1}_{s,t} = 0 \quad \textit{for any } h\geq 1,\ s \textit{ and } t.$$

Proof. Because of Lemma 1.1, this is a consequence of Proposition 2.1 of [2] if $t=0$ and of the idea explained in Lemma 2.3 of [2] in the general case.

Example 1.3. We want to apply this result to the problem of the constructibility of $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$. With m an integer between 1 and n, let us compute $C(m)_*M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$. Studying the $C(m)$-homology exact sequences of the cofibrations defining $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$, we find an additive isomorphism

$$C(m)_*M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n}) \cong \mathbf{Z}/p^i[v_m]/(v_m^{j_m})\otimes\Lambda(w_1, w_2, \cdots, w_{m-1}, w_{m+1}, \cdots, w_n),$$

where Λ denotes an exterior algebra over $\mathbf{Z}$ and $|w_k|=j_k|v_k|+1$. On the other hand, we can calculate $C(m)_*M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$ via the Atiyah-Hirzebruch-Dold spectral sequence for $C(m)$. For this, we first need to know the integral homology of $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$: it is not hard to check that, additively,

$$H_*M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n}) \cong \mathbf{Z}/p^i\cdot a_0\otimes\Lambda(a_1, a_2, \cdots, a_n),$$

where $|a_0|=0$ and $|a_k|=j_k|v_k|+1$ for $k=1, 2, \cdots, n$. But in order to get the right answer for $C(m)_*M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$, there must be a differential which kills $v_m^{j_m}$; more precisely, the differential $d^{j_m|v_m|+1}$ must verify

$$d^{j_m|v_m|+1}_{j_m|v_m|+1,0}(a_m\otimes 1) = \lambda a_0\otimes v_m^{j_m},$$

where λ is a generator of $\mathbf{Z}/p^i$. Now, it follows from Corollary 1.2 that

$$p^{j_m}\lambda a_0\otimes v_m^{j_m} = 0$$

and consequently that $p^{j_m}\lambda$ must vanish in $\mathbf{Z}/p^i$. This implies the following assertion:

If the spectrum $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$ is constructible, then $j_m\geq i$ for $m=1, 2, \cdots, n$.

One should say that this result is not very strong (see Remark 3.6 and notice that in the special case of $M(p^i, v_1^{j_1})$ for an odd prime number p, the exact answer to the question of the constructibility may be deduced from Theorem 12.1 of Adams' [1]: $M(p^i, v_1^{j_1})$ is constructible if and only if j_1 is divisible by p^{i-1}), but it is given here as an example. However, our argument produces the following more general statement: *if X is a connective spectrum and m an integer such that $C(m)_*X\cong\mathbf{Z}/p^i[v_m]/(v_m^j)\otimes A$ for some A (as $C(m)_*$-modules) and $H_*X\cong\mathbf{Z}/p^i\otimes\Lambda(a)\otimes A$, where $|a|=j|v_m|+1$, then $j\geq i$.*

2. A decomposition of $BP\langle m\rangle$

For every integer $m\geq 2$, let $BP\langle m\rangle$ denote as usual the spectrum with $BP\langle m\rangle_*\cong\mathbf{Z}_{(p)}[v_1, v_2, \cdots, v_m]$. Now, for j a positive integer, consider the cofibration

$$\Sigma^{j|v_m|}BP\langle m\rangle \xrightarrow{v_m^j} BP\langle m\rangle \xrightarrow{\sigma_j} BP\langle m\rangle/(v_m^j),$$

where the first arrow indicates a map inducing multiplication by v_m^j on homotopy and where the spectrum $BP\langle m\rangle/(v_m^j)$ is such that $BP\langle m\rangle/(v_m^j)_*\cong$

$\mathbf{Z}_{(p)}[v_1, v_2, \cdots, v_m]/(v_m^j)$; notice the homotopy equivalence $BP\langle m\rangle/(v_m)\simeq BP\langle m-1\rangle$. This section describes a decomposition of $BP\langle m\rangle$ in terms of the spectra $BP\langle m\rangle/(v_m^j)$ for $j=1, 2, \cdots$. Look at the following commutative diagram

$$\begin{array}{ccccc}
\Sigma^{(j+1)|v_m|}BP\langle m\rangle & \xrightarrow{v_m} & \Sigma^{j|v_m|}BP\langle m\rangle & \xrightarrow{\Sigma^{j|v_m|}\sigma_1} & \Sigma^{j|v_m|}BP\langle m-1\rangle \\
\downarrow v_m^{j+1} & & \downarrow v_m^{j} & & \downarrow \\
BP\langle m\rangle & \xrightarrow{=} & BP\langle m\rangle & \longrightarrow & * \\
\downarrow \sigma_{j+1} & & \downarrow \sigma_j & & \downarrow \\
BP\langle m\rangle/(v_m^{j+1}) & \xrightarrow{g_j} & BP\langle m\rangle/(v_m^{j}) & \xrightarrow{l_j} & \Sigma^{j|v_m|+1}BP\langle m-1\rangle\,,
\end{array}$$

in which rows and columns are cofibrations of spectra, the map g_j is determined by the top left square and l_j is the cofiber of g_j. These maps l_j explain how to build the $BP\langle m\rangle/(v_m^{j+1})$'s using $BP\langle m-1\rangle$ as the building blocks (instead of the Eilenberg-MacLane spectra in a Postnikov tower). Notice that l_j is actually an element of $BP\langle m-1\rangle^{j|v_m|+1}(BP\langle m\rangle/(v_m^j))$. Especially interesting is the next result which describes connections between $BP\langle m\rangle$ and $BP\langle m-1\rangle$.

