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HIGH DEGREE ANTI-INTEGRAL EXTENSIONS OF NOETHERIAN DOMAINS

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Introduction. Let R be a Noetherian integral domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and let $\pi: R[X]\to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg\varphi_\alpha(X)=d$ and write $\varphi_\alpha(X)=X^d+\eta_1X^{d-1}+\cdots+\eta_d$. Let $I_{[\alpha]}:=\cap_{i=1}^d(R:_R\eta_i)$. For $f(X)\in R[X]$, let $C(f(X))$ denote the ideal generated by the coefficients of $f(X)$. Let $J_{[\alpha]}:=I_{[\alpha]}C(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called an anti-integral element of degree d over R if $\operatorname{Ker}\pi=I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, $R[\alpha]$ is called an anti-integral extension of R. In the case $K(\alpha)=K$, an anti-integral elemet α is the same as an anti-integral element (i.e., $R=R[\alpha]\cap R[1/\alpha]$) defied in [5]. The element α is called a super-primitive element of degree d over R if $J_{[\alpha]}\not\subset p$ for all primes p of depth one.

For $p\in\operatorname{Spec}(R)$, $k(p)$ denotes the residue field R_p/pR_p and $\operatorname{rank}_{k(p)}R[\alpha]\otimes_R k(p)$ denotes the dimension as a vector space over $k(p)$. We are interested in characterizing the flatness and the integrality of an anti-integral extension $R[\alpha]$ of R. Indeed, among others we obtain the following results:

(i) $R[\alpha]$ is flat over R if and only if $\operatorname{rank}_{k(p)}R[\alpha]\otimes_R k(p)\le d$ for all $p\in\operatorname{Spec}(R)$,

(ii) $R[\alpha]$ is integral over R if and only if $\operatorname{rank}_{k(p)}R[\alpha]\otimes_R k(p)=d$ for all $p\in\operatorname{Spec}(R)$.

Thus if an anti-integral extension $R[\alpha]$ is integral over R, then $R[\alpha]$ is flat over R. Concerning a super-primitive element, we obtain that if R is a Krull domain and α is an algebraic element over R, then α is a super-primitive element. We also obtain that a super-primitive element is an anti-integral element. More precisely, α is super-primitive over R if and only if α is anti-integral over R and $R[\alpha]_p$ is flat over R_p for any prime ideal p of depth one.

Using these results, we obtain the following:

Let $\Delta(S)$ denote the set $\{p\in\operatorname{Spec}(R)\,|\,\operatorname{rank}_{k(p)}S\otimes_R k(p)=d\}$, where S is an extension of R of degree d and let $Dp_1(R)$ denote the set of all prime ideals of R of depth one. Assume that $[L:K]=d$, and that $\alpha_1,\cdots,\alpha_n\in L$ are anti-integral elements of degree d, and let $A=R[\alpha_1,\cdots,\alpha_n]$. If $\Delta(R[\alpha_i])\supset Dp_1(R)$ $(1\le i\le n)$

and $Ur(R[\alpha_i]) \supset Dp_1(R)$, where $Ur(A)$ denotes the set $\{p \in \mathrm{Spec}(R) \mid A_p$ is unramified over $R_p\}$, then A is integral over R, and A_p is etale over R_p for $p \in \Delta(A)$. If $\Delta(A) = \mathrm{Spec}(R)$ in addition to the above assumptions, then A is integral and etale over R.

Notations and Conventions. Throughout this paper, we use the following notations unless otherwise specified.

R: a Noetherian integral domian,
$K := K(R)$: the quotient field of R,
L: an algebraic field extension of K,
α: a non-zero element of L,
$d = [K(\alpha): K]$,
$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, the minimal polynomial of α over K.

