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CENTRALIZER ALGEBRAS OF THE MIXED TENSOR REPRESENTATIONS OF QUANTUM GROUP $U_q(gl(n, C))$*

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Introduction

Let $G=GL_n(C)$ be the group of linear transformations of the n-dimensional C-vector space V_n. Let V_n^* be the dual space of V_n and let $V_n^{(N,M)}:=V_n^{\otimes N}\otimes(V_n^*)^{\otimes M}$ be the (N, M)-mixed tensor power of V_n. We denote the representation of G on $V_n^{(N,M)}$ by $\Phi^{(N,M)}$. The decomposition of $\Phi^{(N,M)}$ into a sum of irreducible representations of G is given in [8] and [13]. This result also indicates the structure of the centralizer algebra of $\Phi^{(N,M)}(G)$.

On the other hand, Jimbo [5, 6] showed that the centralizer algebra of the natural representation of the quantum algebra $U_q(gl(n, C))$ on $V_n^{(N,0)}$ is isomorphic to the Iwahori Hecke algebra if $n\geq N$ and $q\in C$ is generic.

In the present paper we introduce a generalization $H_{N,M}^n(q)$ of the Iwahori Hecke algebra of type A, which is defined by generators and relations. (See Section 2.) Our main result says that the algebra $H_{N,M}^n(q)$ is semisimple and is isomorphic to the centralizer algebra $C_n^{(N,M)}(q)$ of the natural representation of $U_q(gl(n, C))$ on $V_n^{(N,M)}$, if $n\geq N+M$ and q is generic. (See Theorem 6.7).

To prove the above fact, we construct all the irreducible representations of $H_{N,M}^n(q)$ in Section 4, using the Bratteli diagram of the inclusions $C\subseteq H_{1,0}^n(q)\subset H_{2,0}^n(q)\subset\cdots\subset H_{N,0}^n(q)\subset H_{N,1}^n(q)\subset\cdots\subset H_{N,M}^n(q)$ as in [3, 16]. We also give Markov traces of these algebras, which are related to the HOMFLY polynomial of knots and links as in [14]. (See [9].) In the special case $N=M$ or $M\pm1$, the algebra $H_n^{(N,M)}(q)$ is also studied in [2] from a different view point. Some results of this paper have been announced in [9].

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1. Graph $\Gamma_{N,M}$

Let $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ be an integer sequence, and define $|\lambda|=\lambda_1+\lambda_2+\cdots$

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$+\lambda_n$. We call a *partition* of N if the sequence is nonnegative, weakly decreasing, and $|\lambda|=N$. Two partitions $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and $(\lambda_1, \lambda_2, \cdots, \lambda_n, 0)$ are considered to be the same. The *length* $l(\lambda)$ of λ is the number of nonzero terms in λ. Let ϕ be the partition of 0.

Let $\gamma=(\gamma_1, \gamma_2, \cdots, \gamma_n)$ be an integer sequence in which $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$. We will refer to such objects as *staircases of height* n. There are two standard ways in indexing staircases of height n. One is to index them by the pairs (r, λ) of $r \in \boldsymbol{Z}$ and partitions λ with $l(\lambda) \leq n$; the staircase corresponding to (r, λ) is $(\lambda_1+r, \lambda_2+r, \cdots, \lambda_n+r)$ if $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$. The other is to index them by the ordered pairs $[\alpha, \beta]_n$ of partitions with $l(\alpha)+l(\beta) \leq n$; the staircase corresponding to $[\alpha, \beta]_n$ is $(\alpha_1, \alpha_2, \cdots, \cdots, -\beta_2, -\beta_1)$ if $\alpha=(\alpha_1, \alpha_2, \cdots)$ and $\beta=(\beta_1, \beta_2, \cdots)$.

EXAMPLE 1.1. If $n=7$, $\alpha=(3, 2, 1)$ and $\beta=(2, 2, 1)$, then $\gamma=[\alpha, \beta]_n=(3, 2, 1, 0, -1, -2, -2)$.

Staircases of height n are partially ordered by defining $\gamma \supseteq \delta$ if and only if $\gamma_1 \leq \delta_1, \gamma_2 \leq \delta_2, \cdots, \gamma_n \leq \delta_n$.

DEFINITION 1.2. Let N and M be nonnegative integers with $N+M \geq 1$. Let n be a fixed integer such that $n \geq N+M$. An *up down tableau of type* (N, M) *and shape* γ is a sequence $\phi=\gamma^{(0)}, \gamma^{(1)}, \cdots, \gamma^{(N+M)}=\gamma$ of staircases of height n in which $\gamma^{(i)} \supseteq \gamma^{(i-1)}$, $|\gamma^{(i)}|-|\gamma^{(i-1)}|=1$ (if $i \leq N$), $\gamma^{(i)} \subseteq \gamma^{(i-1)}$, $|\gamma^{(i)}|-|\gamma^{(i-1)}|=-1$ (if $i>N$). (Note that the notion of up down tableaux of type (N, M) and shape γ is essentially independent of the choice of $n \geq N+M$.)

DEFINITION 1.3. A *standard tableau of shape* λ with $|\lambda|=N$ is a sequence of $N+1$ nested partitions, $\phi=\lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(N)}=\lambda$, in which $|\lambda^{(i)}|-|\lambda^{(i-1)}|=1$ for $1 \leq i \leq N$. We denote by f^λ the number of standard tableaux of shape λ. In other words, f^λ is the dimension of the irreducible character χ^λ of the symmetric group S_n in n letters.

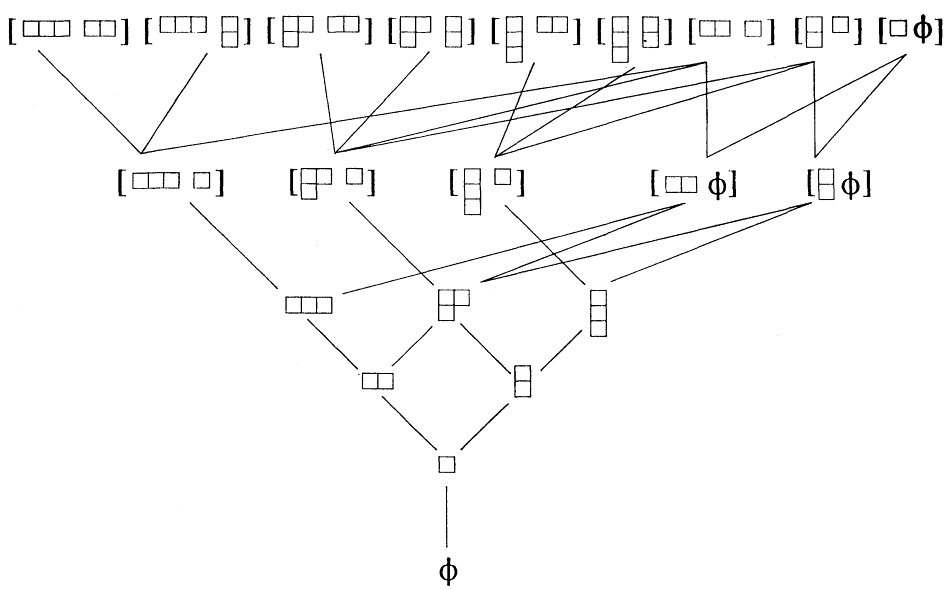
In this paper we consider tableaux of type (N, M) for a fixed (N, M). So, in the following, the terminology "tableaux" always means "up down tableaux of type (N, M)". All the tableaux are conveniently described using the graph $\Gamma_{N,M}$ defined below. Vertices of $\Gamma_{N,M}$ are assigned to $N+M+1$ floors. The vertices in the i-th floor $(0 \leq i \leq N+M)$ of $\Gamma_{N,M}$ are indexed by the staircases which appear in a tableau $(\gamma^{(0)}, \gamma^{(1)}, \cdots, \gamma^{(N+M)})$ as the $(i+1)$-th coordinate $\gamma^{(i)}$. Two vertices indexed by γ and γ' are joined by an edge if and only if there exists a tableau which has both γ and γ' as successive coordinates. If $i \leq N$, the number of vertices on the i-th floor is the same as the number of partitions of i with length less than or equal to n.

Let $\Lambda(N, M)$ be a set of staircases of height n defined by

$$\Lambda(N, M) = \prod_{m=0}^{\min(N,M)} \{[\alpha, \beta]_n;\ \alpha, \beta \text{ partitions, } |\alpha| = N-m,\ |\beta| = M-m\}\ .$$

We can easily see that the vertices on the $(N+M)$-th floor are indexed by the elements of $\Lambda(N, M)$.

EXAMPLE 1.4. When $N=3$, $M=2$ and $n\geq M+N$, the graph $\Gamma_{N,M}$ is:



We can get any tableau of shape γ from the graph $\Gamma_{N,M}$ as an ascending path from the bottom vertex ϕ to the top vertex γ. Conversely, any ascending path from the bottom vertex ϕ to a top vertex γ expresses some tableau. We identify each of these paths with the corresponding tableau. Let γ be a staircase of height n and let $c^{\gamma}_{N,M}$ denote the number of up down tableaux of shape γ and type (N, M). We have an explicit formula for $c^{\gamma}_{N,M}$ (see Proposition 4.8 in [13]):

Propostion 1.5. *Let $\gamma=[\alpha, \beta]_n$ be a staircase of height n. We assume $|N|-|\alpha|=|M|-|\beta|\geq 0$. (Otherwise $c^{\gamma}_{N,M}=0$.) Then*

$$c^{[\alpha,\beta]}_{N,M} = m!\binom{N}{m}\binom{M}{m}f^{\alpha}f^{\beta}\,,$$

where $m=N-|\alpha|=M-|\beta|$.

Let $\Omega=\Omega_{N,M}$ denote the set of tableaux on $\Gamma_{N,M}$. Let $K\Omega$ be the K-vector space with basis Ω over fi a fied K. We define an algebra $A=A_{N,M}\subseteq \mathrm{End}_K(K\Omega)$ as follows. Let $R=\{(\xi, \eta)\in\Omega\times\Omega;$ shape of $\xi=$shape of $\eta\}$. For $(\xi, \eta)\in R$, define $T_{\xi,\eta}\in\mathrm{End}_K(K\Omega)$ by $T_{\xi,\eta}\,\omega=\delta(\eta, \omega)\,\xi\,(\omega\in\Omega)$. Let A be the K-linear span of $\{T_{\xi,\eta}; (\xi, \eta)\in R\}$ in $\mathrm{End}_K(K\Omega)$. Since

$$T_{\xi,\eta}\, T_{\xi',\eta'} = \delta(\eta, \xi')\, T_{\xi,\eta'} \tag{1.6}$$

and $1=\sum_{\xi\in\Omega} T_{\xi,\xi}$, A is a subalgebra of $\mathrm{End}_K(K\Omega)$. For $\gamma\in\Lambda(N, M)$ set $\Omega^\gamma=\{\xi\in\Omega;$ shape of $\xi=\gamma\}$ so that $\Omega=\coprod_{\gamma\in\Lambda(N,M)}\Omega^\gamma$ (disjoint union). It follows from the multiplication law (1.6) for the $T_{\xi,\eta}$ that $A^\gamma=\mathrm{span}\,\{T_{\rho,\eta}; (\xi, \eta)\in\Omega^\gamma\times\Omega^\gamma\}$ is an ideal of A and $A=\oplus_\gamma A^\gamma$ $(\gamma\in\Lambda(N, M))$. Since there exists a natural isomorphicm $A^\gamma\cong\mathrm{End}_K(K\Omega^\gamma)$, we have $A=\oplus_{\gamma\in\Lambda(N,M)} A^\gamma\cong\oplus_{\gamma\in\Lambda(N,M)}\mathrm{End}_K(K\Omega^\gamma)$. Note that the minimal central idempotents in A have the form $z_\gamma=\sum_{\xi\in\Omega^\gamma} T_{\xi,\xi}$.

Lenna 1.7. *Let N and M be nonnegative integers. Then*

$$\sum_{m=0}^{\min(N,M)}(m!)^2\binom{N}{m}^2\binom{M}{m}^2(N-m)!\,(M-m)! = (N+M)!\,.$$

Proof. Consider the group S_{N+M} of permutations of $N+M$ letters $\{1, 2, \cdots, N, \bar{1}, \bar{2}, \cdots, \bar{M}\}$, For $0\leq m\leq\min(N, M)$, let $S_{N+M}^{(m)}=\{\sigma\in S_{N+M};\ \mathrm{Card}\,(\sigma(\{1, 2, \cdots, N\})\cap\{\bar{1}, \bar{2}, \cdots, \bar{M}\})=m\}$. Then $S_{N+M}=\cup_{m=0}^{\min(N,M)} S_{N+M}^{(m)}$ (disjoint). Now the lemma follows by observing that

$$\mathrm{Card}(S_{N+M}^{(m)}) = (m!)^2\binom{N}{m}^2\binom{M}{m}^2(N-m)!\,(M-m)!\,.$$

Corollary 1.8. *The dimension of the algebra $A=A_{N,M}$ is $(N+M)!$.*

Proof. Using Proposition 1.5, we have

$$\begin{aligned}\dim A &= \sum_{\gamma\in\Lambda(N,M)}\dim(K\Omega^\gamma)^2\\ &=\sum_{\gamma\in\Lambda(N,M)}\left\{m!\binom{N}{m}\binom{M}{m}f^\alpha f^\beta\right\}^2\\ &=\sum_{m=0}^{\min(N,M)}\sum_{|\alpha|=N-m}\sum_{|\beta|=M-m}(m!)^2\binom{N}{m}^2\binom{M}{m}^2(f^\alpha)^2(f^\beta)^2\\ &=\sum_{m=0}^{\min(N,M)}(m!)^2\binom{N}{m}^2\binom{M}{m}^2(N-m)!\,(M-m)!\\ &=(N+M)!\,,\end{aligned}$$

where we have used the well known formula $\sum_{|\alpha|=r}(f^\alpha)^2=r!$.

2. Algebra $H_{N,M}^n(q)$

We are going to define a generalization of the Iwahori Hecke algebra of type A (see, e.g. [4]). For an indeterminate q, we define the *q-integer* $[i]$ by

$$[i] = \frac{q^i-q^{-i}}{q-q^{-1}} = q^{i-1}+q^{i-3}+\cdots+q^{1-i}\,.$$

Similarly for nay $q_0 \in K \setminus \{0\}$, $[i]_{q_0}$ is defined by

$$[i]_{q_0} = q_0^{i-1} + q_0^{i-3} + \cdots + q_0^{1-i}\,.$$

In particular we have $[i]_1 = i$. Note that $[0] = [0]_{q_0} = 0$ and $[1] = [1]_{q_0} = 1$ for any q_0.

DEFINITION 2.1. Let K be an arbitrary field and q an indeterminate over K. For integers $N, M \geq 0$ and $n \geq N+M$, we define $H^n_{N,M}(q)$ to be the associative $K(q)$-algebra with unit presented by

generators:

$$\begin{array}{ll} T_1, T_2, \cdots, T_{N-1} & \\ \qquad E & (\text{if } M \geq 1) \\ T_1^*, T_2^*, \cdots, T_{M-1}^* & (\text{if } M \geq 2) \end{array}$$

and

relations:

$$\begin{array}{lll}
(1.\mathrm{a}) & (T_i - q)(T_i + q^{-1}) = 0 & (1 \leq i \leq N-1)\,, \\
(1.\mathrm{b}) & T_i\, T_j = T_j\, T_i & (1 \leq i, j \leq N-1,\ |i-j| \geq 2)\,, \\
(1.\mathrm{c}) & T_i\, T_{i+1}\, T_i = T_{i+1}\, T_i\, T_{i+1} & (1 \leq i \leq N-2)\,, \\
(2.\mathrm{a}) & (T_i^* - q)(T_i^* + q^{-1}) = 0 & (1 \leq i \leq M-1)\,, \\
(2.\mathrm{b}) & T_i^*\, T_j^* = T_j^*\, T_i^* & (1 \leq i, j \leq M-1,\ |i-j| \geq 2)\,, \\
(2.\mathrm{c}) & T_i^*\, T_{i+1}^*\, T_i^* = T_i^{*+1}\, T_i^*\, T_{i+1}^* & (1 \leq i \leq M-2)\,, \\
(3) & E^2 = [n]\, E\,, & \\
(4.\mathrm{a}) & T_i\, T_j^* = T_j^*\, T_i & (1 \leq i \leq N-1,\ 1 \leq j \leq M-1)\,, \\
(4.\mathrm{b}) & E\, T_i = T_i\, E & (1 \leq i \leq N-2)\,, \\
(4.\mathrm{c}) & E\, T_i^* = T_i^*\, E & (2 \leq i \leq M-1)\,, \\
(5) & E\, T_{N-1}\, E = q^n\, E\,, & \\
(6) & E\, T_1^*\, E = q^n\, E\,, & \\
(7) & E\, T_{N-1}^{-1}\, T_1^*\, E(T_{N-1} - T_1^*) = 0\,, & \\
(8) & (T_{N-1} - T_1^*)\, E\, T_{N-1}^{-1}\, T_1^*\, E = 0\,. &
\end{array}$$

Note that $T_i^{-1} = T_i - (q - q^{-1})$ by (1.a). A *monomial* in $H^n_{N,M}(q)$ is a product $\chi_1 \cdots \chi_p$, where $\chi_i \in \{T_1^{-1}, \cdots, T_{N-1}^{-1}, E, T_1^*, \cdots, T_{M-1}^*\}$, $1 \leq i \leq p$.