Theorem 2.1. *Let j be any positive integer and assume that there is an integer e_j such that $v_{m-1}^{e_j}\cdot BP\langle m-1\rangle^{j|v_m|+1}(BP\langle m\rangle/(v_m^j))=0$, then there exists a map*

$$f_j\colon BP\langle m\rangle \to \Sigma^{j|v_m|-e_j|v_{m-1}|}BP\langle m-1\rangle$$

with the property that the homomorphism $(f_j)_*$ induced by f_j on homotopy acts on an element $v_1^{\alpha_1}v_2^{\alpha_2}\cdots v_m^{j+k}\in BP\langle m\rangle_*$ as follows:

$$(f_j)_*(v_1^{\alpha_1}v_2^{\alpha_2}\cdots v_m^{j+k}) = \begin{cases} v_1^{\alpha_1}v_2^{\alpha_2}\cdots v_{m-2}^{\alpha_{m-2}}v_{m-1}^{\alpha_{m-1}+e_j}\,, & \text{if } k=0,\\ 0\,, & \text{if } k>0.\end{cases}$$

Proof. First, if we compose the last two columns of the above diagram with maps inducing multiplication by $v_{m-1}^{e_j}$ in homotopy, we obtain the following diagram in which only the columns are cofibrations:

$$\begin{array}{ccccc}
\Sigma^{j|v_m|}BP\langle m\rangle & \xrightarrow{\Sigma^{j|v_m|}\sigma_1} & \Sigma^{j|v_m|}BP\langle m-1\rangle & \xrightarrow{v_{m-1}^{e_j}} & \Sigma^{j|v_m|-e_j|v_{m-1}|}BP\langle m-1\rangle \\
\downarrow v_m^{j} & & \downarrow & & \downarrow \\
BP\langle m\rangle & \longrightarrow & * & \longrightarrow & * \\
\downarrow \sigma_j & & \downarrow & & \downarrow \\
BP\langle m\rangle/(v_m^{j}) & \xrightarrow{l_j} & \Sigma^{j|v_m|+1}BP\langle m-1\rangle & \xrightarrow{v_{m-1}^{e_j}} & \Sigma^{j|v_m|+1-e_j|v_{m-1}|}BP\langle m-1\rangle.
\end{array}$$

If the hypothesis of the proposition is verified, the bottom composition $v_{m-1}^{e_j} \circ l_j$ is trivial and we get a map

$$f_j : BP\langle m\rangle \to \Sigma^{j|v_m|-e_j|v_{m-1}|} BP\langle m-1\rangle$$

such that $f_j \circ v_m^j = v_{m-1}^{e_j} \circ \Sigma^{j|v_m|}\sigma_1$. The homomorphism $v_{m-1}^{e_j}(\Sigma^{j|v_m|}\sigma_1)_*$ induced by $v_{m-1}^{e_j} \circ \Sigma^{j|v_m|}\sigma_1$ on homotopy satisfies:

$$v_{m-1}^{e_j}(\Sigma^{j|v_m|}\sigma_1)_*(v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_m^{j+k}) = \begin{cases} v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_{m-1}^{\alpha_{m-1}+e_j}, & \text{if } j+k=0, \\ 0, & \text{if } j+k>0. \end{cases}$$

Consequently,

$$(f_j)_*(v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_m^{j+k}) = \begin{cases} v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_{m-1}^{\alpha_{m-1}+e_j}, & \text{if } k=0, \\ 0, & \text{if } k>0. \end{cases}$$

REMARK 2.2. In the case when m is 1, the fact that $p^j\, H\mathbf{Z}_{(p)}^{j|v_1|+1}(BP\langle 1\rangle/(v_1^j))=0$ (follow the argument of the proof of Lemma 1.1) makes it possible to use the decomposition of $BP\langle 1\rangle$ and obtain maps $f_j : BP\langle 1\rangle \to \Sigma^{j|v_1|} H\mathbf{Z}_{(p)}$, for $j \geq 1$, such that

$$(f_j)_*(v_1^{j+k}) = \begin{cases} p^j, & \text{if } k=0, \\ 0, & \text{if } k>0. \end{cases}$$

3. Maps between the Atiyah-Hirzebruch-Dold spectral sequences for $BP\langle 2\rangle$ and $BP\langle 1\rangle$

Now, let us consider the decomposition explained in the previous section in the case $m=2$ and show that the hypothesis of Theorem 2.1 is verified for any positive integer j.