Let $\pi: R[X] \to R[\alpha]$ be an R-algebra homomorphism defined by $X \to \alpha$ and let $A_{[\alpha]} := \mathrm{Ker}\, \pi$. Then $A_{[\alpha]}$ is a prime ideal of $R[X]$ with $A_{[\alpha]} \cap R = (0)$. By definition, $A_{[\alpha]} = \{\psi(X) \in R[X] \mid \psi(\alpha) = 0\}$.
Let $I_{[\alpha]} := \bigcap_{i=1}^{d} (R :_R \eta_i)$, which is an ideal of R.
For $f(X) \in K[X]$,
$C(f(X))$:=the ideal generated by all coefficients of $f(X)$, that is, $C(f(X))$ is the content ideal of $f(X)$.
Let $J_{[\alpha]} := I_{[\alpha]} C(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$.
We also use the following standard notations:

$$k(p) := \text{the residue filed } R_p/pR_p \text{ for } p \in \mathrm{Spec}(R),$$
$$Dp_1(R) := \{p \in \mathrm{Spec}(R) \mid \mathrm{depth}\, R_p = 1\},$$
$$Ht_1(R) := \{p \in \mathrm{Spec}(R) \mid \mathrm{ht}\, p = 1\}.$$

Throughout this paper, all fields, rings and algebras are assumed to be commutative with unity. Our special notations are indicated above and our general reference for unexplained technical terms is [3].

1. Anti-Integral Elements and Super-Primitive Elements

We start with the following definition.

DEFINITION 1.1. Let I be an ideal of $R[X]$ with $I \cap R = (0)$ and let $f(X) = a_0 X^n + a_1 X^{n-1} + \cdots + a_n$ be a polynomial in $R[X]$. We say that $f(X)$ is a *Sharma polynomial* in I if there does not exist $t \in R$ with $t \notin a_0 R$ such that $ta_i \in a_0 R$ for $1 \leq i \leq n$.

We give an equivalent condition for a polynomial to be a Sharma polynomial in the following proposition.

Proposition 1.2. *Let $f(X)$ be a polynomial in $R[X]$. Then $f(X)$ is a Sharma polynomial if and only if $C(f(X))\not\subset p$ for any $p\in Dp_1(R)$.*

Proof. Let $f(X)=a_0X^n+\cdots+a_n(a_i\in R)$.
($\Rightarrow$) Suppose that $C(f(X))\subset p$ for some $p\in Dp_1(R)$. Then $a_0\in p$, and there exists $t\notin a_0R$ such that $p=(a_0R:_R t)$. In this case, $a_i\in p$ implies that $a_it\in a_0R$ $(1\leq i\leq n)$, which asserts that $f(X)$ is not a Sharma polynomial.
($\Leftarrow$) Suppose that $f(X)$ is not a Sharma polynomial. Then there exists $t\in R$ such that $t\notin a_0R$, $ta_i\in a_0R(1\leq i\leq n)$. Since there exists $p\in Dp_1(R)$ such that $(a_0R:_R t)\subset p$, we have $a_i\in(a_0R:_R t)\subset p\,(1\leq i\leq n)$ and obviously $a_0\in p$. So $C(f(X))=(a_0, \cdots, a_n)\subset p$, a contradiction. Q.E.D.

Proposition 1.3. *The following statements are equivalent:*
(i) *$A_{[\alpha]}$ is a principal ideal of $R[X]$,*
(ii) *$I_{[\alpha]}$ is a principal ideal of R,*
(iii) *there exists a Sharma polynomial in $A_{[\alpha]}$ of degree d.*
If one of the above conditions holds, then $A_{[\alpha]}$ is generated by a Sharma polynomial.