Proposition 2.2. $\dim_{K(q)} H^n_{N,M}(q) \leq (N+M)!$.

The proof of this proposition will occupy the remainder of this section. We shall prove it by defining *monomials in normal form in* $H^n_{N,M}(q)$, which will eventually shown to form a basis of $H^n_{N,M}(q)$ as a vector space over $K(q)$. (See Corollary 4.13.) Consider the following sets of monomials.

$$\begin{aligned}
S_1 &= \{1, T_1^{-1}\}, \\
S_2 &= \{1, T_2^{-1}, T_2^{-1} T_1^{-1}\}, \\
\vdots &\quad \vdots \quad \vdots \\
S_i &= \{1, T_i^{-1}, T_i^{-1} T_{i-2}^{-1}, \cdots, T_1^{-1} T_{i-1}^{-1} \cdots T_i^{-1}\}, \\
\vdots &\quad \vdots \quad \vdots \\
S_{N-1} &= \{1, T_{N-1}^{-1}, T_{N-1}^{-1} T_{N-2}^{-1}, \cdots, T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_1^{-1}\}.
\end{aligned}$$

Note that $V_i \in S_i$ implies $T_{i+1}^{-1} V_i \in S_{i+1}$. We shall say that $M_0 = U_1 U_2 \cdots U_{N-1}$ is a *monomial in normal form in* $H^n_{N,0}(q)$, if $U_i \in S_i$ for $i=1, 2, \cdots, N-1$. There are $N!$ of monomials in normal form in $H^n_{N,0}(q)$. As is shown in [4], we have

Lemma 2.3. *The monomials in normal form in $H^n_{N,0}(q)$ generate $H^n_{N,0}(q)$ as a vector space over $K(q)$.*

Next, we define monomials in normal form in $H^n_{0,M}(q)$. Consider the following sets of monomials.

$$\begin{aligned}
S_1^* &= \{1, T_1^*\}, \\
S_2^* &= \{1, T_2^*, T_2^* T_1^*\}, \\
\vdots &\quad \vdots \quad \vdots \\
S_i^* &= \{1, T_i^*, T_i^* T_{i-1}^*, \cdots, T_i^* T_{i-1}^* \cdots T_1^*\}, \\
\vdots &\quad \vdots \quad \vdots \\
S_{M-1}^* &= \{1, T_{M-1}^*, T_{M-1}^* T_{M-2}^*, \cdots, T_{M-1}^* T_{M-2}^* \cdots T_1^*\}.
\end{aligned}$$

We shall say that $M_0 = U_1^* U_2^* \cdots U_{M-1}^*$ is a *monomial in normal form in* $H^n_{N,0}(q)$, if $U_i^* \in S_i^*$ for $i=1, 2, \cdots, M-1$. There are $M!$ of monomials in normal form in $H^n_{0,M}(q)$. Similarly to Lemma 2.3, we have

Lemma 2.4. *The monomials in normal form in $H^*_{0,M}(q)$ generate $H^n_{0,M}(q)$ as a vector space over $K(q)$.*

Now we define monomials in normal form in $H^n_{0,M}(q)$ for general N and M. We shall say that M_0 *is a monomial in normal form in* $H^n_{N,M}(q)$ if

$$M_0 = M_1 T_{i_1}^* T_{i_1-1}^* \cdots T_1^* E T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_{j_1}^{-1} T_{i_2}^* T_{i_2-1}^* \cdots T_1^* E T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_{j_2}^{-1} \\ \cdots T_{i_m}^* T_{i_m-1}^* \cdots T_1^* E T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_{j_m}^{-1} M_2^*$$

for some m $(0 \le m \le \min(N, M))$, where M_1 is a monomial in normal form in $H^n_{N,0}(q)$, M_2^* is a monomial in normal form in $H^n_{0,M}(q)$, and $0 \le i_1 < i_2 < \cdots < i_m \le M-1$, $1 \le j_1 < j_2 < \cdots < j_m \le N$. Here we understand that $T_{i_1}^* \cdots T_1^* = 1$ if $i_1 = 0$, and that $T_{N-1}^{-1} \cdots T_{j_m}^{-1} = 1$ if $j_m = N$. Note that if $m=0$ then $M_0 = M_1 M_2^*$. The number of monomials in normal form in $H^n_{N,M}(q)$ is

$$\sum_{m=0}^{\min(N,M)} N!\,M!\binom{N}{m}\binom{M}{m} = \sum_{m=0}^{\min(N,M)} (m!)^2 \binom{N}{m}^2 \binom{M}{m} (N-m)!\,(M-m)!\,,$$

which is equal to $(N+M)!$ by Lemma 1.7.

Lemma 2.5.

(1) *If* $M>k>i\geq j\geq l\geq 1$ *then*
$(T_i^{*\pm 1})(T_k^* T_{k-1}^* \cdots T_l^*) = (T_k^* T_{k-1}^* \cdots T_l^*)(T_{i+1}^{*\pm 1})$
$(T_i^* T_{i-1}^* \cdots T_j^*)(T_k^* T_{k-1}^* \cdots T_l^*) = (T_k^* T_{k-1}^* \cdots T_l^*)(T_{i+1}^* T_i^* \cdots T_{j+1}^*)$.

(2) *If* $N>i\geq k\geq l>j\geq 1$ *then*
$(T_i^{-1} T_{i-1}^{-1} \cdots T_j^{-1})(T_k^{\pm 1}) = (T_{k-1}^{\pm 1})(T_i^{-1} T_{i-1}^{-1} \cdots T_j^{-1})$,
$(T_i^{-1} T_{i-1}^{-1} \cdots T_j^{-1})(T_k^{-1} T_{k-1}^{-1} \cdots T_l^{-1}) = (T_{k-1}^{-1} T_{k-2}^{-1} \cdots T_{l-1}^{-1})(T_i^{-1} T_{i-1}^{-1} \cdots T_j^{-1})$.

(3) $E\, T_{N-1}^{-1}\, E = q^{-n} E$.

(4) $(T_i^* T_{i-1}^* \cdots T_1^* E\, T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_j^{-1})(T_i^* T_{i-1}^* \cdots T_1^* E)$
$= (T_{N-1})(T_{i-1}^* T_{i-2}^* \cdots T_1^* E\, T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_j^{-1})(T_i^* T_{i-1}^* \cdots T_1^* E)$.

(5) *If* $N-r\geq j$ *then*
$(E\, T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_{N-r}^{-1})(T_i^* T_{i-1}^* \cdots T_1^* E)(T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_j^{-1})$
$= (E\, T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_j^{-1})(T_i^* T_{i-1}^* \cdots T_1^* E)(T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_{N-r+1}^{-1})(T_1^{*-1})$.

Proof. The first two identities follow from Definition 2.1 (1.b), (1.c), (2.b) and (2.c). The identity (3) follows from 2.1 (3), (5). The rests follow from (1), (2) and 2.1 (4a, b, c), (7), (8).

Now we show that monomials in normal form in $H_{N,M}^n(q)$ generate $H_{N,M}^n(q)$ as a vector space over $K(q)$. This will imply Proposition 2.2.

Lemma 2.6. *If M_0 is a monomial in normal form in $H_{N,M}^n(q)$, then $T_i^* M_0$ can be written as a linear combination of monomials in normal form in $H_{N,M}^n(q)$.*

Proof. We first consider:
Case 1. $i_m+1<i$.

$$\begin{aligned} T_i^* M_0 &= (T_i^*)\, M_1\, T_{i_1}^* \cdots T_1^* E\, T_{N-1}^{-1} \cdots T_{j_1}^{-1} \cdot T_{i_m}^* \cdots T_1^* E\, T_{N-1}^{-1} \cdots T_{j_m}^{-1} M_2^* \\ &= M_1\, T_{i_1}^* \cdots T_1^* E\, T_{N-1}^{-1} \cdots T_{j_1}^{-1} \cdot T_{i_m}^* \cdots T_1^* E\, T_{N-1}^{-1} \cdots T_{j_m}^{-1} (T_i^*)\, M_2^*. \end{aligned}$$

Since $T_i^* M_2^*$ can be written as a linear combination of monomials in normal form in $H_{0,M}^n$, we get the lemma in Case 1. Next we consider:

Case 2. $i_k+1<i\leq i_{k+1}+1$ *for some* $k\,(0\leq k\leq m-1)$. We understand that $i_0+1=0$. We further divide this case into 3 cases.

Case 2.1. $i_k+1<i=i_{k+1}+1$ ($\leq i_{k+2}$ if $k\leq m-2$).

$$\begin{aligned} T_i^* M_0 &= T_{i_{k+1}+1}^* M_0 \\ &= M_1\, T_{i_1}^* T_{i_1-1}^* \cdots T_1^* E\, T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_{i_1}^{-1} \cdots \end{aligned}$$

$$(T^*_{i_{k+1}+1})\, T^*_{i_{k+1}} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_{k+1}}\, T^*_{j_{k+2}} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_{k+2}}$$
$$\cdots T^*_{i_m} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_m}\, M^*_2\,.$$

If $i=i_{k+1}+1<i_{k+2}$ or $k=m-1$ then the right hand side is a monomial in normal form. If $i=i_{k+1}+1=i_{k+2}$ then

$$\begin{aligned}
T^*_i\, M_0 &= M_1\, T^*_{i_1}\, T^*_{i_1-1} \cdots T^*_1\, E\, T^{-1}_{N-1}\, T^{-1}_{N-2} \cdots T^{-1}_{j_1} \cdots \\
&\quad \underline{T^*_{i_{k+2}} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_{k+1}}\, T^*_{j_{k+2}} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_{k+2}}} \cdots \\
&= M_1\, T^*_{i_1}\, T^*_{i_1-1} \cdots T^*_1\, E\, T^{-1}_{N-1}\, T^{-1}_{N-2} \cdots T^{-1}_{j_1} \cdots \\
&\quad \underline{(T_{N-1})\, T^*_{i_{k+1}} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_{k+1}}\, T^*_{j_{k+2}} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_{k+2}}}
\end{aligned}$$

by (4) of Lemma 2.5. Using (2) of Lemma 2.5 repeatedly, the above monomials is equal to $M_1(T_{N-(k+1)})\, T^*_{i_1} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_1} \cdots T^*_{i_m} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_m}\, M^*_2$. Since $M_1\, T_{N-(k+1)}$ can be written as a linear combination of monomials in normal form in $H^n_{N,0}(q)$, we get the lemma in Case 2.1.

Case 2.2. $i_k+1<i=i_{k+1}$ for some k.

$$\begin{aligned}
T^*_i\, M_0 &= T^*_{i_{k+1}}\, M_0 \\
&= M_1\, T^*_{i_1}\, T^*_{i_1-1} \cdots T^*_1\, E\, T^{-1}_{N-1}\, T^{-1}_{N-2} \cdots T^{-1}_{j_1} \cdots \\
&\quad (T^*_{i_{k+1}})^2\, T^*_{i_{k+1}-1} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_{k+1}} \cdots T^*_{i_m} \cdots T^{-1}_{j_m}\, M^*_2 \\
&= M_1\, T^*_{i_1}\, T_{i_1-1} \cdots T^*_1\, E\, T^{-1}_{N-1}\, T^{-1}_{N-2} \cdots T^{-1}_{j_1} \cdots \\
&\quad \{(q-q^{-1})\, T^*_{i_{k+1}}+1\}\, T^*_{i_{k+1}-1} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_{k+1}} \cdots T^{-1}_{j_m}\, M^*_2 \\
&= (q-q^{-1})\, M_0 \\
&\quad +M_1\, T^*_{i_1} \cdots T^{-1}_{j_1} \cdots T^*_{i_k} \cdots T^{-1}_{j_k}\, T^*_{i_{k+1}-1}\, T^*_{i_{k+1}-2} \cdots T^{-1}_{j_{k+1}} \\
&\qquad T^*_{i_{k+2}} \cdots T^{-1}_{j_{k+2}} \cdots T^*_{j_m} \cdots T^{-1}_{j_m}\, M^*_2\,.
\end{aligned}$$

Since $i_k<i_{k+1}-1$, the second term is also a monomial in normal form. Therefore $T^*_i\, M_0$ can be written in a desired form.

Case 2.3. $i_k+1<i<i_{k+1}$. By using (1) of Lemma 2.5 repeatedly, we get

$$\begin{aligned}
T^*_i\, M_0 &= M_1\, T^*_{i_1} \cdots T^{-1}_{j_1} \cdots (T^*_i)\, T^*_{j_{k+1}} \cdots T^{-1}_{j_{k+1}} \cdots T^*_{i_m} \cdots T^{-1}_{j_m}\, M^*_2 \\
&= M_1\, T^*_{i_1} \cdots T^{-1}_{j_1} \cdots T^*_{j_m} \cdots T^{-1}_{j_m} (T^*_{i+m-k})\, M^*_2\,.
\end{aligned}$$

Since $T^*_{i+m-k}\, M^*_2$ can be expressed as a linear combination of monomials in normal form, so can $T^*_i\, M_0$.

This completes the proof of Lemma 2.6.

Lemma 2.7. *If M_0 is a monomial in normal form in $H^n_{N,M}(q)$, then $E\, M_0$ can be written as a linear combination of monomials in normal form.*

Proof. We write

$$M_0 = M_1\, T^*_{i_1} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_1} \cdots T^*_{i_m} \cdots T^*_1\, E\, T^{-1}_{N-1} \cdots T^{-1}_{j_m}\, M^*_2\,,$$

where M_1 (resp M_2^*) is a monomial in normal form in $H_{N\,0}^n(q)$ (resp. $H_{0,M}^n(q)$). Note that M_1 involves T_{N-1}^{-1} at most once. We divide the proof into 4 cases.

Case 1. M_1 has no T_{N-1}^{-1} and $i_1=0$ (i.e. $T_{i_1}^* T_{i_1-1}^* \cdots T_1^*=1$). Then $E\,M_0= M_1\,E^2\,T_{N-1}^{-1}\cdots T_{j_1}^{-1}\,T_{i_2}^*\cdots T_{j_m}^{-1}\,M_2^*=[n]\,M_0$. Therefore, $E\,M_0=[n]\,M_0$.

Case 2. M_1 has one T_{N-1}^{-1} and $i_1=0$. We write $M_1=M_1'\,T_{N-1}^{-1}\,T_{N-2}^{-1}\cdots T_{N-r}^{-1}$.

$$\begin{aligned}
E\,M_0 &= E\,M_1'\,T_{N-1}^{-1}\,T_{N-2}^{-1}\cdots T_{N-r}^{-1}\,E\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^* \\
&= M_1'(E\,T_{N-1}^{-1}\,E)\,T_{N-2}^{-1}\cdots T_{N-r}^{-1}\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^* \\
&= q^{-n}\,M_1'\,T_{N-2}^{-1}\cdots T_{N-r}^{-1}\,E\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^*\,.
\end{aligned}$$

Since $q^{-n}\,M_1'\,T_{N-2}^{-1}\cdots T_{N-r}^{-1}$ can be written as a linear combination of monomials in normal form in $H_{N,0}^n(q)$, so can $E\,M_0$ in $H_{N,M}^n(q)$.

Case 3. M_1 has no T_{N-1}^{-1} and $i_1\neq 0$.

$$\begin{aligned}
E\,M_0 &= E\,M_1\,T_{i_1}^*\,T_{i_1-1}^*\cdots T_2^*\,T_1^*\,E\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^* \\
&= M_1\,T_{i_1}^*\,T_{i_1-1}^*\cdots T_2^*(E\,T_1^*\,E)\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^* \\
&= q^n\,M_1\,T_{i_1}^*\,T_{i_1-1}^*\cdots T_2^*\,E\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^* \\
&= q^n\,T_{i_1}^*\,T_{i_1-1}^*\cdots T_2^*\,M_1\,E\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^*\,.
\end{aligned}$$

Since $M_1\,E\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^*$ is a monomial in normal form, iterative use of the previous lemma prove that $E\,M_0$ can be written in a desired form.