Proposition 3.1. *For every positive integer j, the v_1-torsion-free part of $BP\langle 1\rangle^*(BP\langle 2\rangle/(v_2^j))$ is concentrated in even degrees and the v_1-torsion part of $BP\langle 1\rangle^*(BP\langle 2\rangle/(v_2^j))$ is concentrated in positive degrees.*

Proof. Corollary 10 of [6] implies that the assertion holds for $BP\langle 1\rangle^* BP\langle 1\rangle$. Then, use inductively the long exact sequences in $BP\langle 1\rangle$-cohomology associated with the cofibrations

$$BP\langle 2\rangle/(v_2^{j+1}) \xrightarrow{g_j} BP\langle 2\rangle/(v_2^j) \xrightarrow{l_j} \Sigma^{j|v_2|+1} BP\langle 1\rangle$$

given by the bottom sequence in the first diagram of Section 2, for $j=1, 2, \cdots$ (and recall that $BP\langle 2\rangle/(v_2) \simeq BP\langle 1\rangle$):

$$\cdots \to BP\langle 1\rangle^*(BP\langle 2\rangle/(v_2^j)) \to BP\langle 1\rangle^*(BP\langle 2\rangle/(v_2^{j+1})) \to BP\langle 1\rangle^{*-j|v_2|} BP\langle 1\rangle \to \cdots.$$

This produces the statement of the proposition for $BP\langle 1\rangle^*(BP\langle 2\rangle/(v_2^j))$ for all $j\geq 1$.

Corollary 3.2. *For all positive integers k and j, $v_1^{\varepsilon_{k,j}}\cdot BP\langle 1\rangle^{2k+1}(BP\langle 2\rangle/(v_2^j))=0$ if $\varepsilon_{k,j}$ is an integer $>\dfrac{2k+1}{|v_1|}$. In particular, $v_1^{e_j}\cdot BP\langle 1\rangle^{j|v_2|+1}(BP\langle 2\rangle/(v_2^j))=0$ for each integer $e_j>(p+1)j$.*

Proof. An element $x\in BP\langle 1\rangle^{2k+1}(BP\langle 2\rangle/(v_2^j))$ must be v_1-torsion because it is in odd degree and $v_1^{\varepsilon_{k,j}}x=0$ since its degree is negative.

This assertion enables us to apply Theorem 2.1 for $m=2$ and $e_j=(p+1)j+1$: it produces maps

$$f_j\colon BP\langle 2\rangle \to \Sigma^{-|v_1|}BP\langle 1\rangle$$

for all positive integers j. For any bounded below spectrum X, let us look at the Atiyah-Hirzebruch-Dold spectral sequences for $BP\langle 2\rangle$ and $BP\langle 1\rangle$-homology

$$E^2_{s,t}\cong H_s(X;\pi_t BP\langle 2\rangle)\cong H_sX\otimes\pi_t BP\langle 2\rangle\Rightarrow BP\langle 2\rangle_{s+t}(X)$$

and

$$\tilde{E}^2_{s,t}\cong H_s(X;\pi_t BP\langle 1\rangle)\cong H_sX\otimes\pi_t BP\langle 1\rangle\Rightarrow BP\langle 1\rangle_{s+t}(X)$$

respectively. The f_j's induce homomorphisms $E^2_{s,t}\to E^2_{s,t+|v_1|}$ and hence maps between these spectral sequences: we then obtain immediately the next result.

Theorem 3.3. *There are maps of spectral sequences*

$$(f_j)_*\colon E^r_{s,t}\to\tilde{E}^r_{s,t+|v_1|}\,,\quad j\geq 1$$

($r\geq 2, t\geq 0$) with the following property: if $\sum_{k\geq 0}\bar{x}_k\otimes v_1^{d_k}v_2^{h+k}$ belongs to $E^r_{s,t}$ (where the sum is taken over a finite number of k's, $\bar{x}_k$ is represented by an element x_k of H_sX, h is a positive integer and $t=d_k|v_1|+(h+k)|v_2|$), then

$$(f_j)_*(\sum_{k\geq 0}\bar{x}_k\otimes v_1^{d_k}v_2^{h+k})=0\,,\quad \text{if } j<h$$

and

$$(f_h)_*(\sum_{k\geq 0}\bar{x}_k\otimes v_1^{d_k}v_2^{h+k}) \text{ is the class of } x_0\otimes v_1^{d_k+(p+1)h+1} \text{ in } \tilde{E}^r_{s,t+|v_1|}\,.$$