Proof. (iii)$\Rightarrow$(i): Let $f(X)$ be a Sharma polynomial in $A_{[\alpha]}$ of degree d. Since $\deg\varphi_\alpha(X)=d$, this Sharma polynomial has the least degrec. So by [6], $A_{[\alpha]}$ is principal.
(i)$\Rightarrow$(ii): Let $A_{[\alpha]}=f(X)R[X]$. Then $f(X)R[X]\supset I_{[\alpha]}\varphi_\alpha(X)R[X]$. Note that $A_{[\alpha]}\otimes_R K=f(X)K[X]=\varphi_\alpha(X)K[X]$ and hence $\deg f(X)=\deg\varphi_\alpha(X)=d$. Take $a\in I_{[\alpha]}$. Then $a\varphi_\alpha(X)=bf(X)$. Let $f(X)=a_0X^d+\cdots+a_d$ with $a_i\in R$. Then $a=ba_0$, so that $I_{[\alpha]}\supset a_0R$ for some $b\in R$. Since $ba_0\eta_i=a\eta_i=ba_i\,(1\leq i\leq d)$, we have $a_0\eta_i=a_i\in R$. Hence $a_0\in I_{[\alpha]}$, which implies that $I_{[\alpha]}=a_0R$.
(ii)$\Rightarrow$(iii): Let $I_{[\alpha]}=bR$. Then $I_{[\alpha]}\varphi_\alpha(X)R[X]=b\varphi_\alpha(X)R[X]\subset A_{[\alpha]}$ and $b\eta_i\in R\,(1\leq i\leq d)$. Suppose that there exists $t\notin bR$ with $tb\eta_i\in bR\,(1\leq i\leq d)$. Then $t\eta_i\in R$ and hence $t\in I_{[\alpha]}=bR$, a contradiction. Thus $b\varphi_\alpha(X)\in R[X]$ is a Sharma polynomial of degree d. Q.E.D.

For later use, we quote the following.

Lemma 1.4 ([6, Cor. 3]). *Let R be an integral domain and I a non-zero ideal of a plynomial ring $R[X]$ such that $I\cap R=(0)$. If there exists a polynowial $f(X)\in I$ such that $f(X)$ is of the least positive degree in I and $C(f(X))=R$, then I is generated by the polynomial $f(X)$.*

DEFINITION 1.5. i) $\alpha\in L$ is called an *anti-integral element* of degree d over R if $A_{[\alpha]}=I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element, we say that $R[\alpha]$ is an *anti-integral extention* of R.
ii) $\alpha\in L$ is called a *super-primitive element* of degree d over R if $J_{[\alpha]}\not\subset p$ for all $p\in Dp_1(R)$. When α is a super-primitive element, we say that $R[\alpha]$ is a *super-primitive extention* of R.

REMARK 1.6. i) In [5], we studied the anti-integrality which is defined as follows: An element $\alpha\in K$ is called anti-integral over R if $R=R[\alpha]\cap R[1/\alpha]$ $(:=R(\alpha))$. We knew that α is anti-integral over R in this sense if and only if $A_{[\alpha]}$ has a linear basis, that is,

$$A_{[\alpha]}=\sum(c_i X-d_i)R[X]$$

with $d_i/c_i=\alpha$ [5, Proof of (1.9)]. The last condition is equivalent to $A_{[\alpha]}=I_{[\alpha]}\varphi_\alpha(X)R[X]$, where $\varphi_\alpha(X)=X-\alpha$. So $\alpha\in K$ is anti-integral over R in this sense if and only if α is an anti-integral element of degree one over R in the sense of Definition 1.5, that is, the anti-integrality defined in [5] is equivalent to the one defined in (1.5) in the case of degree one.
ii) It is immediate that $\alpha\in L$ is a super-primitive element of degree d over R if and only if α is a super-primitive element of degree d over R_p for any $p\in\mathrm{Spec}(R)$. Thus $R[\alpha]$ is a super-primitive extension if R of and only if $R[\alpha]_p$ is a super-primitive extension of R_p for all $p\in\mathrm{Spec}(R)$, where $R[\alpha]_p$ denotes the localization $S^{-1}R[\alpha]$ with $S=R\backslash p$.

Lemma 1.7. *Let $f(X)$ be an element of a polynomial ring $R[X]$ and let $p\in\mathrm{Spec}(R)$. Then $p\supset C(f(X))$ if and only if $R_p[X]/f(X)R_p[X]$ is not flat over R_p.*

Proof. The implication $(\Leftarrow)$ follows from [3, (20.F)].
$(\Rightarrow)$ Since $C(f(X))\subset p$, $pR[X]$ contains $f(X)$, and hence $Q=pR[X]/f(X)R[X]$ is a prime ideal of $B:=R[X]/f(X)R[X]$. Suppose that $B_p=R_p[X]/f(X)R_p[X]$ is flat over R_p. Then B_Q is obtained from B_p by localizing at QB_p. So depth $B_Q\geq$ depth B_p, and hence depth $B_Q\geq$ depth R_p. It is easy to see that depth B_{pB} $=$depth B_Q and $B_{pB}=R[X]_{pB[X]}/f(X)R[X]_{pR[X]}$. Since R is an integral domian, we have depth $B_{pB}=$depth $R[X]_{pR[X]}-1=$depth R_p-1, which is a contradiction. Q.E.D.