Case 4. M_1 has one T_{N-1}^{-1} and $i_1=0$. This case is rather complicated in comparison with the above 3 cases. We write $M_1=M_1'\,T_{N-1}^{-1}\,T_{N-2}^{-1}\cdots T_{N-r}^{-1}$. By Lemma 2.5 (5), we have

$$\begin{aligned}
E\,M_0 &= E\,M_1'\,T_{N-1}^{-1}\,T_{N-2}^{-1}\cdots T_{N-r}^{-1}\,T_{i_1}^*\,T_{i_1-1}^*\cdots T_1^*\,E\,T_{N-1}^{-1}\cdots T_{j_1}^{-1} \\
&\qquad \cdots T_{i_m}^*\,T_{i_m-1}^*\cdots T_1^*\,E\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^* \\
&= M_1'\,\underline{E\,T_{N-1}^{-1}\,T_{N-2}^{-1}\cdots T_{N-r}^{-1}\cdots T_{i_1}^*\,T_{i_1-1}^*\cdots T_1^*\,E\,T_{N-1}^{-1}\cdots T_{j_1}^{-1}} \\
&\qquad \cdots T_{i_m}^*\,T_{i_m-1}^*\cdots T_1^*\,E\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^* \\
&= M_1'\,\underline{E\,T_{N-1}^{-1}\cdots T_{j_1}^{-1}\cdots T_{i_1}^*\,T_{i_1-1}^*\cdots T_1^*\,E\,T_{N-1}^{-1}\cdots T_{N-r+1}^{-1}(T_1^{*-1})} \\
&\qquad (T_{i_2}^*\,T_{i_2-1}^*\cdots T_1^*)\,E\,T_{N-1}^{-1}\cdots T_{j_2}^{-1}\cdots T_{i_m}^*\,T_{i_m-1}^*\cdots T_1'\,E\,T_{N-1}^{-1}\cdots T_{j_m}^{-1}\,M_2^* \\
&= \cdots \\
&\ \ \vdots \qquad\qquad \vdots \\
&= M_1'\,E\,T_{N-1}^{-1}\cdots T_{j_1}^{-1}\,T_{i_1}^*\cdots T_1^*\,E\,T_{N-1}^{-1}\cdots T_{N-r+1}^{-1}\,T_{i_2}^*\cdots T_{j_m}^{-1}(T_m^{*-1})\,M_2^* \\
&= M_1'\,E\,T_{N-1}^{-1}\,T_{N-2}^{-1}\cdots T_{j_1}^{-1}\,T_{i_1}^*\cdots T_1^*\,\underline{E\,T_{N-1}^{-1}\cdots T_{j_2}^{-1}} \\
&\qquad \underline{T_{i_2}^*\cdots T_1^*\,E\,T_{N-1}^{-1}\cdots T_{N-r+2}^{-1}(T_1^{*-1})}\,T_{i_3}^*\,T_{i_3-1}^*\cdots T_{i_m}^*\cdots T_{j_m}^{-1}\,T_m^{*-1}\,M_2^* \\
&= M_1'\,E\,T_{N-1}^{-1}\cdots T_{j_1}^{-1}\,T_{i_1}^*\cdots T_1^*\,E\,T_{N-1}^{-1}\cdots T_{j_2}^{-1} \\
&\qquad T_{i_2}^*\cdots T_1^*\,E\,T_{N-1}^{-1}\cdots T_{N-r+2}^{-1}\cdots T_{i_m}^*\cdots T_{j_m}^{-1}(T_{m-1}^{*-1})\,T_m^{*-1}\,M_2^* \\
&= \cdots \\
&\ \ \vdots \qquad\qquad \vdots
\end{aligned}$$

$$= M_1' \, E \, T_{N-1}^{-1} \cdots T_{j_1}^{-1} \, T_{i_1}^{*} \cdots T_1^{*} \, E \, T_{N-1}^{-1} \cdots T_{j_2}^{-1} \cdots$$
$$T_{j_{k-1}}^{*} \cdots T_1^{*} \, E \, T_{N-1}^{-1} \cdots T_{j_k}^{-1} \, T_{i_k}^{*} \cdots T_1^{*} \, E \, T_{N-1}^{-1} \cdots T_{N-r+k}^{-1}$$
$$T_{i_{k+1}}^{*} \cdots T_1^{*} \, E \, T_{N-1}^{-1} \cdots T_{j_{k+1}}^{-1} \cdots T_{i_m}^{*} \cdots T_{j_m}^{-1} (T_{m-(k-1)}^{*-1}) \, T_{m-(k-2)}^{*-1} \cdots T_m^{*-1} \, M_2^{*} .$$

Here $T_{m-(k-1)}^{*-1} \, T_{m-(k-2)}^{*-1} \cdots T_m^{*-1} \, M_2^{*}$ can be written as a linear combination of monomials in normal form. If there exists a $k \, (0 \leq k \leq r)$ such that

$$j_k < N-r+k < j_{k+1} \quad (\text{put } j_{m+1} = N, j_0 = 0) , \tag{2.8}$$

then $E \, M_0$ can also be written as a linear combination of monomials in normal form. If there exists no k satisfying (2.8), i.e. $N-r+i$ is always equal to or more than j_{i+1} for $i=0, 1, \cdots, r$, then

$$\begin{aligned}
E \, M_0 &= M_1' \, E \, T_{N-1}^{-1} \cdots T_{j_1}^{-1} \, T_{i_1}^{*} \cdots T_1^{*} \, E \, T_{N-1}^{-1} \cdots T_{j_2}^{-1} \cdots T_{i_{r-1}}^{*} \cdots T_{j_r}^{-1} \\
&\quad T_{i_r}^{*} \, T_{i_r-1}^{*} \cdots T_1^{*} \, \underline{E \, T_{i_{r+1}}^{*}} \, T_{i_{r+1}-1}^{*} \cdots T_1^{*} \, E \, T_{N-1}^{-1} \cdots T_{j_{r+1}}^{-1} \\
&\quad T_{i_m}^{*} \cdots T_{j_m}^{-1} \, T_{m-(r-1)}^{*-1} \, T_{m-(r-2)}^{*-1} \cdots T_m^{*-1} \, M_2^{*} \\
&= q^n \, M_1' \, E \, T_{N-1}^{-1} \cdots T_{j_1}^{-1} \, T_{i_1}^{*} \cdots T_1^{*} \, E \, T_{N-1}^{-1} \cdots T_{j_2}^{-1} \cdots \\
&\quad T_{i_{r-1}}^{*} \cdots T_1^{*} \, E \, T_{N-1}^{-1} \cdots T_{j_r}^{-1} \, T_{i_r}^{*} \cdots T_1^{*} (T_{j_{r+1}}^{*} \cdots T_2^{*}) \, \underline{E \, T_{N-1}^{-1} \cdots T_{j_{r+1}}^{-1}} \cdots \\
&= q^n \, M_1' \, E \, T_{N-1}^{-1} \cdots T_{j_1}^{-1} \, T_{i_1}^{*} \cdots T_{j_2}^{-1} \cdots T_{i_{r-1}}^{*} \cdots T_{j_r}^{-1} \\
&\quad T_{i_r}^{*} \cdots T_1^{*} \, \underline{E \, T_{N-1}^{-1} \cdots T_{j_{r+1}}^{-1}} (T_{i_{r+1}}^{*} \cdots T_2^{*}) \, (T_{i_{r+2}}^{*} \cdots T_1^{*}) \, E \, T_{N-1}^{-1} \cdots T_{j_{r+2}}^{-1} \\
&= \cdots \\
&\;\vdots \qquad \vdots \\
&= q^n \, M_1' \, E \, T_{N-1}^{-1} \cdots T_{j_1}^{-1} \, T_{i_1}^{*} \cdots T_{j_2}^{-1} \cdots (T_{i_m}^{*} \cdots T_1^{*}) \, E \, T_{N-1}^{-1} \cdots T_{j_m}^{-1} \\
&\quad (T_{i_{r+1}+m-r-1}^{*} \, T_{i_{r+1}+m-r-2}^{*} \cdots T_{m-r+1}^{*}) \, T_{m-(r-1)}^{*-1} \, T_{m-(r-2)}^{*-1} \cdots T_m^{*-1} \, M_2^{*} \\
&= q^n \, M_1' \, E \, T_{N-1}^{-1} \cdots T_{j_1}^{-1} \, T_{i_1}^{*} \cdots T_1^{*} \cdots T_{j_2}^{-1} \, T_{i_m}^{*} \cdots T_1^{*} \, E \, T_{N-1}^{-1} \cdots T_{j_m}^{-1} \\
&\quad T_{i_{r+1}+m-r}^{*-1} \, T_{i_{r+1}+m-r+1}^{*-1} \cdots T_m^{*-1} \, M_2^{*} .
\end{aligned}$$

Since $T_{i_{r+1}+m-r}^{*-1} \, T_{i_{r+1}+m-r+1}^{*-1} \cdots T_m^{*-1} \, M_2^{*}$ can be written as a linear combination of monomials in normal form in $H_{0,M}^n(q)$, $E \, M_0$ can be written in a desired form.

Proof of Proposition 2.2. Let M_0 be a monomial in normal form in $H_{N,M}^n(q)$. Then, by Lemma 2.6, Lemma 2.7 and the definition of monomials in normal form, we see that $E \, M_0$, $T_i^{*} \, M_0$, $(1 \leq i \leq M-1)$ and $T_j^{-1} \, M_0 (1 \leq j \leq N-1)$ are all written as linear combinations of monomials in normal forms. Thus we have shown that monomials in normal form generate $H_{N,M}^n(q)$ as a vector space. Hence we get the desired inequality.

3. Weights

We introduce the weights of vertices on $\Gamma_{N,M}$. For any $\gamma \in \boldsymbol{Z}^n$, let $a_\gamma(x_1, x_2, \cdots, x_n)$ denote the monomial antisymmetric function corresponding to γ:

$$a_\gamma(x_1, x_2, \cdots, x_n) = \det [x_i^{\gamma_j}] = \sum_{w\in S_n} (\operatorname{sgn} w)\, x^{w\gamma}\,.$$

DEFINITION 3.1. Let γ be a staircase of height n. The *Schur function* s_γ is defined by $s_\gamma(x_1, x_2, \cdots, x_n)=a_{\gamma+\delta}(x_1, x_2, \cdots, x_n)/a_\delta(x_1, x_2, \cdots, x_n)$ where δ denotes the vector $(n-1, n-2, \cdots, 1, 0)$.

Proposition 3.2. *Let α and β be staircases of height n. Then the following identities hold.*

(a) $(x_1+x_2+\cdots+x_n)\, s_\alpha(x_1, x_2, \cdots, x_n) = \sum_{\substack{\beta\supseteq\alpha \\ |\beta|-|\alpha|=1}} s_\beta(x_1, x_2, \cdots, x_n)$.

(b) $(x_1^{-1}+x_2^{-1}+\cdots+x_n^{-1})\, s_\alpha(x_1, x_2, \cdots, x_n) = \sum_{\substack{\beta\subseteq\alpha \\ |\beta|-|\alpha|=-1}} s_\beta(x_1, x_2, \cdots, x_n)$.

Proof. From the definition of $a_{\alpha+\delta}$, we have

$$(x_1+x_2+\cdots+x_n)\, a_{\alpha+\delta} = \sum_{i=1}^{n} a_{\alpha+\delta+\varepsilon_i}\,,$$
$$(x_1^{-1}+x_2^{-1}+\cdots+x_n^{-1})\, a_{\alpha+\delta} = \sum_{i=1}^{n} a_{\alpha+\delta-\varepsilon_i}\,,$$

where $\varepsilon_i=(0, 0, \cdots, 0, 1, 0, \cdots, 0)$ (only the i-th corodinate is 1). If $\alpha+\varepsilon_i$ (resp. $\alpha-\varepsilon_i$) is not a staircase, then the coordinates of the vector $\alpha+\delta+\varepsilon_i$ (resp. $\alpha+\delta-\varepsilon_i$) are not distinct and hence $a_{\alpha+\delta+\varepsilon_i}=0$ (resp. $a_{\alpha+\delta-\varepsilon_i}=0$). Hence (a) and (b) follow.

Let $\Lambda^k(N, M)$ denote the set of staircases corresponding to the vertices on the k-th floor of $\Gamma_{N,M}$. In particular $\Lambda(N, M)=\Lambda^{N+M}(N, M)$. For each $\gamma\in\Lambda^k(N, M)$, we define its *weight* $w_k(\gamma)$ by $w_k(\gamma)=(1/[n]^k)\, s_\gamma(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1})$. Note that even if the staircase γ of height n is not contained in $\Lambda^k(N, M)$, $w_k(\gamma)$ can also be defined. Putting $x_i=q^{-n+2i-1}$ in Proposition 3.2 we get:

Proposition 3.3. *For $k<N+M$ and $\beta\in\Lambda^k(N, M)$, $w_k(\beta)=\sum_\gamma w_{k+1}(\gamma)$, where γ runs over $\Lambda^{k+1}(N, M)$ connected to β in the graph $\Gamma_{N,M}$.*

By the definition of a_γ, $a_{\gamma+(1,1,\cdots,1)}(x_1, x_2, \cdots, x_n)=x_1 x_2\cdots x_n\, a_\gamma(x_1, x_2, \cdots, x_n)$ and $a_{\gamma-(1,1,\cdots,1)}(x_1, x_2, \cdots, x_n)=x_1^{-1} x_2^{-1}\cdots x_n^{-1}\, a_\gamma(x_1, x_2, \cdots, x_n)$. So $a_{\gamma\pm(1,1\cdots,1)}(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1})=a_\gamma(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1})$. Therefore the following lemma holds.

Lemma 3.4. *Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta=(\beta_1, \beta_2, \cdots, \beta_n)$ be staircases of height n. If $(\alpha_1, \alpha_2, \cdots, \alpha_n)=(\beta_1+r, \beta_2+r, \cdots, \beta_n+r)$ for some $r\in\boldsymbol{Z}$, then $w_k(\alpha)=w_k(\beta)$.*

Lemma 3.5. *Let $\gamma=(\gamma_1, \gamma_2, \cdots, \gamma_n)$ be a staircase. If we put $\gamma^\vee=(-\gamma_n, -\gamma_{n-1}, \cdots, -\gamma_1)$ then $w_k(\gamma)=w_k(\gamma^\vee)$.*

Proof. We have only to show that $a_{\gamma+\delta}(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1})=a_{\gamma^\vee+\delta}(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1})$. From the definition we have $a_{\gamma+\delta}(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1})=$

$\det[q^{(-n+2j-1)(\gamma_i+n-i)}]=\det[q^{(-n+2j-1)(\gamma_i-i)}]\prod_{j=1}^n q^{(-n+2j-1)n}=\det[q^{(-n+2j+1)(\gamma_i-i)}]$. While $a_{\gamma^\vee+\delta}(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1})=\det[q^{(-n+2j-1)(-\gamma_{n-i+1}+n-i)}]$. Reversing the order of columns and rows, we have $a_{\gamma^\vee+\delta}(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1})=\det[q^{(-n+2j-1)(-\gamma_i+i-1)}]$ $=\prod_{j=1}^n q^{(n-2j+1)(-1)}\det[q^{(n-2j+1)(-\gamma_i+i)}]=\det[q^{(n-2j+1)(-\gamma_i+i)}]$. Thus the lemma holds.

From the previous two lemmas we get the following proposition.

Proposition 3.6. *Let $\gamma=(\gamma_1, \gamma_2, \cdots\gamma_n)$ be a staircase. If $\lambda=\{\gamma-(r, r, \cdots, r)\}^\vee$ then $w_k(\gamma)=w_k(\lambda)$.*

Now we assume that λ is a partition with $l(\lambda)\leq n$. Then λ is identified with the set $\{(i, j)\in \boldsymbol{Z};\ 1\leq j\leq\lambda_i\}$, which may be viewed as a collection of boxes called Young diagram. Let λ_j^* be the number of boxes in the j-th column of λ, i.e. $\lambda_j^*=\mathrm{Card}\{i;\ \lambda_i\geq j\}$. The *hook-length* of λ at $(i, j)\in\lambda$ is defined to be $h_\lambda(i, j)=\lambda_i+\lambda_j^*-i-j+1$.

Lemma 3.7. *Let λ be a partition, then*

$$\sum_{(i,j)\in\lambda} h_\lambda(i, j)=\sum_{(i,j)\in\lambda}(i+j-1).$$

The proof is easy, so we omit it.

Proposition 3.8. *Let λ be a partition with $|\lambda|=N$. Then*

$$w_k(\lambda)=\frac{1}{[n]^k}\prod_{(i,j)\in\lambda}\frac{[n-i+j]}{[h_\lambda(i, j)]}.$$

Proof. By Example 1 in [11], p.9 the following identity holds.

$$\prod_{i<j}(1-t^{\lambda_i-\lambda_j-i+j})=\frac{\prod_{i\geq1}\prod_{j=1}^{\lambda_i+n-j}(1-t^j)}{\prod_{(i,j)\in\lambda}(1-t^{h_\lambda(i,j)})}$$

Putting $t=q^2$ we have

$$\prod_{i<j}(1-q^{2(\lambda_i-\lambda_j-i-j)})=\frac{\prod_{i\geq1}\prod_{j=1}^{\lambda_i+n-j}(1-q^{2j})}{\prod_{(i,j)\in\lambda}(1-q^{2h_\lambda j(i,j)})}. \tag{3.9}$$

Moreover putting $\lambda=\phi$, we get

$$\prod_{i<j}(1-q^{2(-i+j)})=\prod_{i\geq1}\prod_{j=1}^{\lambda_i+n-j}(1-q^{2j}). \tag{3.10}$$

We note that

$$\begin{aligned}&a_{\lambda+\delta}(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1})\\&=\det[q^{(-n+2j-1)(\lambda_i+n-i)}]_{1\leq i,j\leq n}\\&=\det[q^{-(n-1)(\lambda_i+n-i)}\cdot q^{2(j-1)(\lambda_i+n-i)}]\\&=\prod_{i=1}^n q^{-(n-1)(\lambda_i+n-i)}\prod_{i<j}(q^{2(\lambda_j+n-j)}-q^{2(\lambda_i+n-i)})\end{aligned}$$

$$= \prod_{i=1}^{n} q^{-(n-1)(\lambda_i+n-i)} \prod_{i<j} q^{2(\lambda_j+n-j)} (1-q^{2(\lambda_i-\lambda_j-i+j)})$$

and

$$a_\delta(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1}) = \prod_{i=1}^{n} q^{-(n-1)(n-i)} \prod_{i<j} q^{2(n-j)} (1-q^{2(-i+j)}) .$$

So, using (3.9) and (3.10) and Lemma 3.7.

$$\begin{aligned}
& s_\lambda(q^{-n+1}, q^{-n+3}, \cdots, q^{n-1}) \\
&= \prod_{i=1}^{n} q^{-(n-1)\lambda_i} \prod_{i<j} \frac{q^{2\lambda_j}(1-q^{2(\lambda_i-\lambda_j-i+j)})}{(1-q^{2(-i+j)})} \\
&= \prod_{(i,j)\in\lambda} q^{-(n-1)} \prod_{i<j} q^{2\lambda_j} \frac{\prod_{i\geq 1}\prod_{j=n-i+1}^{\lambda_i+n-i}(1-q^{2j})}{\prod_{(i,j)\in\lambda}(1-q^{2h_\lambda(i,j)})} \\
&= \prod_{(i,j)\in\lambda} q^{-(n-1)} \prod_{(i,j)\in\lambda} q^{2(i-1)} \prod_{(i,j)\in\lambda} \frac{q^{n-i+j}[n-i+j]}{q^{h_\lambda(i,j)}[h_\lambda(i,j)]} \\
&= \prod_{(i,j)\in\lambda} \frac{[n-i+j]}{[h_\lambda(i,j)]} .
\end{aligned}$$

This completes the proof.