REMARK 3.4. This theorem provides infinitely many ways to compare the Atiyah-Hirzebruch-Dold spectral sequence for $BP\langle 2\rangle$ with that for $BP\langle 1\rangle$. If one is dealing with a specific problem, it is generally advantageous to use several of the maps $(f_j)_*$. This method may be of special interest in order to understand the differentials $d^r_{s,t}\colon E^r_{s,t}\to E^r_{s-r,t+r-1}$ in the spectral sequence for $BP\langle 2\rangle$: for instance, if we choose $j<h$, the vanishing of $(f_j)_*(\sum_{k\geq 0}\bar{x}_k\otimes v_1^{d_k}v_2^{h+k})$ gives the

equality

$$(f_j)_* \circ d^r_{s,t}\Big(\sum_{k\geq 0} \bar{x}_k \otimes v_1^{d_k} v_2^{h+k}\Big) = 0 \quad \text{in} \quad \tilde{E}^r_{s-r,t+|v_1|+r-1} \,.$$

EXAMPLE 3.5. We present here a more explicit application of our comparison method. We have seen in Section 1 that if the spectrum $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$ is constructible, then j_1 and j_2 are forced to be $\geq i$. But now, we can prove that there must also exist a relation between j_1 and j_2.

If we compute $BP\langle 2\rangle_* M(p^i, v_1^{j_1}, v_2^{j_2})$ via the $BF\langle 2\rangle$-homology exact sequences associated with the cofibrations defining $M(p^i, v_1^{j_1}, v_2^{j_2})$, we find that $BP\langle 2\rangle_* M(p^i, v_1^{j_1}, v_2^{j_2}) \cong \mathbf{Z}/p^i[v_1, v_2]/(v_1^{j_1}, v_2^{j_2})$. On the other hand, if we perform this calculation via the Atiyah-Hirzebruch-Dold spectral sequence

$$E^2_{s,t} \cong H_s(M(p^i, v_1^{j_1}, v_2^{j_2}); \pi_t BP\langle 2\rangle) \cong H_s M(p^i, v_1^{j_1}, v_2^{j_2}) \otimes \pi_t BP\langle 2\rangle \Rightarrow BP\langle 2\rangle_{s+t} M(p^i, v_1^{j_1}, v_2^{j_2})$$

(recall that $H_* M(p^i, v_1^{j_1}, v_2^{j_2}) \cong \mathbf{Z}/p^i \cdot a_0 \otimes \Lambda(a_1, a_2)$ with $|a_0| = 0$ and $|a_k| = j_k|v_k| + 1$ for $k=1$ and 2), we observe as in Example 1.3 that

$$d^{j_2|v_2|+1}_{j_2|v_2|+1,0}(a_2 \otimes 1) = \lambda a_0 \otimes v_2^{j_2} \,,$$

where λ is a generator of $\mathbf{Z}/p^i$. Then, take the map $f_{j_2}\colon BP\langle 2\rangle \to \Sigma^{-|v_1|} BP\langle 1\rangle$ and notice that the homomorphism $(f_{j_2})_*$ induced by f_{j_2} on homotopy maps 1 onto a multiple of v_1, say μv_1 with $\mu \in \mathbf{Z}_{(p)}$, since $(f_{j_2})_*(1)$ belongs to $\pi_{|v_1|} BP\langle 1\rangle$. The homomorphism $(f_{j_2})_*\colon E^{j_2|v_2|+1}_{*,*} \to \tilde{E}^{j_2|v_2|+1}_{*,*+|v_1|}$ given by Theorem 3.3 provides the diagram

$$\begin{array}{cccc} \lambda a_0 \otimes v_2^{j_2} \in & E^{j_2|v_2|+1}_{0,j_2|v_2|} & \xrightarrow{(f_{j_2})_*} & \tilde{E}^{j_2|v_2|+1}_{0,j_2|v_2|+|v_1|} \\ & \Big\uparrow d^{j_2|v_2|+1}_{j_2|v_2|+1,0} & & \Big\uparrow \tilde{d}^{j_2|v_2|+1}_{j_2|v_2|+1,|v_1|} \\ a_2 \otimes 1 \quad \in & E^{j_2|v_2|+1}_{j_2|v_2|+1,0} & \xrightarrow{(f_{j_2})_*} & \tilde{E}^{j_2|v_2|+1}_{j_2|v_2|+1,|v_1|} \,. \end{array}$$