Our almost all main results are based on the following theorem.

Theorem 1.8. *Assume that α is an anti-integral element of degree d over R. Then for $p\in\mathrm{Spec}(R)$, the following are equivalent:*
(i) $\mathrm{rank}_{k(p)}R[\alpha]\otimes_R k(p)\leq d$,
(ii) $\mathrm{rank}_{k(p)}R[\alpha]\otimes_R k(p)<\infty$,
(iii) *$R[\alpha]\otimes_R k(p)$ is not isomorphic to a polynomial ring $k(p)[T]$,*
(iv) $J_{[\alpha]}\not\subset p$,
(v) $pR[X]\not\supset A_{[\alpha]}$,
(vi) *$R[\alpha]_p$ is flat over R_p.*

Proof. Since α is anti-integral, $A_{[\alpha]}=I_{[\alpha]}\varphi_\alpha(X)R[X]$.
(iv)$\Rightarrow$(vi): Since $R_p=(J_{[\alpha]})_p=(I_{[\alpha]})_pC(\varphi_\alpha(X))_p$, $(I_{[\alpha]})_p$ is a principal ideal bR_p

for some $b\in I_{[\alpha]}$. So $(A_{[\alpha]})_p=b\varphi_\alpha(X)R_p[X]$. It follows that $R[\alpha]_p\simeq R_p[X]/(A_{[\alpha]})_p=R_p[X]/b\varphi_\alpha(X)R_p[X]$. Thus $R[\alpha]_p$ is flat over R_p by Lemma 1.7 because $R_p=(J_{[\alpha]})_p=C(b\varphi_\alpha(X))_p$.

(iv)$\Rightarrow$(i): By the same argument as above, we have $R[\alpha]_p\simeq R_p[X]/(A_{[\alpha]})_p=R_p[X]/b\varphi_\alpha(X)R_p[X]$. Since $R_p=(J_{[\alpha]})_p=C(b\varphi_a(X))_p$, there exists $i\,(0\leq i\leq d)$ such that $b\eta_i\notin pR_p[X]$. We take i minimal among such ones. Then $b\varphi_\alpha(X)=bX^d+b\eta_1X^{d-1}+\cdots+b\eta_d\equiv b\eta_iX^{d-1}+\cdots+b\eta_d\equiv 0 \pmod{pR_p[X]}$, which means that $\mathrm{rank}_{k(p)}R[\alpha]\otimes_R k(p)\leq d-i\leq d$.

(i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iv): Note that $R[\alpha]_p/pR[\alpha]_p\simeq R_p[X]/(pR[X]+A_{[\alpha]})_p$. Since $\mathrm{rank}_{k(p)}R[\alpha]\otimes_R k(p)<\infty$, $(pR[X]+A_{[\alpha]})_p$ contains an element $f(X)\in R[X]$ such that $C(f(X))_p=R_p$. Indeed, if not, we conclude that $R[\alpha]\otimes_R k(p)\simeq k(p)[T]$, a polynomial ring, a contradiction. We may asume that $f(X)\in A_{[\alpha]}$. So the equality $(A_{[\alpha]})_p=I_{[\alpha]}\varphi_\alpha(X)R_p[X]$ yields that $(J_{[\alpha]})_p=(I_{[\alpha]})_pC(\varphi_\alpha(X))_p=R_p$.