Let $t=(\gamma^0, \gamma^1, \cdots, \gamma^{(k-1)}, \gamma^{(k)}, \gamma^{(k+1)}, \cdots)$ be a tableau. In case $k<N$ we put $\gamma^{(k-1)}=\nu$, $\gamma^k=\mu$ and $\gamma^{(k+1)}=\lambda$. In case $k>N$ we put $\nu=\{\gamma^{(k-1)}-(r, r, \cdots, r)\}^\vee$, $\mu=\{\gamma^{(k)}-(r, r, \cdots, r)\}^\vee$ and $\lambda=\{\gamma^{(k+1)}-(r, r, \cdots, r)\}^\vee$, where $r=\gamma_1^{(k-1)}$(see Proposition 3.6 for the notation $\surd$). Under the above notation we define an integer $d(t, k)$ as follows:

$$d(t, k) = (c_m-r_m)-(c_l-r_l) = \begin{cases} 1-h_\lambda(r_m, c_l) & (\text{if } c_l<c_m) \\ h_\lambda(r_l, c_m)-1 & (\text{if } c_l\geq c_m) , \end{cases}$$

where (r_l, c_l) and (r_m, c_m) are the coordinate of the box $\mu\backslash\nu$ and $\lambda\backslash\mu$ respectively. For $d\in \boldsymbol{Z}\backslash\{0\}$, we put $a_d(q)=q^d/[d]$.

The following proposition, which is a version of Proposition 3.3 in [16], will be used in Section 4 to construct irreducible representations of $H_{N,M}^n(q)$ and in Section 5 to get Markov traces.

Proposition 3.11. *Let $\alpha\in\Lambda^{k-1}(N, M)$ and $\beta\in\Lambda^k(N, M)$. If we assume that α and β are connected in $\Gamma_{N,M}$, then*

$$\sum_\gamma \frac{w_{k+1}(\gamma)}{w_k(\beta)} a_{d(t(\gamma),k)}(q) = \frac{q^n}{[n]} .$$

Here γ runs over all the vertices on the $(k+1)$-th floor which are connected to the vertex β, and $t(\gamma)$ is a tableau whose k-th, $(k+1)$-th and $(k+2)$-th coordinates are α, β and γ respectively.

To prove this, we first show the following two lemmas by using Proposition 3.3.

Lemma 3.12. *Let μ be a Young diagram with $l(\mu)<n$ and $\lambda^1, \lambda^2, \cdots, \lambda^{p(\mu)}$, $\lambda^l=(\lambda_1^l, \lambda_2^l, \cdots, \lambda_n^l)$ $(l=1, 2, \cdots, p(\mu))$ be all the Young diagrams obtained from μ by adding a box (r_l, c_l) $(l=1, 2, \cdots, p(\mu))$. Then.*

$$\sum_{l=1}^{p(\mu)} \frac{w_{k+1}(\lambda^l)}{w_k(\mu)} \frac{[n-r_l+c_l+1]}{[n-r_l+c_l]} = \frac{[n+1]}{[n]}.$$

Proof. Observe that $\lambda_{r_l}^l=\mu_{r_l}+1$, $(\lambda^{l*})_{c_l}=(\mu^*)_{c_l}+1$, $\lambda_j^l=\mu_j$ $(j\neq r_l)$ and $(\lambda^{l*})_j=(\mu^*)_j$ $(j\neq c_l)$. So the hook-lengths at (i,j) are the same in λ^l and in μ if $i\neq r_l$ and $j\neq c_l$. From Proposition 3.3 we obtain

$$\begin{aligned}(3.13) \qquad 1 &= \sum_{l=1}^{p(\mu)} \frac{w_{k+1}(\lambda^l)}{w_k(\mu)} \\ &= \sum_{l=1}^{p(\mu)} \frac{(1/[n]^{k+1}) \prod_{(i,j)\in\lambda^l}([n-i+j]/[h_{\lambda^l}(i,j)])}{(1/[n]^k) \prod_{(i,j)\in\mu}([n-i+j]/[h_\mu(i,j)])} \\ &= \sum_{l=1}^{p(\mu)} \frac{[n-r_l+c_l]}{[n]} \prod_{j=1}^{c_l-1} \frac{[h_\mu(r_l,j)]}{[h_\mu(r_l,j)+1]} \prod_{i=1}^{r_l-1} \frac{[h_\mu(i,c_l)]}{[h_\mu(i,c_l)+1]}.\end{aligned}$$

For any partition $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ we define $\bar{\lambda}$ to be $(\lambda_1, \lambda_2, \cdots, \lambda_n, 0)$. Then we have $w_k(\bar{\lambda})=(1/[n+1]^k) \prod_{(i,j)\in\lambda}[n+1-i+j]/[h_\lambda(i,j)]$ and

$$1=\sum_{l=1}^{p(\mu)} \frac{w_{k+1}(\bar{\lambda}^l)}{w_k(\bar{\mu})} = \sum_{l=1}^{p(\mu)} \frac{[n+1-r_l+c_l]}{[n+1]} \prod_{j=1}^{c_l-1} \frac{[h_\mu(r_l,j)]}{[b_\mu(r_l,j)+1]} \prod_{i=1}^{r_l-1} \frac{[h_\mu(i,c_l)]}{[h_\mu(i,c_l)+1]}.$$

Hence by (3.13), $\sum_{l=1}^{p(\mu)}(w_{k+1}(\lambda^l)/w_k(\mu))([n-r_l+c_l+1]/[n-r_l+c_l])([n]/[n+1])=1$.

Lemma 3.14. *Let μ and λ^l $(l=1, 2, \cdots, p(\mu))$ be Young diagram as in Lemma 3.12 and v a Young diagram obtained from μ by removing one box. Then*

$$\sum_{l=1}^{p(\mu)} \frac{w_{k+1}(\lambda^l)}{w_k(\mu)} \frac{[n-r_l+c_l+1]}{[n-r_l+c_l]} \frac{[d(t^l, k)-1]}{[d(t^l, k)]} = 1,$$

where t^l is a tableau whose k-th, $(k+1)$-th and $(k+2)$-th coordinates are v, μ, λ^l respectively.

Proof. Let $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ be a partition with the length $l(\lambda)$ and let $i_0\in\boldsymbol{N}$, $1\leq i_0\leq l(\lambda)$. We will modify λ so that the lemma can be reduced to the special case of Proposition 3.3. We define the Young diagram $\lambda(i_0)$ such that $\lambda(i_0)_i=\lambda_i+1$ for $i<i_0$ and $\lambda(i_0)_i=\lambda_{i+1}$ for $i\geq i_0$. From this definition we find $\lambda(i_0)_i^*=\lambda_i^*-1$ for $i\leq\lambda_{i_0}$ and $\lambda(i_0)_i^*=\lambda_{i-1}^*$ for $i>\lambda_{i_0}$. Let $h_\lambda(i,j)$ and $h_{\lambda(i_0)}(i,j)$ denote the hook lengths corresponding to the box (i,j) in λ and $\lambda(i_0)$ respectively. It follows from the above remarks that $h_\lambda(i,j)=h_{\lambda(i_0)}(i,j+1)$ for $i<i_0$ and $j>\lambda_{i_0}$, and $h_\lambda(i,j)=h_{\lambda(i_0)}(i-1,j)$ for $i>i_0$. If $i<i_0$ and $j\leq\lambda_{i_0}$ then $h_\lambda(i,j)=h_{\lambda(i_0)}(i,j)$. Consider $w_{k+1}(\lambda^l)/w_k(\mu)$ and $w_{k+1}(\lambda^l(i_0))/w_k(\mu(i_0))$. From the proof of the previous lemma

$$\frac{w_{k+1}(\lambda^l)}{w_k(\mu)}=\frac{[n-r_l+c_l]}{[n]}\prod_{j=1}^{c_l-1}\frac{[h_\mu(r_l,j)]}{[h_\mu(r_l,j)+1]}\prod_{i=1}^{r_l-i}\frac{[h_\mu(i,c_l)]}{[h_\mu(i,c_l)+1]},$$

where $\lambda^l-\mu=\{(r_l,c_l)\}$. Note that "the first factor" of $w_{k+1}(\lambda^l)/w_k(\mu)$ is $[n-r_l+c_l]/[n]$. On the other hand, since $\lambda^l(i_0)-\mu(i_0)=\{(r_l,c_l+1)\}$ (if $i_0<r_l$) or $\{(r_l-1,c_l)\}$ (if $i_0>r_l$), the first factor of $w_{k+1}(\lambda^l(i_0))/w_k(\mu(i_0))$ is $[n-r_l+c_l]/[n]$. From the consideration of hook lengths in μ and $\mu(i_0)$, the remaining factors of $w_{k+1}(\lambda^l)/w_k(\mu)$ and $w_{k+1}(\lambda^l(i_0))/w_k(\mu(i_0))$ differ only by the factor belonging either to $(r^l,\mu_{i_0}+1)$ in $w_{k+1}(\lambda^l(i_0))/w_k(\mu(i_0))$ (if $r_l<i_0$) or to (r_l,μ_{i_0}) in $w_{k+1}(\lambda^l)/w_k(\mu)$ (if $i_0<r_l$). Hence

$$\frac{w_{k+1}(\lambda^l(i_0))}{w_k(\mu(i_0))}=\frac{[n-r_l+c_l+1]}{[n-r_l+c_l]}\frac{[c_l-\mu_{i_0}+i_0-r_l-1]}{[c_l-\mu_{i_0}+i_0-r_l]}\frac{w_{k+1}(\lambda^l)}{w_k(\mu)},$$

and so

$$1=\sum_{l=1}^{p(\mu(i_0))}\frac{w_{k+1}(\lambda^l(i_0))}{w_k(\mu(i_0))}=\sum_{l=1}^{p(\mu)}\frac{w_{k+1}(\lambda^l)}{w_k(\mu)}\frac{[n-r_l+c_l+1]}{[n-r_l+c_l]}\frac{[c_l-\mu_{i_0}+i_0-r_l-1]}{[c_l-\mu_{i_0}+i_0-r_l]}.$$

Taking i_0 so that $\mu-\nu=\{(i_0,\mu_{i_0})\}$, we find $c_l-\mu_{i_0}+i_0-r_l=d(t^l,k)$. Hence the lemma follows.

Proof of Proposition 3.11. From the definition of $d=d(t,k)$ and Proposition 3.6, we have only to consider the case α, β and γ are partitions. So we use the same notation as in the previous two lemmas. Lemma 3.12 and Lemma 3.14 say that

$$\sum_{l=1}^{p(\mu)}\frac{w_{k+1}(\lambda^l)}{w_k(\mu)}\frac{[n-r_l+c_l+1]}{[n-r_l+c_l]}=\frac{[n+1]}{[n]},$$

$$\sum_{l=1}^{p(\mu)}\frac{w_{k+1}(\lambda^l)}{w_k(\mu)}\frac{[n-r_l+c_l+1]}{[n-r_l+c_l]}\frac{[d(t^l,k)-1]}{[d(t^l,k)]}=1,$$

where $d(t^l,k)=c_l-\mu_{i_0}+i_0-r_l$. We can check that

$$\frac{[i_0-\mu_{i_0}-n-2]}{[i_0-\mu_{i_0}-n-1]}\frac{[n-r_l+c_l+1]}{[n-r_l+c_l]}+\frac{[i_0-\mu_{i_0}-n]}{[i_0-\mu_{i_0}-n-1]}\frac{[n-r_l+c_l+1]}{[n-r_l+c_l]}\frac{[d(t^l,k)-1]}{[d(t^l,k)]}=\frac{[d(t^l,k)-1]}{[d(t^l,k)]}$$

using $d(t^l,k)=c_l-\mu_{i_0}+i_0-r_l$. Therefore

$$\sum_{l=1}^{p(\mu)} \frac{w_{k+1}(\lambda^l)}{w_k(\mu)} \frac{[d(t^l,k)-1]}{[d(t^l,k)]} = \frac{[i_0-\mu_{i_0}-n-2]}{[i_0-\mu_{i_0}-n-1]} - \frac{[n+1]}{[n]} + \frac{[i_0-\mu_{i_0}-n]}{[i_0-\mu_{i_0}-n-1]}$$
$$= [2] - \frac{[n+1]}{[n]} = \frac{[n-1]}{[n]}.$$

Hence we have

$$\sum_{l=1}^{p(\mu)} \frac{w_{k+1}(\lambda^l)}{w_k(\mu)} \left(q - \frac{a^{d(t^l,k)}}{[d(t^l,k)]}\right) = \left(q - \frac{q^n}{[n]}\right).$$

Hence

$$\sum_{l=1}^{p(\mu)} \frac{w_{k+1}(\lambda^l)}{w_k(\mu)} a_{d(t^l,k)}(q) = \frac{q^n}{[n]}.$$

4. Representations of $H^n_{N,M}(q)$ on $K(q)\,\Omega$

We are now going to construct the irreducible representations of $H^n_{N,M}(q)$. For $\gamma \in \Lambda(N, M)$ let Ω^γ be the set of all tableaux of shape γ and let Ω be $\coprod_{\gamma\in\Lambda(N,M)}\Omega^\gamma$ as in Section 1. Let V^γ be the $K(q)$-vector space with basis $\{v_t; t\in\Omega^\gamma\}$. We now define endomorphisms $\pi_\gamma(T_i)$, $\pi_\gamma(T_j^*)$ and $\pi_\gamma(E)$ $(i=1, 2, \cdots, N-1)$, $(j=1, 2, \cdots, M-1)$ of V^γ. For each tableau $t=(\gamma^{(0)}=\phi, \gamma^{(1)}, \cdots, \gamma^{(N+M)}=\gamma)$, we have to define $\pi_\gamma(T_i)\,v_t$ $(i=1, 2, \cdots, N-1)$, $\pi_\gamma(T_j^*)\,v_t$ $(j=1, 2, \cdots, M-1)$ and $\pi_\gamma(E)\,v_t$.

Definition of $\pi_\gamma(T_i)\,v_t$ and $\pi_\gamma(T_j^*)\,v_t$.

As for $\pi_\gamma(T_i)\,v_t$, our definition is almost the same as the one given in [3, 16]. Let $i\in\{1, 2, \cdots, N-1\}$ (resp. $j\in\{1, 2, \cdots, M-1\}$). The partition $\gamma^{(i+1)}$ (resp. $\gamma^{(N+j-1)}$) is obtained from $\gamma^{(i-1)}$ (fresp. $\gamma^{(N+j-1)}$) in one of the following three ways:

(a) By adding two boxes to (resp. removing two boxes from) the same row of $\gamma^{(i-1)}$ (resp. $\gamma^{(N+j-1)}$). In this case $\pi_\gamma(T_i)\,v_t=qv_t$ (resp. $\pi_\gamma(T_j^*)\,v_t=qv_t$).

(b) By adding two boxes to (resp. removing two boxes from) the same column of $\gamma^{(i-1)}$ (resp. $\gamma^{(N+j-1)}$). In this case $\pi_\gamma(T_i)\,v_t=-q^{-1}\,v_t$(resp. $\pi_\gamma(T_j^*)\,v_t=-q^{-1}\,v_t$).

(c) By adding (resp. removing) boxes in different rows and columns. There is precisely one tableau which differs from t only in its $(i+1)$-th (resp. $(N+j+1)$-th) coordinate. We call this tableau $\hat{t}$. Define

$$\pi_\gamma(T_i)\,v_t = \frac{q^d}{[d]}\,v_t + \frac{[d-1]}{[d]}\,v_t \left(\text{resp. } \pi_\gamma(T_j^*)\,v_t = \frac{q^d}{[d]}\,v_t + \frac{[d+1]}{[d]}\,v_{\hat{t}}\right),$$

where $d=d(t, i)$ (resp. $d=d(t, N+j)$).

Definition of $\pi_\gamma(E)\,v_t$.

In case $\gamma^{(N+1)}\neq\gamma^{(N-1)}$, we define $\pi_\gamma(E)=0$. In case $\gamma^{(N+1)}=\gamma^{(N-1)}$, it is necessarily a partition. So we put $\gamma^{(N+1)}=\gamma^{(N-1)}=\mu$ and $\gamma^{(N)}=\lambda^1$. Let $p(\mu)-1$

be the number of tableaux $t^2, t^3, \cdots, t^{p(\mu)}$ which differ from $t=t^1$ only in their $(N+1)$-th coordinates. We denote their $(N+1)$-th coordinates by $\lambda^2, \lambda^3, \cdots, \lambda^{p(\mu)}$ respectively. Define

$$\pi_\gamma(E)\, v_t = [n] \sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)} v_{t^l}\,.$$

One can verify that $\pi_\gamma(T_i)$, $\pi_\gamma(T_j^*)$ and $\pi_\gamma(E)$ satisfy the relations (1)a, b, (2)a, b, (3) and (4) in Definition 2.1. (Use $\sum_{l=1}^{p(\mu)} w_N(\lambda^l)/w_{N-1}(\mu)=1$ for (3).) We have to verify that they also satisfy (1)c, (2)c, (5), (6) and (7), (8) in Definition 2.1.