Consequently, the commutativity of the diagram shows that $\tilde{d}^{j_2|v_2|+1}_{j_2|v_2|+1,|v_1|}(a_2 \otimes \mu v_1)$ is exactly the class of $\lambda a_0 \otimes v_1^{(p+1)j_2+1}$ in $\tilde{E}^{j_2|v_2|+1}_{0,j_2|v_2|+|v_1|}$. But recall that $BP\langle 1\rangle_* M(p^i, v_1^{j_1}, v_2^{j_2}) \cong \mathbf{Z}/p^i[v_1]/(v_1^{j_1}) \otimes \Lambda(w_2)$ (see Example 1.3). Therefore, there are only two systems of non-trivial differentials in the spectral sequence $\tilde{E}^*_{*,*}$:

$$\tilde{d}^{j_1|v_1|+1}_{j_1|v_1|+1,k|v_1|}(a_1 \otimes v_1^k) = \lambda_k a_0 \otimes v_1^{j_1+k}$$

and

$$\tilde{d}^{j_1|v_1|+1}_{j_1|v_1|+j_2|v_2|+2,k|v_1|}(a_1 a_2 \otimes v_1^k) = \lambda'_k a_2 \otimes v_1^{j_1+k}$$

for all $k \geq 0$, where λ_k and λ'_k are generators of $\mathbf{Z}/p^i$. Thus, we conclude that $\tilde{d}^{j_2|v_2|+1}_{j_2|v_2|+1,|v_1|}$ is trivial and that the class of $\lambda a_0 \otimes v_1^{(p+1)j_2+1}$ in $\tilde{E}^{j_2|v_2|+1}_{0,j_2|v_2|+|v_1|}$

is 0. However, it turns out that the only differential which may kill it is $\tilde{d}^{j_1|v_1|+1}_{j_1|v_1|+1,((p+1)j_2+1-j_1)|v_1|}$ starting from $a_1 \otimes v_1^{(p+1)j_2+1-j_1}$ (provided $(p+1)j_2+1-j_1 \geq 0$). This produces the following inequality in order to be in the right $\tilde{E}^r$:

$$j_2|v_2|+1 > j_1|v_1|+1 ,$$

or in other words,

$$j_2 > \frac{j_1|v_1|}{|v_2|} = \frac{j_1}{p+1}$$

(observe that the condition $(p+1)j_2+1-j_1 \geq 0$ is then trivially verified). Therefore we have deduced very easily from our general argument the following condition on the constructibility of $M(p^i, v_1^{j_1}, v_2^{j_2})$:

If the specrtum $M(p^i, v_1^{j_1}, v_2^{j_2})$ is constructible, then $j_1 \geq i$, $j_1 \geq i$ and

$$j_2 > \frac{j_1}{p+1} .$$

REMARK 3.6. Stronger results may be deduced from the fact that for any ideal $I \subset BP_*$, the realizability of BP_*/I as the BP-homology of a spectrum implies that I is an invariant ideal (see pages 138 and 319 of [8]). Thus the full power of BP_*BP cooperations can be brought to bear on the problem. Notice also that Lin gives conditions for the realizability of $M(p^i, v_1^{j_1}, v_2^{j_2})$ for the case $p \geq 5$ in [7].

REMARK 3.7. The method presented in the above example provides in general new information on the existence of connective spectra X such that $BP\langle 2\rangle_* X \cong \mathbf{Z}/p^i[v_1, v_2]/(v_1^{j_1}, v_2^{j_2}) \otimes A$ for some A.

REMARK 3.8. In the case when m is 1, Remark 2.2 allows us to compare the Atiyah-Hirzebruch-Dold spectral sequence for $BP\langle 1\rangle$-homology with that for ordinary homology with $\mathbf{Z}_{(p)}$-coefficients (whose E^2-term has only one non-trivial line).

REMARK 3.9. Of course, we would like to generalize our argument and obtain maps between the Atiyah-Hirzebruch-Dold spectral sequences for $BP\langle m\rangle$ and $BP\langle m-1\rangle$ for any $m \geq 1$. In particular, if we were able to check the hypothesis of Theorem 2.1, i.e., to show that for any $m \geq 2$ some power of v_{m-1} annihilates $BP\langle m-1\rangle^{j|v_m|+1}(BP\langle m\rangle/(v_m^j))$ for appropriate j's, then we could conclude that

$$j_m > \frac{j_{m-1}|v_{m-1}|}{|v_m|} = j_{m-1} \frac{p^{m-2}+p^{m-3}+\cdots p+1}{p^{m-1}+p^{m-2}+\cdots p+1}$$

for $2 \leq m \leq n$ in a constructible $M(p^i, v_1^{j_1}, v_2^{j_2}, \cdots, v_n^{j_n})$.

4. A modified Bockstein spectral sequence for $BP\langle 2\rangle$

In Proposition 5.14 of [5], Johnson and Wilson introduce a spectral sequence coming from the BP Bockstein cofibration sequence

$$\Sigma^{|v_n|}BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \to BP\langle n-1\rangle\,.$$

A similar spectral sequence can be obtained by gluing together the cofibrations in the bottom row of the first diagram in Section 2 to get the following tower of cofibrations.