(vi)$\Rightarrow$(iv): Suppose that $J_{[\alpha]}\subset p$. Localizing at p, we may assume that R is a local ring (R, m). Consider the exact sequence:

$$0\rightarrow A_{[\alpha]}\rightarrow R[X]\rightarrow R[\alpha]\rightarrow 0\,.$$

Then $A_{[\alpha]}$ is flat over R because $R[X]$ and $R[\alpha]$ are flat over R. The isomorphism $A_{[\alpha]}=I_{[\alpha]}\varphi_\alpha(X)R[X]\simeq I_{[\alpha]}R[X]$ yields that $I_{[\alpha]}R[X]$ is flat over $R[X]$ and hence $I_{[\alpha]}$ is flat over R. Since R is local, $I_{[\alpha]}=bR$ for some $b\in I_{[\alpha]}$. So $J_{[\alpha]}=bC(\varphi_\alpha(X))$ and $A_{[\alpha]}=b\varphi_\alpha(X)R[X]$. So $C(b\varphi_\alpha(X))\subset m$, and hence $R[\alpha]$ is not flat over R by Lemma 1.7.

(iv)$\Rightarrow$(v): Since $J_{[\alpha]}=I_{[\alpha]}C(\varphi_\alpha(X))\not\subset p$, there exists $a\in I_{[\alpha]}$ such that $aC(\varphi_\alpha(X))=C(a\varphi_\alpha(X))\not\subset p$. Thus $a\varphi_\alpha(X)\notin pR[X]$ and hence $A_{[\alpha]}\not\subset pR[X]$.

(v)$\Rightarrow$(iv): Since $A_{[\alpha]}=I_{[\alpha]}\varphi_\alpha(X)R[X]$, there exists $a\in I_{[\alpha]}$ such that $C(a\varphi_\alpha(X))\not\subset p$. So $J_{[\alpha]}=J_{[\alpha]}C(\varphi_\alpha(X))\not\subset p$.

(v)$\Rightarrow$(iii): There exists $f(X)\in A_{[\alpha]}$ with $f(X)\notin pR[X]$. So $R[\alpha]/pR[\alpha]=(R/p)[\alpha']$, where α' denotes the residue class of α in $R[\alpha]/pR[\alpha]$, and $f(\alpha')=0$. Thus α' is algebraic over R/p.

(iii)$\Rightarrow$(v): Suppose that $A_{[\alpha]}\subset pR[X]$. Then $R[\alpha]/pR[\alpha]=(R]X]/A_{[\alpha]})/p(R[X]/A_{[\alpha]})=R[X]/pR(X)=(R/p)[X]$, which is a polynomial ring over R/p.

Q.E.D.

After the definition in [5], we employ the following.

DEFINITION 1.9. Let A be an extension of R and let $p\in\mathrm{Spec}(R)$. We say that A is a *blowing-up at* p or p is a *blowing-up point* of A/R if the following two conditions are satisfied:

(i) $pA_p\cap R_p=pR_p$ (equivalently $pA\cap R=p$),

(ii) A_p/pA_p is isomorphic to a polynomial ring $(R_p/pR_p)[T]$.

Making use of the above definition, we get the following corollary to Theorem 1.8.

Corollary 1.10. *When α is an anti-integral element over R, the blowing-up locus $\{p\in\mathrm{Spec}(R)\mid p$ is not a blowing-up point of $R[\alpha]\}$ is given by $V(J_{[\alpha]})$, and is the same as the non-flat locus $\{p\in\mathrm{Spec}(R)\mid R[\alpha]_p$ is not flat over $R_p\}$.*

Proof. This follows from Theorem 1.8 and Lemma 1.7.

The next proposition gives rise to the relation between Sharma polynomials and the ideal $A_{[\alpha]}$.

Proposition 1.11.
(*a*) *$R[\alpha]$ is not a blowing-up at any point in $Dp_1(R)$ if and only if $A_{[\alpha]}$ contains a Sharma polynomial.*
(*b*) *$R[\alpha]$ is not a blowing-up at any point in* Spec(R) *if and only if there exists a polynomial $f(X)$ in $A_{[\alpha]}$ such that $C(f(X))=R$.*