Proposition 4.1.

(1) $$\pi_\gamma(T_i)\, \pi_\gamma(T_{i+1})\, \pi_\gamma(T_i) = \pi_\gamma(T_{i+1})\, \pi_\gamma(T_i)\, \pi_\gamma(T_{i+1})\,,$$

(2) $$\pi_\gamma(T_j^*)\, \pi_\gamma(T_{j+1}^*)\, \pi_\gamma(T_j^*) = \pi_\gamma(T_{j+1}^*)\, \pi_\gamma(T_j^*)\, \pi_\gamma(T_{j+1}^*)\,.$$

These relations are proved by modifying the argument in [16], pp. 361–364 and we omit it.

Proposition 4.2.

(1) $$\pi_\gamma(E)\, \pi_\gamma(T_{N-1})\, \pi_\gamma(E) = q^n\, \pi_\gamma(E)\,,$$

(2) $$\pi_\gamma(E)\, \pi_\gamma(T_1^*)\, \pi_\gamma(E) = q^n\, \pi_\gamma(E)\,.$$

Proof. Let $t_\nu^1=(\gamma^{(0)}, \cdots, \gamma^{(N-2)}=\nu, \gamma^{(N-1)}, \gamma^{(N)}=\lambda^1, \gamma^{(N+1)}, \gamma^{(N+2)}, \cdots)$ be a staircase. If $\gamma^{(N-1)} \neq \gamma^{(N+1)}$, then $\pi_\gamma(E)\, v_{t_\nu^1}=0$. So both (1) and (2) hold. We assume that $\gamma^{(N-1)}=\gamma^{(N+1)}=\mu$. Let $t_\nu^2, t_\nu^3, \cdots, t_\nu^{p(\mu)}$ be distinct tableaux which have the same i-th coordinates as t_ν^1 for any $i \neq N+1$ and which have $\lambda^2, \lambda^3, \cdots, \lambda^{p(\mu)}$ respectively as the $(N+1)$-th coordinates. Let $\lambda^l \backslash \mu=\{(r_l, c_l)\}$, $\mu \backslash \nu=\{(r_0, c_0)\}$ and $d_l=d(t_\nu^l, N-1)=c_l-r_l-(c_0-r_0)$. If we write $\pi_\gamma(T_{N-1})\, v_{t_\nu^l}=a_{d_l}\, v_{t_\nu^l} +\hat{a}_{d_l}\, v_{\hat{t}_\nu^l}$, then

$$\begin{aligned}
\pi_\gamma(E)\, \pi_\gamma(T_{N-1})\, \pi_\gamma(E)\, _{t_\nu^1} &= \pi_\gamma(E)\, \pi_\gamma(T_{N-1})\, [n] \sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)} v_{t_\nu^l} \\
&= \pi_\gamma(E)\, [n] \sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)} (a_{d_l}\, v_{t_\nu^l} + \hat{a}_{d_l}\, v_{\hat{t}_\nu^l}) \\
&= [n] \sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)}\, a_{d(t_\nu^l, N-1)}\, \pi_\gamma(E)\, v_{t_\nu^l} \\
&= q^n\, \pi_\gamma(E)\, v_{t_\nu^1} \quad \text{(Proposition 3.11)}.
\end{aligned}$$

Hence (1) holds. Similarly we obtain

$$\pi_\gamma(E)\, \pi_\gamma(T_1^*)\, \pi_\gamma(E) = [n] \sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)}\, a_{d(t_\nu^l, N+1)}\, \pi_\gamma(E)\,.$$

We have to show

$$\sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)} a_{d(t_\nu^l, N+1)} = \frac{q^n}{[n]}. \tag{4.3}$$

The $(N+3)$-th coordinates of t_ν^l $(l=1, 2, \cdots, p(\mu))$ are $\gamma^{(N+2)}$, which is either the partition ν' or the staircase of the form $\mu-(0, 0, \cdots, 0, 1)$.

If $\gamma^{(N+2)}$ is the partition ν' with $\mu\backslash\nu'=\{(r_0', c_0')\}$, then $d(t_\nu^l, N+1)=c_l-r^l-(c_0'-r_0')=d(t_{\nu'}^l, N-1)$. Hence we have

$$\sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)} a_{d(t_\nu^l, N-1)} = \sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)} a_{d(t_{\nu'}^l, N+1)} = \frac{q^n}{[n]}.$$

If $\gamma^{(N+2)}$ is the staircase of the form $\mu-(0, 0, \cdots, 0, 1)$, then $d(t_\nu^l, N+1)=c_l-r_l+n$. If we put $\tilde{\nu}=\mu+(1, 1, \cdots, 1, 0)$, $\tilde{\mu}=\mu+(1, 1, \cdots, 1, 1)$ and $\tilde{\lambda}^l=\lambda^l+(1, 1, \cdots, 1, 1)$ $(l=1, 2, \cdots, p(\mu))$ and apply Proposition 3.11 to the sequence $\tilde{\nu}\subset\tilde{\mu}\subset\tilde{\lambda}^l$ $(l=1, 2, \cdots, p(\mu))$, then we have

$$\sum_{l=1}^{p(\mu)} \frac{w_{n+N}(\tilde{\lambda}^l)}{w_{n+N-1}(\tilde{\mu})} \frac{q^{n+c_l-r_l}}{[n+c_l-r_l]} = \frac{q^n}{[n]}.$$

Using the definition of $w_k(\gamma)$ and Lemma 3.4 we have

$$\frac{w_{n+N}(\tilde{\lambda}^l)}{w_{n+N-1}(\tilde{\mu})} = \frac{w_N(\tilde{\lambda}^l)}{w_{N-1}(\tilde{\mu})} = \frac{w_N(\lambda^l)}{w_{N-1}(\mu)}.$$

Hence

$$\sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)} \frac{q^{n+c_l-r_l}}{[n+c_l-r_l]} = \frac{q^n}{[n]}.$$

Since $d(t_\nu^l, N+1)=c_l-r_l+n$,

$$\sum_{l=1}^{p(\mu)} \frac{w_N(\lambda^l)}{w_{N-1}(\mu)} a_{d(t_\nu^l, N+1)} = \frac{q^n}{[n]}.$$

Thus we have shown (4.3). Hence (2) is proved.

Proposition 4.4.

(1) $\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,(\pi_\gamma(T_{N-1})-\pi_\gamma(T_1^*)) = 0.$

(2) $(\pi_\gamma(T_{N-1})-\pi_\gamma(T_1^*))\,\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E) = 0.$

Before proving this we shall show the following lemma.

Lemma 4.5. *Let* $t=(\gamma^{(0)}, \cdots, \gamma^{(N-2)}, \gamma^{(N-1)}, \gamma^{(N)}, \gamma^{(N+1)}, \gamma^{(N+2)}, \cdots)$ *be a tableau. If* $\gamma^{(n-2)}\neq\gamma^{(N+2)}$ *or* $\gamma^{(N-1)}\neq\gamma^{(N+1)}$ *then*

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_t = 0.$$

Proof. If $\gamma^{(N-1)}\neq\gamma^{(N+1)}$ then the claim is obvious. We assume that $\gamma^{(N-1)}=\gamma^{(N+1)}=\mu^0$, $\gamma^{(N)}=\lambda^{0,1}$ and $\gamma^{(N-2)}=\nu\neq\gamma^{(N+2)}$. Let $t_j=(\gamma^{(0)},\cdots,\nu,\mu^0,\lambda^{0,j},\mu^0,\gamma^{(N+2)},\cdots)$, $(j=1,2,\cdots,p(\mu^0))$ be tableaux, where $\lambda^{0,j}$, $(j=1,2,\cdots,p(\mu^0))$ are all the partitions obtained from μ^0 by adding one box. Moreover let $t'_j=(\gamma^{(0)},\cdots,\nu,\mu^j,\lambda^{0,j},\mu^0,\gamma^{(N+2)},\cdots)$, $t''_j=(\gamma^{(0)},\cdots,\nu,\mu^0,\lambda^{0,j},\beta^j,\gamma^{(N+2)},\cdots)$, $t'''_j=(\gamma^{(0)},\cdots,\nu,\mu^j,\lambda^{0,j},\beta^j,\gamma^{(n+2)},\cdots)$, where μ^j is the partition obtained from ν by adding the box $\lambda^{0,j}\backslash\mu^0$ instead of $\mu^0\backslash\nu$ and $\beta^j=\lambda^{0,j}-(0,\cdots,0,1)$. Even if there does not exist μ^j we define μ^j, t'_j, t'''_j formally. Note that $\mu^j\neq\beta^j$ for any j and $\mu^0\neq\beta^j$. For brevity, we write v_j for v_{t_j} and v'_j, v''_j, v'''_j denote $v_{t'_j}$, $v_{t''_j}$, $v_{t'''_j}$ respectively. And we write $a_{j,k}$ for $a_{d(t_j,k)}$. Since $\pi_\gamma(E)\,v_j=\pi_\gamma(E)\,v''_j=\pi_\gamma(E)\,v'''_j=0$ for any j, we have

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_t=\pi_\gamma(E)\,\pi_\gamma(T_{N-1})\,\pi_\gamma(T_1^*)\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}v_j$$
$$=\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}a_{j,N+1}\,a_{j,N-1}\,\pi_\gamma(E)\,v_j=\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}a_{j,N+1}\,a_{j,N-1}\,\pi_\gamma(E)\,v_t\,.$$

From the definition of $d(t,k)$, $d(t_j,N-1)-d(t_j,N+1)$ is determined only by ν, μ^0 and $\gamma^{(N+2)}$ and they do not depend on j. Since $\mu^j\neq\beta^j$, $d(t_j,N-1)-d(t_j,N+1)\neq0$. So $Q=(1/(q-q^{-1}))\,(q^{d(t_j,N-1)-d(t_j,N+1)}/[d(t_j,N-1)-d(t_j,N+1)])$ also does not depend on j. We can easily check that the following identity: $(q-q^{-1})\,(a_{j,N+1}\,Q+a_{j,M-1}(1-Q))=a_{j,N+1}\,a_{j,N-1}$. From this, Proposition 3.11 and (4.3), we have

$$\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}a_{j,N+1}\,a_{j,N-1}=(q-q^{-1})\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\{a_{j,N+1}\,Q+a_{j,N-1}(1-Q)\}$$
$$=(q-q^{-1})\frac{q^n}{[n]}\,.$$

Hence $\pi_\gamma(E)\,\pi_\gamma(T_{N-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_t=(q-q^{-1})\,q^n\,\pi_\gamma(E)\,v_t$. Finally, using Proposition 4.3(2) we have

$$\begin{aligned}&\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_t\\&=\pi_\gamma(E)\,\pi_\gamma(T_{N-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_t-(q-q^{-1})\,\pi_\gamma(E)\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_t\\&=(q-q^{-1})\,q^n\,\pi_\gamma(E)\,v_t-(q-q^{-1})\,q^n\,\pi_\gamma(E)\,v_t\\&=0\,.\end{aligned}$$

Lemma 4.6. *Let $t_{0,j}$ $(j=1,2,\cdots,p(\mu^0))$ be tableaux whose coordinates are given by $(\gamma^{(0)},\cdots,\gamma^{(N-2)}=\nu,\mu^0,\lambda^{0,j},\mu^0,\nu,\cdots)$, where $\lambda^{0,j}$ $(j=1,2,\cdots,p(\mu^0))$ are all the partitions obtained from μ^0 by adding one box. Let μ^j be the partition obtained from ν by adding the box $\lambda^{0,j}\backslash\mu^0$, instead of $\mu^0\backslash\nu$. Even if μ^j is not a partition, we define it formally and define $w_{N-1}(\mu^j)$ to be zero for the μ^j. Then*

$$\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\cdot\frac{[d_j+1][d_j-1]}{[d_j]^2}=\frac{w_{N-1}(\mu^j)}{w_{N-2}(\nu)}$$

and

$$\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\cdot\frac{1}{[d_j]^2}=\frac{w_{N-1}(\mu^0)}{w_{N-2}(\nu)},$$

where $d_j=d(t_{0,j},N-1)=d(t_{0,j},N+1)$.

Proof. If $\lambda^{0,j}$ is obtained from ν by adding two boxes in the same row or column, then we cannot get another tableau by changing the order of adding these boxes. This means $w_{N-1}(\mu^j)=0$. On the other hand $[d_j+1]\,[d_j-1]$ also becomes zero. So in this case the upper identity holds.

We assume that $\lambda^{0,j}$ is obtained from ν by adding two boxes in different rows and columns. Let $\lambda^{0,j}\backslash\mu^0=\mu^j\backslash\nu=\{(r_j,c_j)\}$ and $\lambda^{0,j}\backslash\mu^j=\mu^0\backslash\nu=\{(r_0,c_0)\}$. Then

$$\begin{aligned} d_j &= c_j-r_j-(c_0-r_0)=(c_j-c_0)+(r_0-r_j)\\ &=\begin{cases} h_\nu(r_j,c_0)+1=h_{\mu^0}(r_j,c_0) & (\text{if } c_j>c_0)\\ -(h_\nu(r_0,c_j)+1)=-h_{\mu^0}(r_0,c_j) & (\text{if } c_0<c_j)\,. \end{cases}\end{aligned}$$

In case $c_j>c_0$ we have

$$\begin{cases} h_{\mu^0}(r_j,c_0)=h_\nu(r_j,c_0)+1 & (k=c_0)\\ h_{\mu^0}(r_j,k)=h_\nu(r_j,k) & (k\neq c_0)\,. \end{cases}$$

Hence in this case we obtain

$$\begin{aligned} &\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\cdot\frac{[d_j+1][d_j-1]}{[d_j]^2}\\ &=\frac{[n-r_j+c_j]}{[n]}\prod_{k=1}^{c_j-1}\frac{[h_{\mu^0}(r_j,k)]}{[h_{\mu^0}(r_j,k)+1]}\prod_{k=1}^{r_j-1}\frac{[h_{\mu^0}(k,c_j)]}{[h_{\mu^0}(k,c_j)+1]}\\ &\quad\times\frac{[h_{\mu^0}(r_j,c_0)+1]}{[h_{\mu^0}(r_j,c_0)]}\cdot\frac{[h_\nu(r_j,c_0)]}{[h_\nu(r_j,c_0)+1]}\\ &=\frac{[n-r_j+c_j]}{[n]}\prod_{k=1}^{c_j-1}\frac{[h_\nu(r_j,k)]}{[h_\nu(r_j,k)+1]}\prod_{k=1}^{r_j-1}\frac{[h_\nu(k,c_j)]}{[h_\nu(k,c_j)+1]}\\ &=\frac{w_{N-1}(\mu^j)}{w_{N-2}(\nu)}\end{aligned}$$

Similarly, in case $c_0>c_j$ we obtain the same identity. Using the above consequence we have

$$\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\frac{1}{[d_j]^2}=1-\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\left(1-\frac{1}{[d_j]^2}\right)$$

$$=\sum_{j=0}^{p(\mu^0)}\frac{w_{N-1}(\mu^j)}{w_{N-2}(\nu)}-\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\cdot\frac{[d_j+1][d_j-1]}{[d_j]^2}$$

$$=\sum_{j=0}^{p(\mu^0)}\frac{w_{N-1}(\mu^j)}{w_{N-2}(\nu)}-\sum_{j=1}^{p(\mu^0)}\frac{w_{N-1}(\mu^j)}{w_{N-2}(\nu)}$$

$$=\frac{w_{N-1}(\mu^0)}{w_{N-2}(\nu)}.$$

Lemma 4.7. *Let μ^j be the previous ones. Let $t_{j,k}=(\gamma^{(0)},\cdots,\gamma^{(N-2)}=\nu, \mu^j, \lambda^{j,k}, \mu^j, \nu, \cdots)$ be tableaux, where $\lambda^{j,k}$ $(k=1,2,\cdots,p(\mu^j))$ are all the partitions obtained from μ^j by adding one box. We label them so that $\lambda^{j,1}$ becomes $\lambda^{0,j}$ for each j. Then*

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_{0,1}=[n]^2\sum_{j=0}^{p(\mu^0)}\sum_{k=1}^{p(\mu^j)}\frac{w_N(\lambda^{j,k})}{w_{N-2}(\nu)}v_{j,k}\,.$$

Proof. We write d_j and $v_{j,k}$ for $d(t_{0,j}, N-1)$ and $v_{j,k}$ respectively. We can easily check that

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_{0,1}$$
$$=[n]\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\left\{\frac{1}{[d_j]^2}\pi_\gamma(E)\,v_{0,j}+\frac{[d_j+1][d_j-1]}{[d_j]^2}\pi_\gamma(E)\,v_{j,1}\right\}.$$