$$\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
BP\langle 2\rangle/(v_2^{s+2}) & & \\
\downarrow g_{s+1} & \nwarrow & \\
BP\langle 2\rangle/(v_2^{s+1}) & \xrightarrow{l_{s+1}} & \Sigma^{(s+1)|v_2|+1}BP\langle 1\rangle \\
\downarrow g_s & \nwarrow & \\
BP\langle 2\rangle/(v_2^{s}) & \xrightarrow{l_s} & \Sigma^{s|v_2|+1}BP\langle 1\rangle \\
\downarrow & & \\
\vdots & & \\
\downarrow g_1 & & \\
BP\langle 2\rangle/(v_2) & \xrightarrow{l_1} & \Sigma^{|v_2|+1}BP\langle 1\rangle \\
\downarrow g_0 & \nwarrow & \\
* & \longrightarrow & \Sigma BP\langle 1\rangle
\end{array}$$

The diagonal maps, Δ_s, are the cofibers of the l_s's and have degree one. If we smash everything with a spectrum X and then take homotopy, we get a spectral sequence with

$$E^1_{s,t} = (\Sigma^{s|v_2|+1}BP\langle 1\rangle)_t(X)$$
$$d^r_{s,t}\colon E^r_{s,t} \to E^r_{s+r,t-1}$$

(see Boardman's [3], Section 4). We call this the *modified Bockstein spectral sequence* for $BP\langle 2\rangle_* X$, because, as we will show later, its E^∞-term is analogous to that of the ordinary Bockstein spectral sequence. It has the same E^1-term, up to a change of grading, as the spectral sequence of [5]. Note that the inverse limit of the vertical maps in the tower is $BP\langle 2\rangle$.

Let us analyze the convergence of this spectral sequence. For convenience we will assume that X is (-1)-connected and we set $M=BP\langle 2\rangle_* X$ and write

$A_{s,t}$ for $BP\langle 2\rangle/(v_2^s)_t X$, and for any graded module D, write D_t to mean the elements of degree t in D, so that we have the short exact sequence

$$0\to (M/v_2^s\cdot M)_t \xrightarrow{i_s} A_{s,t} \xrightarrow{p_s} (\operatorname{Ker}\{\cdot v_2^s\colon M\to M\})_{t-s|v_2|-1}\to 0\,.$$

In the terminology of [3], this is a right half plane spectral sequence so that, by Theorem 9.2 of [3], if $\operatorname{R}\varprojlim_r Z^r_{s,t}=0$, where $Z^r_{s,t}=\Delta_s^{-1}(\operatorname{Im}\{g_{s+1}\circ\cdots\circ g_{s+r-1}\colon A_{s+r,t}\to A_{s+1,t}\})$, this spectral sequence converges strongly to $\Sigma\varprojlim_j(BP\langle 2\rangle/(v_2^j)_*X)$, filtered by $F_s=\operatorname{Ker}\{\varprojlim_j A_j\to A_s\}$. We have two jobs, show that $\operatorname{R}\varprojlim_r Z^r_{s,t}=0$ and determine F_s/F_{s+1}, which is of more interest than the group being filtered, as is usual for Bockstein spectral sequences.

The whole strategy for both computations, is to work with the much simpler systems, $C_s=M/v_2^s\cdot M$ and $B_s=\operatorname{Ker}\{\cdot v_2^s\colon M\to M\}$ and then use the short exact sequence of inverse systems

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ 0 & \to & C_{s+1} & \xrightarrow{i_{s+1}} & A_{s+1} & \xrightarrow{p_{s+1}} & B_{s+1} & \to & 0\\ & & \downarrow g'_s & & \downarrow g_s & & \downarrow g''_s & & \\ 0 & \to & C_s & \xrightarrow{i_s} & A_s & \xrightarrow{p_s} & B_s & \to & 0\\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

to derive the results for A_s.

Lemma 4.1. $\varprojlim_s A_s\cong\varprojlim_s C_s$.

Proof. If x is an element of B_{s+1}, then $g''_s(x)=v_2x\in B_s$ because of the definition of g_s given by the first diagram of Section 2. Now use the exact sequence (see 1.8 of [3])

$$0\to\varprojlim_s B_s\to\Pi_s B_s\xrightarrow{d}\Pi_s B_s\to\operatorname{R}\varprojlim_s B_s\to 0\,,\qquad (*)$$

where $(dx)_s=x_s-g''_s(x_{s+1})=x_s-v_2x_{s+1}$. But since $M_*=0$ for $*<0$, for any $x\in M_t$ there exists an integer k (with $0\le k\le\frac{t}{|v_2|}$) and an element $y\in M_{t-k|v_2|}$ such that $x=v_2^k y$ and $y\notin\operatorname{Im}\{\cdot v_2\}$. This shows that d is injective and that $\varprojlim_s B_s=0$. Finally, from the six term exact sequence

$$0\to\varprojlim_s C_s\to\varprojlim_s A_s\to\varprojlim_s B_s\to\operatorname{R}\varprojlim_s C_s\to\operatorname{R}\varprojlim_s A_s\to\operatorname{R}\varprojlim_s B_s\to 0\,,$$

we get the desired isomorphism.

Now we want to identify F_s/F_{s+1}. First, define $F'_s=\operatorname{Ker}\ \{\varprojlim_j C_j\to C_s\}$.