Proof. (a) Take $g_0(X)\in A_{[\alpha]}\backslash(0)$. If $g_0(X)$ is a Sharma polynomial, then we are done. Suppose that $g_0(X)$ is not a Sharma polynomial. Let $\{p_1, \cdots, p_t\}$ be the set of all elements in $Dp_1(R)$ satisfying $C(g_0(X))\subset p_i$. Such p_i exists by Proposition 1.2. Since $A_{[\alpha]}\not\subset pR[X]$ for any $p\in Dp_1(R)$, there are $g_i(X)\in A_{[\alpha]}$ such that $C(g_i(X))\not\subset p_i$ $(1\leq i\leq t)$. Put $N(0):=\deg(g_0(X))$ and $N(i):=N(i-1)+\deg(g_i(X))+1$ inductively. Let $f(X):=\sum g_i(X)\,X^{N(i)}$. Then $C(f(X))=C(g_0(X))+\cdots+C(g_t(X))$. By the choice of p_i, there does not exist $p\in Dp_1(R)$ such that $C(f(X))\subset p$. Hence $f(X)$ is a Sharma polynomial. Assume that $A_{[\alpha]}$ contains a Sharma polynomial. Then $A_{[\alpha]}\not\subset pR[X]$ for any $p\in Dp_1(R)$ by Proposition 1.2. So a blowing-up does not occur for $R[\alpha]/R$ on $Dp_1(R)$.
(b) Let $A_{[\alpha]}=(f_1(X), \cdots, f_n(X))\,R[X]$. Take $p\in\mathrm{Spec}(R)$. Then $A_{[\alpha]}\not\subset pR[X]$. So there exists i such that $C(f_i(X))\not\subset p$. Put $N(0)=0$ and $N(i)=N(i-1)+\deg(f_i(X))+1$, and let $f(X)=\sum f_i(X)\,X^{N(i)}$. Then $C(f(X))=C(f_1(X))+\cdots+C(f_n(X))=R$. The converse is obvious. Q.E.D.

By the following theorem, we see that a super-primitive element is an anti-integral element.

Theorem 1.12. *Under the above notations, the following statements are equivalent:*
(i) *α is a super-primitive element of degree d,*
(ii) *α is an anti-integral element of degree d over R and $R_p[\alpha]$ is flat over R_p for all $p\in Dp_1(R)$,*
(iii) *α is an anti-integral element of degree d over R and $pR[X]\not\supset A_{[\alpha]}$ for all $p\in Dp_1(R)$,*
(iv) *α is an anti-integral element of degree d over R and there exists a Sharma polynomial in $A_{[\alpha]}$,*

(v) $J_{[\alpha]}{}^{-1}=R$, *where* $J_{[\alpha]}{}^{-1}:=(R:_K J_{[\alpha]})$.

Proof. (i)$\Rightarrow$(ii): It is clear that $I_{[\alpha]}\varphi_\alpha(X)R[X]\subset A_{[\alpha]}$, and hence $I_{[\alpha]}R[X]\subset\varphi_\alpha(X)^{-1}A_{[\alpha]}$. Put $J=\varphi_\alpha(X)^{-1}A_{[\alpha]}$. Let $I_{[\alpha]}R[X]=Q_1\cap\cdots\cap Q_n$ be an irredundant primary decomposition of the ideal $I_{[\alpha]}R[X]$ and let $P_i=\sqrt{Q_i}$ $1\leq i\leq n$). Assume that Q (resp. P) represents some Q_i (resp. P_i). Since $I_{[\alpha]}$ is a divisorial ideal of R, $I_{[\alpha]}R[X]$ is a divisorial ideal of $R[X]$, and hence depth $R[X]_P=1$. Put $p=P\cap R$. As $p\supset I_{[\alpha]}$, we see that $p\neq(0)$. Thus we have $P=pR[X]$ and depth$(R_p)=1$. Since α is a super-primitive elemnt, $J_{[\alpha]}\not\subset p$ by definition. Therefore there exists an element $a\in I_{[\alpha]}$ such that $(A_{[\alpha]})_p=a\varphi_\alpha(X)R_p[X]$. Hence we have $J_p=aR_p[X]\subset I_{[\alpha]}R_p[X]\subset QR_p[X]$. Thus we get $J\subset R[X]\cap QR_p[X]=Q$, that is, $J\subset I_{[\alpha]}R[X]$ because Q (resp. P, p) is any Q_i (resp. $P_i, p_i:=P_i\cap R$) for $1\leq i\leq n$. This implies that α is an anti-integral element. Hence the assertion follows from Theorem 1.8.
(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv): It is immediate from Theorem 1.8 and Proposition 1.11.
(iv)$\Rightarrow$(i): Since α is an anti-integral element, $A_{[\alpha]}=I_{[\alpha]}\varphi_\alpha(X)R[X]$. By Proposition 1.11(a), $A_{[\alpha]}\not\subset pR[X]$ for all $p\in Dp_i(R)$. Hence there exists an element $a(p)\in I_{[\alpha]}$ such that $f(X)=a(p)\varphi_\alpha(X)$ and $C(f(X))\not\subset p$. Thus $J_{[\alpha]}\not\subset p$ for any $p\in Dp_1(R)$. Therefore α is a super-primitive element.
(i)$\Rightarrow$(v): Assume that $J_{[\alpha]}\not\subset p$ for any $p\in Dp_1(R)$. Then $(J_{[\alpha]}{}^{-1})_p=(R:_K J_{[\alpha]})_p=(R_p:_K(J_{[\alpha]})_p)=(R_p:_K R_p)=R_p$ for any $p\in Dp_1(R)$. Since $J_{[\alpha]}{}^{-1}$ is a divisorial ideal of R, we have $R=\cap R_p=\cap(J_{[\alpha]}{}^{-1})_p\supset J_{[\alpha]}{}^{-1}$, where p ranges over prime ideals of depth one. Thus $R=J_{[\alpha]}{}^{-1}$. Conversely, suppose that $R=J_{[\alpha]}{}^{-1}$ and $J_{[\alpha]}\subset p$ for some $p\in Dp_1(R)$. Then $J_{[\alpha]}{}^{-1}\supset p^{-1}$ and hence $R=(J_{[\alpha]}{}^{-1})^{-1}\subset(p^{-1})^{-1}=p$, a contradiction. Q.E.D.