Hence

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_{0,1}$$
$$=[n]^2\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\left\{\frac{1}{[d_j]^2}\sum_{k=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,k})}{w_{N-1}(\mu^0)}v_{0,k}\right.$$
$$\left.+\frac{[d_j+1][d_j-1]}{[d_j]^2}\sum_{k=1}^{p(\mu^j)}\frac{w_N(\lambda^{j,k})}{w_{N-1}(\mu^j)}v_{j,k}\right\}$$
$$=[n^2]\left\{\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\cdot\frac{1}{[d_j]^2}\sum_{k=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,k})}{w_{N-1}(\mu^0)}v_{0,k}\right.$$
$$\left.+\sum_{j=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,j})}{w_{N-1}(\mu^0)}\cdot\frac{[d_j+1][d_j-1]}{[d_j]^2}\sum_{k=1}^{p(\mu^j)}\frac{w_N(\lambda^{j,k})}{w_{N-1}(\mu^j)}v_{j,k}\right\}$$
$$=[n]^2\left\{\frac{w_{N-1}(\mu^0)}{w_{N-2}(\nu)}\sum_{k=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,k})}{w_{N-1}(\mu^0)}v_{0,k}+\sum_{j=1}^{p(\mu^0)}\frac{w_{N-1}(\mu^j)}{w_{N-2}(\nu)}\sum_{k=1}^{p(\mu^j)}\frac{w_N(\lambda^{j,k})}{w_{N-1}(\mu^j)}v_{j,k}\right\}$$
$$=[n]^2\left\{\sum_{k=1}^{p(\mu^0)}\frac{w_N(\lambda^{0,k})}{w_{N-2}(\nu)}v_{0,k}+\sum_{j=1}^{p(\mu^0)}\sum_{k=1}^{p(\mu^j)}\frac{w_N(\lambda^{j,k})}{w_{N-2}(\nu)}v_{j,k}\right\}$$
$$=[n]^2\sum_{j=0}^{p(\mu^0)}\sum_{k=1}^{p(\mu^j)}\frac{w_N(\lambda^{j,k})}{w_{N-2}(\nu)}v_{j,k}\,.$$

Proof of Proposition 4.4 (1). Let $t=(\cdots,\gamma^{(N-2)},\gamma^{(N-1)},\gamma^{(N)},\gamma^{(N+1)},\gamma^{(N+2)},\cdots)$ be a tableau. Put $t'=(\cdots,\gamma^{(N-2)},\gamma^{(N-1)},\gamma^{(N)},\hat{\gamma}^{(N+1)},\gamma^{(N+2)},\cdots)$ and $t''=$

$(\cdots, \gamma^{(N-2)}, \hat{\gamma}^{(N-1)}, \gamma^{(N)}, \gamma^{(N+1)}, \gamma^{(N+2)}, \cdots)$, where $\gamma^{(N)}\supseteq\hat{\gamma}^{(N+1)}\supset\gamma^{(N+2)}$, $\hat{\gamma}^{(N+1)}\neq\gamma^{(N+1)}$ and $\gamma^{(N-2)}\subseteq\hat{\gamma}^{(N-1)}\subseteq\gamma^{(N)}$, $\hat{\gamma}^{(N-1)}\neq\gamma^{(N-1)}$. Note that if there exists such a $\hat{\gamma}^{(N+1)}$ or $\hat{\gamma}^{(N-1)}$ it is uniquely determined. Even if there does not exist such a $\hat{\gamma}^{(N+1)}$ or $\hat{\gamma}^{(N-1)}$, we define the tableaux t' and t'' formally.

If $\gamma^{(N+2)}\neq\gamma^{(N-2)}$ then, by Lemma 4.5, $\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_t=0$ for $v=v_t$, $v_{t'}$, and $v_{t''}$. Since $\pi_\gamma(T_1^*)\,v_t$ is contained in $K(q)\,v_t\oplus K(q)\,v_t'$ and $\pi_\gamma(T_{N-1})\,v_t$ is in $K(q)\,v_t\oplus K(q)\,v_{t''}$, we have $\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,(\pi_\gamma(T_1^*)-\pi_\gamma(T_{N-1}))\,v_t=0$.

We assume that $\gamma^{(N+2)}=\gamma^{(N-2)}=\nu$ and $\gamma^{(N)}=\lambda$. There are at most four possibilities which tableaux satisfying the above assumption can take. We classify them and label them as follows: $t_1=(\cdots, \gamma^{(N-2)}=\nu, \mu, \lambda, \mu, \nu, \cdots)$, $t_2=(\cdots, \gamma^{(N-2)}=\nu, \mu, \lambda, \mu', \nu, \cdots)$, $t_3=(\cdots, \gamma^{(N-2)}=\nu, \mu', \lambda, \mu, \nu, \cdots)$, $t_4=(\cdots, \gamma^{(N-2)}=\nu, \mu', \lambda, \mu', \nu, \cdots)$, where $\mu\neq\mu'$. We write v_i for v_{t_i} $(i=1, 2, 3, 4)$. Since $\pi_\gamma(T_1^*)\,v_1=(q^d/[d])\,v_1+([d+1]/[d])\,v_2$ and $\pi_\gamma(T_{N-1})\,v_1=(q^d/[d])\,v_1+([d-1]/[d])\,v_3$ for suitable d, and since

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_2=\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_3=0\,, \tag{4.8}$$

we have

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,(\pi_\gamma(T_1^*)-\pi_\gamma(T_{N-1}))\,v_1=0\,,$$
$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,(\pi_\gamma(T_1^*)-\pi_\gamma(T_{N-1}))\,v_4=0\,.$$

Now we shall show the same identity for v_2. Since $\pi_\gamma(T_1^*)\,v_2=(q^{-d}/[-d])\,v_2+([d-1]/[d])\,v_1$, $\pi_\gamma(T_{N-1})\,v_2=(q^d/[d])\,v_2+([d-1]/[d])\,v_4$ for suitable d, using (4.8) we find it is sufficient to show

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_1=\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_4\,.$$

But by Lemma 4.7 the above identity holds. So we have

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,(\pi_\gamma(T_1^*)-\pi_\gamma(T_{N-1}))\,v_2=0\,.$$

Similarly we have the same idnetity for v_3. This completes the proof of (1).

Proof of Proposition 4.4 (2). By Lemma 4.5 we can assume a tableau $t=t_{0,1}$ is the following form: $t_{0,1}=(\cdots, \gamma^{(N-2)}=\nu, \mu^0, \lambda^{0,1}, \mu^0, \nu, \cdots)$. Using the notation of Lemma 4.7, we have

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_{0,1}=[n]^2\sum_{j=0}^{p(\mu^0)}\sum_{k=1}^{p(\mu^j)}\frac{w_N(\lambda^{j,k})}{w_{N-2}(\nu)}\,v_{j,k}\,.$$

For a tableau $t_{j,k}=(\cdots, \nu, \mu^j, \lambda^{j,k}, \mu^j, \nu, \cdots)$ we put $t'_{j,k}=(\cdots, \nu, \mu^{j'}, \lambda^{j,k}, \mu^{j'}, \nu, \cdots)$. If there exist such $t'_{j,k}$, it is unique. Otherwise we put $t'_{j,k}=t_{j,k}$. We write $v'_{j,k}$ for $u_{t'_{j,k}}$. Using these notations, we have

$$\pi_\gamma(E)\,\pi_\gamma(T_{N-1}^{-1})\,\pi_\gamma(T_1^*)\,\pi_\gamma(E)\,v_{0,1}=\frac{[n]^2}{2}\sum_{j,k}\frac{w_N(\lambda^{j,k})}{w_{N-2}(\nu)}(v_{j,k}+v'_{j,k})\,.$$

We can easily check $\pi_\gamma(T_{N-1})\,(v_{j,k}+v'_{j,k})=\pi_\gamma(T_1^*)\,(v_{j,k}+v'_{j,k})$. Hence 4.4 (2) follows.

Finally we show that the representation π_γ is irreducible and $H^n_{N,M}(q)$ is semisimple. The following facts will be important in the proof of this. Observe that if $M=0$, then the map $t=(\gamma^{(0)},\cdots,\gamma^{(N-1)},\gamma^{(N)}=\lambda)\mapsto t'=(\gamma^{(0)},\cdots,\gamma^{(N-1)})$ defines a bijection between $\Omega^\lambda_{N,0}$ and $\bigcup_{\substack{\lambda'\in\Lambda(N-1,0)\\ \lambda'\subseteq\lambda}}\Omega^{\lambda'}_{N-1,0}$, and that if $M\geq 1$, then the map $t=(\gamma^{(0)},\cdots,\gamma^{(N+M-1)},\gamma^{(N+M)}=\lambda)\mapsto t'=(\gamma^{(0)},\cdots,\gamma^{(N+M-1)})$ defines a bijection between $\Omega^\gamma_{N,M}$ and $\bigcup_{\substack{\beta\in\Lambda(N,M-1)\\ \beta\subseteq\gamma}}\Omega^\beta_{N,M-1}$. The case $M=0$ is treated in [16]. So we concentrate on the case $M\geq 1$. In this case,

$$K(q)\,\Omega^\gamma_{N,M}\cong\bigoplus_{\substack{\beta\in\Lambda(N,M-1)\\ \beta\subseteq\gamma}}K(q)\,\Omega^\beta_{N,N-1}.\quad\text{(as } K(q)\text{-vector spaces)}\tag{4.9}$$

from the above argument. $K(q)\,\Omega^\gamma_{N,M}$ is an $H^n_{N,M}(q)$-module and this decomposition is compatible with the action of the subalgebra $H^n_{N,M-1}(q)$ of $H^n_{N,M}(q)$. So we have

$$\pi_\gamma\big|_{H^n_{N,M-1}(q)}\cong\bigoplus_{\substack{\beta\in\Lambda(N,M-1)\\ \beta\subseteq\gamma}}\pi_\beta\,.\tag{4.10}$$

Using this, we can show the following theorem.

Theorem 4.11. *Let $\gamma\in\Lambda(N,M)$ be a staircase. Then π_γ is an irreducible representation of $H^n_{N,M}(q)$ on $K(q)\,\Omega^\gamma_{N,M}$. If $\tilde\gamma\in\Lambda(N,M)$, then $\pi_{\tilde\gamma}$ is isomorphic to π_γ if and only if $\gamma=\tilde\gamma$. The map π: $x\in H^n_{N,M}(q)\mapsto\oplus_{\gamma\in\Lambda(N,M)}\pi_\gamma(x)$ define a faithful representation of $H^n_{N,M}(q)$. In particular, the algebra $H^n_{N,M}(q)$ is semisimple.*

Before proving this theorem we shall show the following lemma, which is the extended version of Lemma 2.11(b) in [16]. Consider the graph $\Gamma_{N,M}$ whose vertices are parametrized by the multiset $\bigcup_{k=0}^{N+M}\Lambda^k(N,M)$. Let γ,γ' be staircases, we call γ' is a *substaircase* of γ if there exists a tableau such that $t=(\gamma^{(0)},\cdots,\gamma^{(i)}=\gamma',\cdots,\gamma^{(j)}=\gamma,\cdots,\gamma^{(N+M)})$ $(1\leq i<j\leq N+M)$.

Lemma 4.12. (1) *Let λ and $\tilde\lambda$ be two distinct partitions with $|\lambda|=|\tilde\lambda|=N$, $N\geq 3$. Then at least one of them has a subpartition μ with $|\mu|=N-1$ which is not a subpartition of the other one.*

(2) *Let $\gamma=[\alpha,\beta]_n$, $\tilde\gamma=[\tilde\alpha,\tilde\beta]_n$ be two distinct staircases with $|\gamma|=|\tilde\gamma|=N-M$ and $|\alpha|<N$, $|\tilde\alpha|<N$. Suppose that $M\geq 1$, $N\geq 1$. Then at least one of them has a substaircase γ' with $|\gamma'|=N-(M-1)$ which is not a substaircase*

of the other one.

Proof. The proof of (1) is given in [16]. We show (2) in similar way to the proof of Lemma 2.11(b) in [16]. Let $\gamma=(\gamma_1, \gamma_2, \cdots, \gamma_n)$ and let $\tilde{\gamma}=(\tilde{\gamma}_1, \tilde{\gamma}_2, \cdots, \tilde{\gamma}_n)$. Let s be the largest index for which $\tilde{\gamma}_2 \neq \tilde{\gamma}_s$. We may assume $\gamma_s < \tilde{\gamma}_s$. If $s \neq n$, take γ' to be $\gamma'=(\tilde{\gamma}_1, \cdots, \tilde{\gamma}_s, \tilde{\gamma}_{s+1}+1, \cdots \tilde{\gamma}_n)$. If $s=n$, take γ' to be $\gamma'=(\gamma_1+1, \gamma_2, \cdots, \gamma_n)$. These γ's have desired property.

Proof of Theorem 4.11. As is well known (see, e.g. [16]), $H^n_{N,0}(q)$ is a semisimple algebra. We assume that $H^n_{N,M-1}(q)$ is a semisimple algebra with minimal central idempotent z_β labeled by staircases $\beta \in \Lambda(N, M-1)$. Let $0 \neq W \subseteq K(q)\,\Omega^\gamma_{N,M}$ be an $H^n_{N,M}(q)$ submodule. As $K(q)\,\Omega^\gamma_{N,M}$ decomposes as an $H^n_{N,M-1}(q)$ module into the direct sum of irreducible modules $K(q)\,\Omega^\beta_{N,M-1}$ by (4.10), there exists a $\beta \in \Lambda(N, M-1)$ such that $\beta \supseteq \gamma$ and $K(q)\,\Omega^\beta_{N,M-1} \subset W$. Let $\tilde{\beta} \neq \beta$ be another staircase in $\Lambda(N, M-1)$ such that $\tilde{\beta} \supseteq \gamma$. There is exactly one $\alpha \in \Lambda(N, M-2)$ (resp. $\alpha \in \Lambda(N-1, 0)$ if $M=1$) connected to both β and $\tilde{\beta}$. Let $t=(\gamma^{(0)}, \gamma^{(1)}, \cdots, \gamma^{(N+M)}=\gamma)$ be a tableau such that $\gamma^{(N+M-1)}=\beta$ and $\gamma^{(N+M-2)}=\alpha$. Then $\pi_\gamma(z_{\tilde{\beta}})\,\pi_\gamma(T^*_{M-1})\,v_t=([d+1]/[d])\,v_{\tilde{t}}$ (resp. $\pi_\gamma(z_{\tilde{\beta}})\,\pi_\gamma(E)\,v_t=(w_N(\tilde{\beta})/w_{N-1}(\alpha))\,v_{\tilde{t}}$), where $d=d(t, N+M-1)$ and $\tilde{t}=(\gamma^{(0)}, \gamma^{(1)}, \cdots \alpha\tilde{\beta}, \gamma)$. Hence the irreducible $H^n_{N,M-1}(q)$ module $K(q)\,\Omega^{\tilde{\beta}}_{N,M-1}$ is contained in W. Since $\tilde{\beta}$ was arbitrary, $W \supset \bigoplus_{\substack{\gamma \subseteq \beta \\ \beta \in \Lambda(N,M-1)}} K(q)\,\Omega^\beta_{N,M-1}=K(q)\,\Omega^\gamma_{N,M}$.

Next we shall show that $K(q)\,\Omega^\gamma_{N,M}$s are mutually non isomorphic $H^n_{N,M}(q)$ modules. We write $\gamma=[\alpha, \beta]_n$ and $\tilde{\gamma}=[\tilde{\alpha}, \tilde{\beta}]_n$ and assume that $\gamma \neq \tilde{\gamma}$. If $|\alpha|<N$ and $|\tilde{\alpha}|<N$ then, by (4.10) and the previous lemma, $K(q)\,\Omega^\gamma$ and $K(q)\,\Omega^\gamma$ already differ as $H^n_{N,M-1}(q)$ modules. If $|\alpha|<N$ and $|\tilde{\alpha}|=N$ or $\alpha \neq \tilde{\alpha}$ with $|\alpha|=|\tilde{\alpha}|=N$, then we can get a substaircase of γ by removing one box from the partition β, which never become a substaircase of $\tilde{\gamma}$. So we have $\pi_\gamma \not\cong \pi_{\tilde{\gamma}}$ again by (4.10). If $\alpha=\tilde{\alpha}$ with $|\alpha|=|\tilde{\alpha}|=N$ then we apply the same argument with respect to β and $\tilde{\beta}$ as in [16] using Lemma 4.12 (1).

Finnaly we can show the representation π is faithful. In fact, by Proposition 2.2 and Corollary 1.8, $\dim_{K(q)} H^n_{N,M}(q) \leq (N+M)!$ and $\dim_{K(q)} H^n_{N,M}(q) \geq \dim(\oplus_{\gamma \in \Lambda(N,M)} \pi_\gamma(H^n_{N,M}(q)))=\sum_{\gamma \in \Lambda(N,M)} \dim(K(q)\,\Omega^\gamma)^2=(N+M)!$. Therefore, we have $\dim_{K(q)} H^n_{N,M}(q)=\sum_{\gamma \in \Lambda(N,M)} \dim(K(q)\,\Omega^\gamma)^2$, and π is faithful.

The above equality and Lemma 2.6 implies the following.

Corollay 4.13. *The monomials in normal form defined in section* 2 *form a basis of* $K(q)\,H^n_{N,M}(q)$.

If we take $q_0 \in K \setminus \{0\}$ so that $[n]_{q_0} \neq 0$, then we can define K-algebra $H^n_{N,M}(q_0)$ taking q_0 for q. Moreover if $[N+M+n]_{q_0}!=[N+M+n]_{q_0}[N+M+n-1]_{q_0} \cdots [1]_{q_0} \neq 0$, then π_γ is also defined for $H^n_{N,M}(q)$ taking q_0 for q. So we have the following theorem.