A diagram chase around

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ F'_s & \hookrightarrow & \varprojlim_j C_j & \longrightarrow & C_s \\ \downarrow & & \downarrow \cong & & \downarrow \\ F_s & \hookrightarrow & \varprojlim_j A_j & \longrightarrow & A_s \end{array}$$

(where the two rows and the last column are exact), shows that $F_s \cong F'_s$: so it is sufficient to identify F'_s/F'_{s+1}. If $x \in M$, let us call x'_s the class of x in C_s: it is then clear that $g'_s(x'_{s+1}) = x'_s$. Now from the sequence $(*)$ for $\varprojlim_s C_s$, we see that $\varprojlim_s C_s$ is the usual (v_2)-adic completion of M. Thus $F'_s/F'_{s+1} = v_2^s \cdot M/v_2^{s+1} \cdot M$ and we have proved the following

Lemma 4.2. $F_s/F_{s+1} \cong v_2^s \cdot M/v_2^{s+1} \cdot M$.

Finally we have to show that the spectral sequence converges. Although we only really need weak convergence, Theorem 9.2 of [3] implies that our filtration makes weak convergence equivalent to strong convergence and that we may verify the convergence of the spectral sequence to $\Sigma \varprojlim_j (BP\langle 2\rangle/(v_2^j)_* X) \cong \Sigma BP\langle 2\rangle_* X$ (see Lemma 4.1) by checking that $\operatorname{Rlim}_{\leftarrow r} Z^r_{s,t} = 0$. Here we definitely need to fix a value of t before taking these limits. Since the maps in the B system change t, we are going to regrade B, taking $B_{s,t}$ to mean the image of $A_{s,t}$, i.e. $B_{s,t}$ means elements of degree $t - s|v_2| - 1$ annihilated by multiplication by v_2^s so that $B_{s,t} = 0$ for $t \leq s|v_s|$. Having fixed s and t, pick r to be any integer with $(r+s)|v_2| \geq t+1$. Now, a non-zero element x in $Z^r_{s,t}$ corresponds to an element x_{s+r} in $A_{s+r,t}$ with $y = g_{s+1} \circ \cdots \circ g_{s+r-1}(x_{s+r}) \neq 0$ but $g_s(y) = 0$. Let $Z'^r_{s,t}$ be defined similarly: $Z'^r_{s,t} = \Delta_s^{-1}(\operatorname{Im}\{i_{s+1} \circ g'_{s+1} \circ \cdots \circ g'_{s+r-1} \colon C_{s+r,t} \to C_{s+1,t}\})$. For $x' \neq 0 \in Z'^r_{s,t}$ we have an element x'_{s+r} in $C_{s+r,t}$ with analogous conditions about the action of the g'_j's: $y' = i_{s+1} \circ g'_{s+1} \circ \cdots \circ g'_{s+r-1}(x'_{s+r}) \neq 0$ but $g_s(y') = 0$. But since $B_{s+r,t} = 0$, we deduce that $i_{s+r} \colon C_{s+r} \to A_{s+r}$ is an isomorphism and it follows from the fact that $g_j \circ i_{j+1} = i_j \circ g'_j$ for all j that

$$Z^r_{s,t} \cong Z'^r_{s,t} \text{ for all } r \geq \frac{t+1}{|v_2|} - s\,.$$

Now, because C_{r+1} maps surjectively onto C_r, $Z'^{r+1}_{s,t}$ maps onto $Z'^r_{s,t}$ so that from $(*)$ we get

Lemma 4.3. $\operatorname{Rlim}_{\leftarrow r} Z^r_{s,t} = \operatorname{Rlim}_{\leftarrow r} Z'^r_{s,t} = 0$.

Finally, what we have proved is

Theorem 4.4. *The modified Bockstein spectral sequence for $BP\langle 2\rangle_* X$, where X is (-1)-connected, converges strongly to $BP\langle 2\rangle_* X$ filtered by* $F_{s,t}=\mathrm{Ker}\{\lim\limits_{\longleftarrow j} BP\langle 2\rangle/(v_2^j)_t X \to BP\langle 2\rangle/(v_2^s)_t X\}$ *with*

$$E^\infty_{s,t+1} \cong F_{s,t}/F_{s+1,t} = v_2^s \cdot BP\langle 2\rangle_{t-s|v_2|} X / v_2^{s+1} \cdot BP\langle 2\rangle_{t-(s+1)|v_2|} X\,.$$

Being a kind of Bockstein spectral sequence, although it is meant to give us a schema for computing $BP\langle 2\rangle$-homology from $BP\langle 1\rangle$-homology, in practice what is of interest are the differentials. Since

$$v_1^{(p+1)j+1} \circ l_j = 0$$

(see Corollary 3.2), we get:

Theorem 4.5. *In the modified Bockstein spectral sequence for $BP\langle 2\rangle_* X$,*

$$v_1^{(p+1)(r+s)+1} d^r_{s,t} = 0\,.$$

Proof. Notice that the horizontal arrows, l_j, in the tower of cofibrations are elements of the groups which Corollary 3.2 tells us are killed by appropriate powers of v_1. Since the definition of the diffeιential $d^r_{s,t}$ involves composing a certain class of maps in $BP\langle 2\rangle/(v_2^{s+r})_*(X)$ with the homomorphism induced by

$$l_{s+r}\colon BP\langle 2\rangle/(v_2^{s+r}) \to \Sigma^{(s+r)|v_2|+1} BP\langle 1\rangle\,,$$

we get the desired result. Notice that the same result holds for the cohomology version of this spectral sequence.