More equivalent conditions will be seen in the section 2.

By the following result, we see that a super-primitive element is not so special.

Theorem 1.13. *Assume that R is a Krull domain, then any element α whcih is algebraic over R is a super-primitive element over R.*

Proof. Since R is a Krull domian, $Dp_1(R)=Ht_1(R)$. Take $p\in Ht_1(R)$. Then R_p is a DVR. Let v denote the valuation corresponding to R_p. Let $\varphi_\alpha(X)=X^d+\eta_1X^{d-1}+\cdots+\eta_d$ be the minimal polynomial of α. Put $\eta_0=1$. Then there exists j such that $v(\eta_j)\leq v(\eta_i)$ for all i. Thus $\eta_i/\eta_j=a_i/b\in R_p$, where $b\in R\backslash p$, $a_i\in R$. In particular, $a_j=b\notin p$. Hence

$$\varphi_\alpha(X)=\eta_j(a_0/\eta_j)X^d+\cdots+\eta_j(a_d/\eta_j)\eta_d\,.$$

Hence $f(X):=(b/\eta_j)\varphi_\alpha(X)=a_0X^d+\cdots+a_d\in\varphi_\alpha(X)K[X]$. Since $a_j=b\notin p$, we have $C(f(X))\not\subset p$. Since deg $f(X)=d$, we conclude that α is a super-primitive element over R by Theorem 1.10. Q.E.D.

Once we find one super-primitive element, we can get many such elements. Indeed we obtain the following.

Proposition 1.14. *Assume that α is a super-primitive element of degree d over R. Then for any unit u of R and any element $b\in R$, $\beta=u\alpha+b$ is a super-primitive element of degree d over R.*

Proof. We may assume that $u=1$. It is clear that $\varphi_\beta(X)=\varphi_\alpha(X-b)$ because $K(\beta)=K(\alpha)$, $d=\deg\varphi_\alpha(X-b)$ and $\varphi_\alpha(X-b)$ is monic in $K[X]$. We see that $I_{[\alpha]}\subset I_{[\alpha]}$ and $C(\varphi_\alpha(X))=C(\varphi_\alpha(X-b))=C(\varphi_\beta(X))$. Since $(J_{[\alpha]})_p=(I_{[\alpha]})_p C(\varphi_\alpha(X))_p=R_p$ for any $p\in Dp_1(R)$ by Theorem 1.12, $R_p=(J_{[\alpha]})_p\subset(J_{[\beta]})_p$ and hence $(J_{[\beta]})_p=R_p$ for any $p\in Dp_1(R)$. Thus β is a super-primitive element of degree d over R by Theorem 1.12. Q.E.D,