Theorem 4.14. *If* $[N+M+n]_{q_0}!\neq 0$, *then* $H^n_{N,M}(q_0)\simeq H^n_{N,M}(q)$.

REMARK 4.15. If we take algebraically closed field as underlying field of $H^n_{N,M}(q)$. (e.g. $\overline{K(q)}$), then we can define π_γ as follows:

$$\pi_\gamma(T_i)\, v_t = \frac{q^d}{[d]} v_t + \sqrt{\frac{[d+1]\,[d-1]}{[d]^2}}\, v_{\hat{t}},\quad \pi_\gamma(T_j^*)\, v_t = \frac{q^d}{[d]} v_t + \sqrt{\frac{[d+1]\,[d-1]}{[d]^2}}\, v_{\hat{t}}\,,$$

$$\pi_\gamma(E)\, v_t = \sum_l \sqrt{\frac{w_N(\lambda^1)\, w_N(\lambda^l)}{w_{N-1}(\mu)}}\, v_{t^l}\,.$$

This means that π_γ is expressed as symmetric matrices.

REMARK 4.16. The mapping π_γ defined in the beginning of this section also defines an irreducible representation of $H^n_{N,M}(q)$ in the case $n<N+M$. However, the mapping π in Theorem 4.11 is no longer injective.

5. Markov trace

Let M be a finite dimensional split semisimple K-algebra. A *trace* on M is a linear map Tr: $M\to K$ such that $\mathrm{tr}(xy)=\mathrm{tr}(yx)$ for all $x, y\in M$. Let M be written as $M=\oplus_{i=1}^m z_i\, M$ with the central minimal idempotents z_i $(i=1, 2, \cdots, m)$. To a given trace tr, we associate the vector $w=(\mathrm{tr}(p_i))_{1\le i\le m}$, where p_i is a minimal idempotent in $z_i\, M$. Conversely, the vector $w\in K^m$ determines a unique trace:

$$\mathrm{tr}_w = \sum_{i=1}^m w_i\, \chi^i\,,$$

where χ^i is the diagonal sum on the full matrix ring $z_i\, M$. A trace tr on M is faithful if and only if the associated vector w has no zero entries. If we associate the vector $w=(w_{N+M}(\gamma))_{\gamma\in\Lambda(N+M)}$ on $A_{N,M}$ in Section 1, then we have the tract $\mathrm{Tr}^{(N,M)}$ on $H^n_{N,M}(q)$:

$$\mathrm{Tr}^{(N,M)} = \mathrm{tr}_w\circ\pi = \sum_{\gamma\in\Lambda(N,M)} w_{N+M}(\gamma)\, \chi^\gamma\circ\pi_\gamma\,.$$

Any element $x\in H^n_{N,M-1}(q)$ (resp. $x\in H^n_{N-1,0}(q)$) can be considered as an element in $H^n_{N,M}(q)$ (resp. $H^n_{N,0}(q)$). Proposition 3.11 asserts that there is the following relation between $\mathrm{Tr}^{(N,M)}$ and $\mathrm{Tr}^{(N,M-1)}$ (resp. $\mathrm{Tr}^{(N,0)}$ and $\mathrm{Tr}^{(N-1,0)}$). (See Proposition 2.5.1 of [3]).

Lemma 5.1.

(a) *If* $x\in H^n_{N,M-1}(q)$ $(M\ge 1)$, *then* $\mathrm{Tr}^{(N,M-1)}(x) = \mathrm{Tr}^{(N,M)}(x)$.

(b) *If* $x\in H^n_{N-1,0}(q)$ $(M\ge 1)$, *then* $\mathrm{Tr}^{(N-1,0)}(x) = \mathrm{Tr}^{(N,0)}(x)$.

For the proof of this lemma, see for example [3]. The next proposition characterizes $\mathrm{Tr}^{(N,M)}$.

Proposition 5.2. *Let* $\mathrm{Tr}=\mathrm{Tr}^{(N,M)}$ *be the trace defined above, then* Tr *has the following properties.*

(1) $\mathrm{Tr}(1)=1$.
(2) $\mathrm{Tr}(ab)=\mathrm{Tr}(ba)$ *for* $\forall a, \forall b \in H^n_{N,N}(q)$.
(3) *If* $x \in \mathrm{alg}\{T_1, T_2, \cdots, T_{k-1}\}$ $(0 \le k-1 < N-1)$, *then* $\mathrm{Tr}(xT_k)=(q^n/[n])\,\mathrm{Tr}(x)$.
(4) *If* $x \in \mathrm{alg}\{T_1, T_2, \cdots, T_{N-1}\}$, *then* $\mathrm{Tr}(xE)=(1/[n])\,\mathrm{Tr}(x)$.
(5) *If* $x \in \mathrm{alg}\{T_1, T_2, \cdots, T_{N-1}, E, T_1^*, \cdots, T_{k-1}^*\}$ $(0 \le k-1 < M-1)$, *then* $\mathrm{Tr}(xT_k^*)$ $=(q^n/[n])\,\mathrm{Tr}(x)$.

Proof. By the definition of Tr and the previous lemma, (1) and (2) are clear. For the proof of (3), (4) and (5), we can prove them in similar way to [16]. So we shall prove only (4). In the following proof, we use the representation of $H^n_{N,M}(q)$ on $\Omega_{N,M}$ constructed in Section 4. Let $t \in \Omega_{N,M}$ be a tableau. We can define the unique minimal idempotent $p_t \in H^n_{N,M}(q)$ by $\pi(p_t)=T_{t,t} \in \mathrm{End}_{K(q)}(K(q)\,\Omega)$. (For $T_{t,t}$ see Section 1.) Note that if $t \in \Omega_{N,M-1}$, then $p_t \in H^n_{N,M-1}(q)$ by the definition. We can also consider p_t as an element of $H^n_{N,M}(q)$ by $p_t=\sum_s p_s$, where s runs over all the tableaux in $\Omega_{N,M}$ obtained from t by adding its $(N+M)$-th coordinate. As $\sum_{t \in \Omega_{N,0}} p_t=1$, the property (4) is equivalent to $\mathrm{Tr}(x\,E\,p_t)=(1/[n])\,\mathrm{Tr}(x\,p_t)$ for all $t \in \Omega_{N,0}$ and for all $x \in H^n_{N,0}(q)$. Since p_t is a minimal projection in $H^n_{N,0}(q)$, there is an $\alpha(x) \in K(q)$ such that $p_t\,x\,p_t=\alpha(x)\,p_t$. So if $t=(\gamma^{(0)}, \cdots, \gamma^{(N-1)}=\mu, \gamma^{(N)}=\lambda)$, then we have

$$\mathrm{Tr}(x\,p_t)=\alpha(x)\,w_N(\lambda)\,. \tag{5.3}$$

Let z_λ be the central minimal idempotent in $H^n_{N,0}(q)$ indexed by λ and let $\hat{t}=(\gamma^{(0)}, \cdots, \gamma^{(N-1)}=\mu, \gamma^{(N)}=\lambda, \gamma^{(N+1)}=\mu)$. We write $p_t=\sum_s p_s$ as we noted. Considering the action of $\pi(z_\lambda\,E\,p_t)$ on $\Omega_{N,1}$ we obtain

$$z_\lambda\,E\,p_t=z_\lambda\,E\sum_s p_s=[n]\,\frac{w_N(\lambda)}{w_{N-1}(\mu)}\,p_{\hat{t}}=[n]\frac{w_N(\lambda)}{w_{N-1}(\mu)}\,p_t\,p_{\hat{t}}\,.$$

Hence we have

$$\begin{aligned} p_t\,x\,E\,p_t &= z_\lambda\,p_t\,x\,E\,p_t=p_t\,x\,z_\lambda\,E\,p_t=[n]\,p_t\,x\,p_t\,\frac{w_N(\lambda)}{w_{N-1}(\mu)}\,p_{\hat{t}} \\ &= [n]\,\alpha(x)\,\frac{w_N(\lambda)}{w_{N-1}(\mu)}\,p_t\,p_{\hat{t}}=[n]\,\alpha(x)\,\frac{w_N(\lambda)}{w_{N-1}(\mu)}\,p_{\hat{t}}\,. \end{aligned}$$

It follows $\mathrm{Tr}(x\,E\,p_t)=[n]\,\dfrac{w_N(\lambda)}{w_{N-1}(\mu)}\,\alpha(x)\,w_{N+1}(\mu)=(1/[n])\,\alpha(x)\,w_N(\lambda)$. Comparing with (5.3) we obtain $\mathrm{Tr}(E\,x\,p_t)=(1/[n])\,\mathrm{Tr}(x, p_t)$ as desired.

Proposition 5.4. *If a trcae* tr *on* $H^n_{N,M}(q)$ *satisfies the properties* (1)-(5)

in Proposition 5.2, *then it coincides with* Tr.

Proof. It is sufficient to prove that, for any $x\in H^n_{N,M}(q)$, $\mathrm{tr}(x)$ is uniquely determined using the properties (1)-(5) in Proposition 5.2. We can assume that x is a monomial in normal form. If $x=\alpha\in K(q)$ then $\mathrm{tr}(x)=\mathrm{tr}(\alpha\cdot 1)=\alpha\,\mathrm{tr}(1)=\alpha$ by the property (1).

If $x\in\mathrm{alg}\{T_1, \cdots, T_k\}$ $(1\leq k\leq N-1)$ and x involves T_k^{-1}, then x is of the form $y_1 T_k^{-1} y_2$ $(y_1, y_2\in\mathrm{alg}\{T_1, \cdots, T_{k-1}\})$. Hence $\mathrm{tr}(x)=\mathrm{tr}(y_1 T_k^{-1} y_2)=(q^{-n}/[n])$ $\mathrm{tr}(y_2 y_1)$. By the induction on k, $\mathrm{tr}(y_2 y_1)$ is uniquely determined. So is $\mathrm{tr}(x)$.

If $x\in\mathrm{alg}\{T_1, \cdots, T_{N-1}, E\}$ and x involves E, then x is of the form $M_1 E T_{N-1}^{-1}$ $T_{N-2}^{-1}\cdots T_{j_1}^{-1}$ $(M_1\in\mathrm{alg}\{T_1, \cdots, T_{N-1}\})$. Hence $\mathrm{tr}(M_1 E T_{N-1}^{-1} T_{N-2}^{-1}\cdots T_{j_1}^{-1})=(1/[n])$ $\mathrm{tr}(T_{N-1}^{-1} T_{N-2}^{-1}\cdots T_{j_1}^{-1} M_1)$. By the previous argument $\mathrm{tr}(T_{N-1}^{-1} T_{N-2}^{-1}\cdots T_{j_1}^{-1} M_1)$ is uniquely determined. So is $\mathrm{tr}(x)$.

Assume that x is in $\mathrm{alg}\{T_1, \cdots, T_{N-1}, E, T_1^*, \cdots, T_k^*\}$ and x involves T_k^*. If we write $x=M_1 T_{i_1}^*\cdots T_1^* E T_{N-1}^{-1}\cdots T_{j_1}^{-1}\cdots T_{i_m}^*\cdots T_1^* E T_{N-1}^{-1}\cdots T_{j_m}^{-1} M_2^*$, then T_k^* is in M_2^* or $T_k^*=T_{i_m}^*$ or both. If T_k^* is not in M_2^* or $T_k^*\neq T_{i_m}^*$, then we can write $x=y_1 T_k^* y_2$ $(y_1, y_2\in\mathrm{alg}\{T_1, \cdots, T_{N-1}, E, T_1^*, \cdots T_{k-1}^*\})$. So we have $\mathrm{tr}(x)=\mathrm{tr}(y_1 T_k^* y_2)=(q^n/[n])\,\mathrm{tr}(y_2 y_1)$. Again by the induction on k, we can uniquely determine $\mathrm{tr}(x)$.

Finally consider the case M_2^* involves T_k^* and $T_{i_m}^*=T_k^*$. Using the relation of T_i^*, we can write $M_2^*=T_r^* T_{r+1}^*\cdots T_k^* M_3^*$ for some monomial in normal form M_3^* in $\mathrm{alg}\{T_1^*, T_2^*, \cdots, T_{k-1}^*\}$. Hence we have either $x=M_1 T_k^*\cdots T_1^* E T_{N-1}^{-1}\cdots$ $T_{j_m}^{-1}\times T_s^*\cdots T_k^* M_3^*$ or $x=y_0 E T_{N-1}^{-1}\cdots T_{j_m-1}^{-1} T_k^*\cdots T_1^* E T_{N-1}^{-1}\cdots T_{j_m}^{-1}(T_s^* T_{s+1}^*\cdots T_k^*$ $M_3^*)$ $(y_0\in\mathrm{alg}\{T_1, \cdots, T_{N-1}, E, T_1^*, \cdots, T_{k-1}^*\})$.

Case 1. If x is of the first form, then

$$\begin{aligned}\mathrm{tr}(x) &= \mathrm{tr}(M_1 T_k^*\cdots T_1^* E T_{N-1}^{-1}\cdots T_{j_m}^{-1} T_s^*\cdots T_k^* M_3^*)\\ &= \mathrm{tr}(M_1 T_s^*\cdots T_k^* M_3^* T_k^*\cdots T_1^* E T_{N-1}^{-1}\cdots T_{j_m}^{-1})\,.\end{aligned}$$

Since $T_s^*\cdots T_k^* M_3^* T_k^*\cdots T_1^*$ can be written as a linear combination of monomials in normal form. We can reduce the number of T_k^* to one. Using (2), (5) and the induction on k we can determine $\mathrm{tr}(x)$.

Case 2.1. If x is of the second form and $s>1$, then

$$\begin{aligned}x &= y_0 E T_{N-1}^{-1}\cdots T_{j_m-1}^{-1} T_k^*\cdots T_1^* E T_{N-1}^{-1}\cdots T_{j_m}^{-1}(T_s^* T_{s+1}^*\cdots T_k^* M_3^*)\\ &= y_1(T_k^*\cdots T_1^* T_s^* T_{s+1}^*\cdots T_k^*) y_2 \quad (y_2, y_1\in\mathrm{alg}\{T_1, \cdots, T_{N-1}, E, T_1^*, \cdots, T_{k-1}^*\})\,.\end{aligned}$$

Similarly to Case 1 we can determine $\mathrm{tr}(x)$.

Case 2.2. If x is of the second form and $s=1$, then

$$\begin{aligned}x &= y_0 E T_{N-1}^{-1}\cdots T_{j_m-1}^{-1} T_k^*\cdots T_1^* E T_{N-1}^{-1}\cdots T_{j_m}^{-1}(T_1^* T_2^*\cdots T_k^* M_3^*)\\ &= y_0 T_k^*\cdots T_2^* E T_{N-1}^{-1} T_1^* E T_1^* T_{N-2}^{-1}\cdots T_{j_m-1}^{-1} T_{N-1}^{-1}\cdots T_{j_m}^{-1} T_2^*\cdots T_k^* M_3^*\\ &= y_0 T_k^*\cdots T_2^* E T_{N-1}^{-1} T_1^* E T_{N-1}^{-1} T_{N-2}^{-1}\cdots T_{j_m-1}^{-1} T_{N-1}^{-1}\cdots T_{j_m}^{-1} T_2^*\cdots T_k^* M_3^*\end{aligned}$$

$$= y_1 T_k^* \cdots T_2^* T_1^* T_2^* \cdots T_k^* y_2 \quad (y_1, y_2 \in \mathrm{alg}\{T_1, \cdots, T_{N-1}, E T_1^*, \cdots, T_{k-1}^*\}).$$

Similarly to Case 1 we can determine $\mathrm{tr}(x)$. The proof is completed.

We call the property (1)-(5) in Proposition 5.2 *Markov property*, and if a trace has Markov property then we call it a *Markov trace*.

REMARK 5.5. In the case $n<N+M$, we can define a trace Tr of $H_{N,M}^n(q)$ satisfying the properties (1)-(5) in Proposition 5.2. However, this trace is not faithful, i.e. there is $x \in H_{N,M}^n(q)$ such that $\mathrm{Tr}(x, y)=0$ for all $y \in H_{N,M}^n(q)$.

6. Centralizer algebra

In this section we shall give a representation of $H_{N,M}^n(q)$ acting on the mixed tensor space, and we shall show that it is naturally isomorphic the centralizer algebra of $U_q(gl(n, \boldsymbol{C}))$ on the mixed tensor space. In the classical case $q=1$, such centralizer algebra is studied in Section 1 of [8]. In this section we put $K=\boldsymbol{C}$.