EXAMPLE 4.6. Again we apply this result to the study of the constructibility of $M(p^i, v_1^{j_1}, v_2^{j_2})$ and not surprisingly get a similar result. First, setting $J=\mathbf{Z}/p^i[v_1]/(v_1^{j_1})$ so that $BP\langle 1\rangle_* M(p^i, v_1^{j_1}, v_2^{j_2}) \cong J \oplus \Sigma^{j_2|v_2|+1} J$, we find that

$$E^1_{s,*} = \Sigma^{s|v_2|+1} J\cdot \alpha_s \oplus \Sigma^{(s+j_2)|v_2|+2} J\cdot \beta_s\,.$$

Then, for degree reasons, $d^r_{s,*}(\alpha_s)=0$ for all r and s as well as $d^r_{*,*}=0$ for all $r>j_2$. Since $BP\langle 2\rangle_* M(p^i, v_1^{j_1}, v_2^{j_2}) \cong \mathbf{Z}/p^i[v_1, v_2]/(v_1^{j_1}, v_2^{j_2})$, we get:

$$E^\infty_{s,*} \cong v_2^s \cdot BP\langle 2\rangle_* M(p^i, v_1^{j_1}, v_2^{j_2})/v_2^{s+1}\cdot BP\langle 2\rangle_* M(p^i, v_1^{j_1}, v_2^{j_2}) \cong \begin{cases} \Sigma^{s|v_2|+1} J, & \text{if } s<j_2 \\ 0 &, \text{if } s\geq j_2. \end{cases}$$

The first case tells us that $d^r_{s-r,*}=0$ for $s<j_2$ (i.e. no multiple of α_s is in the image of a differential when $s<j_2$), while the second case coupled with the fact that $d^r_{*,*}=0$ for $r>j_2$ tells us that $d^{j_2}_{0,*}(\beta_0)=\lambda\alpha_{j_2}$ where λ generates $\mathbf{Z}/p^i$. Finally, by Theorem 4.5 we have

$$\begin{array}{ccc} d_{0,*}^{j_2}(\beta_0) & = & \lambda\alpha_{j_2} \\ \Big\downarrow \cdot v_1^{(p+1)j_2+1} & & \Big\downarrow \cdot v_1^{(p+1)j_2+1} \\ 0 & = & \lambda v_1^{(p+1)j_2+1}\alpha_{j_2} \in \mathbf{Z}/p^i[v_1]/(v_1^{j_1}). \end{array}$$

Therefore, we must have $(p+1)j_2+1>j_1-1$, i.e., $j_2>\dfrac{j_1-2}{p+1}$ in a constructible $M(p^i, v_1^{j_1}, v_2^{j_2})$.

References

[1] J.F. Adams: *On the groups J(X)-IV*, Topology **5** (1966), 21–71.

[2] D. Arlettaz: *The order of the differentials in the Atiyah-Hirzebruch spectral sequence*, K-Theory **6** (1992), 347–361.

[3] M. Boardman: *Conditionally convergent spectral sequences*, preprint.

[4] H. Cartan: *Algèbres d'Eilenberg-MacLane et homotopie, exposé* 11, Séminaire H. Cartan Ecole Norm. Sup. (1954/1955).

[5] D.C. Johnson and W.S. Wilson: *Projective dimension and Brown-Peterson homology*, Topology **12** (1973), 327–353.

[6] J. Klippenstein: *The bu cohomology of BP⟨n⟩*, J. Pure Appl. Algebra **57** (1989), 127–140.

[7] J. Lin: *Detection of second periodicity families in stable homotopy of spheres*, Amer. J. Math. **112** (1990), 595–610.

[8] D.C. Ravenel: Complex cobordism and stable homotpy groups of spheres, Academic Press, 1986.

[9] N. Yagita: *Equivariant BP-cohomology for finite groups*, Trans. Amer. Math. Soc. **317** (1990), 485–499.

Dominique Arlettaz
Institut de mathématiques
Université de Lausanne
CH-1015 Lausanne
Switzerland

John Klippenstein
Department of Mathematics
University of British Columbia
Vancouver, B.C. V6T 1Z2
Canada

Current address:
John Klippenstein
Didatech Software
200–4250 Dawson Street
Burnaby, B.C.
Canada