We define $U_q(gl(n, \boldsymbol{C}))$ to be the associative algebra over $\boldsymbol{C}(q)$ generated by the symbols $q^{\pm\varepsilon_i}$, X_i, Y_i $(1\leq i\leq n-1)$ under the following relations:

$$q^{\varepsilon_i}\cdot q^{-\varepsilon_i} = q^{-\varepsilon_i}\cdot q^{\varepsilon_i} = 1, \quad q^{\varepsilon_i}\cdot q^{\varepsilon_j} = q^{\varepsilon_j}\cdot q^{\varepsilon_i},$$

$$q^{\varepsilon_i} X_j q^{-\varepsilon_i} = \begin{cases} q X_j & \text{if } j=i, \\ q^{-1} X_j & \text{if } j=i-1, \\ X_j & \text{if otherwise,} \end{cases} \qquad q^{\varepsilon_i} Y_j q^{-\varepsilon_i} = \begin{cases} q^{-1} Y_j & \text{if } j=i, \\ q Y_j & \text{if } j=i-1, \\ Y_j & \text{if otherwise,} \end{cases}$$

$$[X_i, Y_j] = \delta_{ij} \frac{q^{\varepsilon_i-\varepsilon_{i+1}} - q^{-\varepsilon_i+\varepsilon_{i+1}}}{q-q^{-1}},$$

$$X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i, \quad (|i-j|\geq 2),$$

$$X_i^2 X_{i\pm1} - (q+q^{-1}) X_i X_{i\pm1} X_i + X_{i\pm1} X_i^2 = 0 \quad (1\leq i, i\pm1 \leq n-1),$$

$$Y_i^2 Y_{i\pm1} - (q+q^{-1}) Y_i Y_{i\pm1} Y_i + Y_{i\pm1} Y_i^2 = 0 \quad (1\leq i, i\pm1 \leq n-1).$$

We put $q^{H_i}=q^{\varepsilon_i-\varepsilon_{i+1}}$. We have an algebra homomorphism $\Delta^{(N)}$: $U_q \to U_q \otimes \cdots \otimes U_q$ (N-fold) such that

$$\Delta^{(N)}(q^{\pm\varepsilon_i}) = q^{\pm\varepsilon_i} \otimes \cdots \otimes q^{\pm\varepsilon_i},$$

$$\Delta^{(N)}(X_i) = \sum_{j=1}^{N} q^{H_i} \otimes \cdots \otimes q^{H_i} \otimes X_i \otimes 1 \otimes \cdots \otimes 1,$$

$$\Delta^{(N)}(Y_i) = \sum_{j=1}^{N} 1 \otimes \cdots \otimes 1 \otimes Y_i \otimes q^{-H_i} \otimes \cdots \otimes q^{-H_i},$$

and an algebra antihomomorphism S: $U_q \to U_q$ such that

$$S(X_i) = -q^{-H_i} X_i, \quad S(Y_i) = -Y_i q^{H_i}, \quad S(q^{\varepsilon_i}) = q^{-\varepsilon_i},$$

The Lie algebra $gl(n, \boldsymbol{C})$ naturally acts on $V_n = \boldsymbol{C}^n$. This representation, called the vector representation of $gl(n, \boldsymbol{C})$, can be deformed into the *vector representation* $(\phi, V_n \otimes \boldsymbol{C}(q))$ of U_q as follows:

$$\phi\colon X_i \mapsto E_{i,i+1}\,, \quad Y_i \mapsto E_{i+1,i}\,, \quad q^{\varepsilon_i} \mapsto q^{E_{i,i}}\,,$$

where $E_{i,j}$ denotes the matrix unit.

Let V_n^* be the dual space of V_n. We can define the contragradient representation ϕ^* on $V_n^* \otimes \boldsymbol{C}(q)$ by $\phi^* = {}^t\phi \circ S$. Using ϕ, ϕ^* and $\Delta^{(N+M)}$ we can define the mixed tensor representation $\Phi^{(N+M)}$ on $V_n^{(N+M)}$ by $\Phi^{(N+M)} = (\phi^{\otimes N} \otimes \phi^{*\otimes M}) \circ \Delta^{(N+M)}$. The generators X_i, Y_i and $q^{\pm\varepsilon_i}$ are mapped into $\mathrm{End}(V_n^{(N,M)}) \cong \mathrm{End}(V_n)^{\otimes N} \otimes \mathrm{End}(V_n^*)^{\otimes M}$ by Φ as follows:

$$\begin{aligned}
\Phi(X_i) &= \sum_{j=1}^{N} \underbrace{K_i \otimes \cdots \otimes K_i}_{j-1} \otimes E_{i,i+1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{} \\
&\quad + \sum_{j=1}^{M} \underbrace{K_i \otimes \cdots \otimes K_i}_{N} \otimes \underbrace{K_i^{-1} \otimes \cdots \otimes K_i^{-1}}_{j-1} \otimes (-q^{-1})\, E_{i+1,i} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{M-j}\,, \\
\Phi(Y_i) &= \sum_{j=1}^{N} \underbrace{1 \otimes \cdots \otimes 1}_{j-1} \otimes E_{i+1,i} \otimes \underbrace{K_i^{-1} \otimes \cdots \otimes K_i^{-1}}_{N-j} \otimes \underbrace{K_i \otimes \cdots \otimes K_i}_{M} \\
&\quad + \sum_{j=1}^{M} \underbrace{1 \otimes \cdots \otimes 1}_{N+j-1} \otimes (-q)\, E_{i,i+1} \otimes \underbrace{K_i \otimes \cdots \otimes K_i}_{M-j}\,, \\
\Phi(q^{\pm\varepsilon_i}) &= \underbrace{q^{\pm E_{i,i}} \otimes \cdots \otimes q^{\pm E_{i,i}}}_{N} \otimes \underbrace{q^{\mp E_{i,i}} \otimes \cdots \otimes q^{\mp E_{i,i}}}_{M}\,,
\end{aligned}$$

where K_i denotes $q^{E_{i,i} - E_{i+1,i+1}}$.

DEFINTION 6.1. Let V be a vector space over a field K and let A be a subalgebra of $\mathrm{End}_K(V)$. Then we define A' by

$$A' = \mathrm{End}_A(V) = \{b \in \mathrm{End}_K(V);\, b\,a = a\,b \quad \text{for all} \quad a \in A\}\,.$$

We consider the subalgebra

$$C_n^{(N,M)}(q) = \{x \in \mathrm{End}(V_n^{(N,M)});\, x\,\Phi(g) = \Phi(g)\,x \quad \text{for any} \quad g \in U_q\}\,.$$

of $\mathrm{End}(V_n^{(N,M)})$, the *centralizer algebra* of U_q on $V_n^{(N,M)}$.

Proposition 6.2. *The following linear map σ defines a homomorphism from $H_{N,M}^n(q)$ to $C_n^{(N,M)}(q)$:*

$$\sigma(T_i) = \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes T \otimes \underbrace{1 \otimes \cdots \otimes 1}_{N+M-i-1} \quad (1 \le i \le N-1)\,,$$

where $T = q \sum_{j=1}^{n} E_{j,j} \otimes E_{j,j} + \sum_{j \neq k} E_{j,k} \otimes E_{k,j} + (q-q^{-1}) \sum_{j<k} E_{j,j} \otimes E_{k,k}\,;$

$$\sigma(T_i^*) = \underbrace{1 \otimes \cdots \otimes 1}_{N+i-1} \otimes T^* \otimes \underbrace{1 \otimes \cdots \otimes 1}_{M-i-1} \quad (1 \le i \le M-1)\,,$$

where $T^* = q\sum_{j=1}^{n} E_{j,j}\otimes E_{j,j} + \sum_{j\neq k} E_{j,k}\otimes E_{k,j} + (q-q^{-1})\sum_{j>k} E_{j,j}\otimes E_{k,k}$;

$$\sigma(E) = \underbrace{1\otimes\cdots\otimes 1}_{N-1}\otimes\Big(\sum_{j,k} q^{-n+2k-1} E_{j,k}\otimes E_{j,k}\Big)\otimes\underbrace{1\otimes\cdots\otimes 1}_{M-1}.$$

Moreover of $n\geq N+M$, *then* σ *is injective.*

Proof. It is easy to check that σ defines a representation of $H^n_{N,M}(q)$ and that $\sigma(T_i)$, $\sigma(T_j^*)$ and $\sigma(E)$ commute with $\Phi(X_i)$, $\Phi(Y_i)$ and $\Phi(q^{\varepsilon_i})$. We show that if $n\geq N+M$, then σ is faithful. To prove this, we have only to define the linear map τ: $C_n^{(N,M)}(q)\mapsto \boldsymbol{C}(q)$ such that $\mathrm{Tr}(x)=\tau(\sigma(x))$. In fact, if there exists such a τ, then by the faithfulness of Tr, σ has no kernel. From the uniqueness of the Markov trace on $A_{N,M}$ the proof of this proposition attributes to the following lemma.

Lemma 6.3. *Let* χ_k *be the diagonal sum on* $\mathrm{Mat}(n, \boldsymbol{C}(q))^{\otimes k}$, *and put* $d=\mathrm{diag}(q^{-n+1}, q^{-n+3}, q^{-n+5}, \cdots, q^{n-1})$. *If we define the linear map* τ *on* $\sigma(H^n_{N,M}(q))$ *by* τ: $x\mapsto(1/[n]^{N+M})\,\chi_{N+M}(d\otimes\cdots\otimes d\otimes d^{-1}\otimes\cdots\otimes d^{-1}\,x)$, *then* $\tau\circ\sigma$ *satisfies the Markov property.*

Proof. We can easily check that $\tau\circ\sigma$ satisfies the Markov property (1), (3), (4) and (5) by direct calculation. Note that $d\otimes\cdots\otimes d\otimes d^{-1}\otimes\cdots\otimes d^{-1}$ is the element of $\Phi(U_q)$ and that $\Phi(U_q)$ and $\sigma(H^n_{N,M}(q))$ commutes each other. Hence

$$\tau(\sigma(a\,b))\,\frac{1}{[n]^{N+M}}\,\chi_{N+M}(d\otimes\cdots\otimes d\otimes d^{-1}\otimes\cdots\otimes d^{-1}\,\sigma(a)\,\sigma(b))$$
$$=\frac{1}{[n]^{N+M}}\,\chi_{N+M}(d\otimes\cdots\otimes d\otimes d^{-1}\otimes\cdots\otimes d^{-1}\,\sigma(b)\,\sigma(a)) = \tau(\sigma(b\,a)).$$

This implies that $\tau\circ\sigma$ satisfies the Markov property (2).

Corollary 6.4. *Let* q_0 *be a complex number. Under the notation of Proposition* 6.2, *if* $n\geq N+M$ *and* $[N+M+n]_{q_0}!\neq 0$ *then* σ *defines the faithful representation of* $H^n_{N,M}(q_0)$ *into* $C_n^{(N,M)}(q_0)$.

Finally we shall show that in case $n\geq N+M$, σ maps $H^n_{N,M}(q)$ onto $C_n^{(N,M)}(q)$. Hence $H^n_{N,M}(q)$ and $C_n^{(N,M)}(q)$ are isomorphic to each other. Since the irreducible rational representations of $GL_n(\boldsymbol{C})$ is indexed by staircases of height n (see, e.g. Propostion 2.1 in [13]), we can label each of them by ρ_γ using the corresponding staircase γ of height n. The following lemma is combinatorially proved and is essential for the following theorem. (See [8] or Corollary 4.7 in [13]).

Lemma 6.5. *Let* γ *be a staircase of height n and let* ρ_γ *denote the irreducible* *rational representation of $GL_n(\boldsymbol{C})$ corresponding to γ. Then the multiplicity of ρ_γ in the decomposition of $V_n^{(N,M)}$ is $c_{N,M}^\gamma$.*

This lemma implies that the dimension of centralizer algebra of $GL_n(\boldsymbol{C})$ on $V_n^{(N,M)}$ is equal to $\sum_{\gamma\in\Lambda}(c_{N,M}^\gamma)^2$, where Λ denotes the set of all the staircases of height n. If $n\geq N+M$, then using Proposition 1.5 and Corollary 1.8 we have

$$\sum_{\gamma\in\Lambda}(c_{N,M}^\gamma)^2=\sum_{\gamma\in\Lambda(N,M)}\left\{m!\binom{N}{m}\binom{M}{m}f^\alpha f^\beta\right\}^2=(N+M)!\,.$$

We can easily ckeck that the centralizer algebra of $GL_n(\boldsymbol{C})$ on the mixed tensor and that of $gl(n,\boldsymbol{C})$ (hence that of $U(gl(n,\boldsymbol{C}))$) coincide. If $n\geq N+M$, then by the faithfulness of σ, we have the following theorem.

Theorem 6.6. *If $n\geq N+M$, then the centralizer algebra $C_n^{(N,M)}(1)$ is isomorphic to the generalized Hecke algebra $H_{N,M}^n(1)$.*

We extend this result to the case when the parameter q takes a generic complex value.

Theorem 6.7. *Assume $n\geq N+M$. Then $C_n^{(N,M)}(q_0)\cong H_{N,M}^n(q_0)$ for generic $q_0\in\boldsymbol{C}$, i.e., there exists a finite subset S of $\boldsymbol{C}$ such that $C_n^{(N,M)}(q_0)\cong H_{N,M}^n(q_0)$ holds for $q_0\in\boldsymbol{C}\backslash S$.*

Proof. Define the $\boldsymbol{C}(q)$-subalgebras $\tilde{H}$ and $\tilde{G}$ of $\mathrm{End}_{\boldsymbol{C}(q)}(V_n^{(N,M)})$ by $\tilde{H}=\sigma(H_{N,M}^n(q))$ and $\tilde{G}=\Phi(U_q(gl(n,\boldsymbol{C})))$. For a complex number $q_0\in\boldsymbol{C}\backslash\{0\}$, we can define $\boldsymbol{C}$-algebras $H(q_0)=\sigma(H_{N,M}^n(q_0))$ and $G(q_0)=\Phi(U_{q_0}(gl(n,\boldsymbol{C})))$. In particular $H(1)=\sigma(H_{N,M}^n(1))$ and $G(1)=\Phi(U(gl(n,\boldsymbol{C})))$. Let $H(q_0)'$ be the centralizer algebra of $H(q_0)$ in $\mathrm{End}_{\boldsymbol{C}}(V_n^{(N,M)})$. Corollary 6.4 implies $G(q_0)\subset H(q_0)'$. Since $H(q_0)$ is semisimple under the condition $[n+N+M]_{q_0}!\neq 0$, we have only to show $\dim_{\boldsymbol{C}} G(q_0)=\dim_{\boldsymbol{C}} H(q_0)'$.

We consider $\tilde{H}'=\{X\in\mathrm{End}_{\boldsymbol{C}(q)}(V_n^{(N,M)});\ XY=YX \text{ for all } Y\in\tilde{H}\}$.

First we show that for an arbitrary complex number $q_0\in\boldsymbol{C}\backslash\{0\}$,

$$\dim_{\boldsymbol{C}} G(q_0)\leq\dim_{\boldsymbol{C}(q)}\tilde{G}\,. \tag{6.8}$$

In fact, if we take a $\boldsymbol{C}(q)$-basis of $\tilde{G}$, $\{X_j(q)\}_{1\leq j\leq r}$, then $G(q_0)=\sum\boldsymbol{C}X_j(q_0)$. Moreover the value of q_0 which makes $\{X_j(q_0)\}$ linearly dependent can be presented as the zeros of some polynomials. It follows that the equality holds in (6.8) for any generic number q_0.

Next we show that for an arbitary complex number $q_0\in\boldsymbol{C}\backslash\{0\}$,

$$\dim_{\boldsymbol{C}(q)}\tilde{H}'\leq\dim_{\boldsymbol{C}(q)}(H(q_0))'\,. \tag{6.9}$$

In fact, if we write $X\in\mathrm{End}(V_n^{(N,M)})$ as $X=\sum X_{I,J}\,e_{i_1}\otimes\cdots\otimes e_{i_N}\otimes e_{j_1}^*\otimes\cdots\otimes e_{j_M}^*$ $(X_{I,J}\in\boldsymbol{C}(q),\ I=(i_1,\cdots,i_N),\ J=(j_1,\cdots,j_M))$ using the natural basis of $\mathrm{End}(V_n^{(N,M)})$, the condition $X\in\tilde{H}'$ can be expressed by linear equations with $\boldsymbol{C}(q)$-coefficients. The condition $X\in(H(q_0))'$ can be expressed by the same set of equations, taking q_0 instead of q. The rank of this system of linear equations reduces only in the case that q takes some values which make some nonzero minors zero. It follows that the equality holds in (6.9) for any generic number q_0.

Sice $\tilde{G}\subset\tilde{H}'$, we have $\dim_{\boldsymbol{C}} G(q_0)\leq\dim_{\boldsymbol{C}(q)}\tilde{G}\leq\dim_{\boldsymbol{C}(q)}\tilde{H}'\leq\dim_{\boldsymbol{C}}(H(q_0))'$. Take $q_0=1$. Then, since $\dim_{\boldsymbol{C}} G(1)=\dim_{\boldsymbol{C}}(H(1))'$, we get $\dim_{\boldsymbol{C}(q)}\tilde{G}=\dim_{\boldsymbol{C}(q)}\tilde{H}'$. It follows that if q_0 is generic, then $\dim_{\boldsymbol{C}} G(q_0)=\dim_{\boldsymbol{C}(q)}\tilde{G}=\dim_{\boldsymbol{C}(q)}\tilde{H}'=\dim_{\boldsymbol{C}}(H(q_0))'$. This completes the proof of Theorem 6.7.

REMARK 6.10. In the case of $n<N+M$, Theorem 6.7 does not hold. However, as in [10], the representation theory for finite dimensional irreducible representations of $U_q(gl(n,\boldsymbol{C}))$ is similar to that of $gl(n,\boldsymbol{C})$ and so $C_n^{(N,M)}(q)$ is isomorphic to $C_n^{(N,M)}(1)$ if q is generic. By using this and representation theory of $gl(n,\boldsymbol{C})$, we can show that the mapping σ in Proposition 6.2 is factored by π in Remark 4.16 and $C_n^{(N,M)}(q)$ is isomorphic to the image $\pi(H_{N,M}^n(q))$.

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