Proposition 1.15. *Assume that R is a local ring containing an infinite field k and that $J_{[\alpha]}=R$. Then there exists an element $\lambda\in k$ which satisfies that*
(*a*) $1/(\alpha-\lambda)$ *belongs to* $R[\alpha]$,
(*b*) $1/(\alpha-\lambda)$ *is a super-primitive element of degree d over R,*
(*c*) $1/(\alpha-\lambda)$ *is integral over R.*

Proof. Since R is local, there exists an element λ in k such that $I_{[\alpha]}\varphi_\alpha(X+\lambda)$ contains a degree d polynomial $g(X)$ in $R[X]$ of which constant term is 1. Put $\beta=\alpha-\lambda$. Then $g(\beta)=0$. Let $h(X)=X^d g(1/X)\in R[X]$. Then $h(1/\beta)=(1/\beta)^d g(\beta)=0$. So $1/\beta$ is integral over R. Since $[K(\alpha):K]=[K(\beta):K]=d$, we conclude that $\varphi_{1/\beta}(X)=h(X)\in R[X]$. Thus $I_{[1/\beta]}=R$ and hence $J_{[1/\beta]}=I_{[1/\beta]}C(\varphi_{1/\beta}(X))=R$. In particular, $1/\beta$ is a super-primitive element of degree d over R by Theorem 1.12. Q.E.D.

2. Integrality and Flatness of Anti-Integral Extensions

The following result asserts that the integrality of an extension of R is determined by localizing at prime ideals in $Dp_1(R)$.

Proposition 2.1. *Let A be an integral domain containing R. Then A is integral over R if and only if A_p $(:=A\otimes_R R_p)$ is integral over R_p for any $p\in Dp_1(R)$.*

Proof. The implication ($\Rightarrow$) is trivial. Consider the converse and assume that A_p is integral over R_p for any $p\in Dp_1(R)$. We have only to show that α is integral over R. Let R' be the integral closure of R in K. Then R' is a Krull domain [3, p.144]. It suffices to show that α is integral over R'. Let R'' be the integral closure of R in $K(A)$ and let $C=R'':_{R''}\alpha$, a denominator ideal of R''. Then $K(R'')=K(A)$ and C is a divisorial ideal of R''. There exists $P\in Dp_1(R'')=Ht_1(R'')$ such that $C\subset P$. Since R''/R' is integral and R' is integrally closed in K, the Going-Down Theorem holds for R''/R'. Thus $P\cap R'\in$

$Ht_1(R')=Dp_1(R')$. In particular, $P\cap R'$ is a divisorial ideal of R'. So $R'':_{R'}\alpha=C\cap R'\subset P\cap R'\in Dp_1(R')$. By [2, (4.6)], $(P\cap R')\cap R$ is a divisorial ideal of R. Hence $R'':_R\alpha=(C\cap R')\cap R\subset(P\cap R')\cap R\in Dp_1(R)$. Put $p=(P\cap R')\cap R$. Then we have $p\in Dp_1(R)$ and $R'':_R\alpha\subset p$, which is a contradiction. Q.E.D$_3$

The integrality of anti-integral extensions is characterized as follows:

Theorem 2.2. *Assume that α is an anti-integral element of degree d over R. Then the following are equivalent:*

(i) *$R[\alpha]$ is integral over R,*
(ii) $\varphi_\alpha(X)\in R[X]$,
(iii) $I_{[\alpha]}=R$,
(iv) $\mathrm{rank}_{k(q)} R[\alpha]\otimes_R k(q)=d$ *for any* $q\in Dp_1(R)$,
(v) $\mathrm{rank}_{k(q)} R[\alpha]\otimes_R k(q)=d$ *for any* $q\in Spec(R)$.

Proof. Since α is anti-integral, $A_{[\alpha]}=I_{[\alpha]}\varphi_\alpha(X)R[X]$. So the equivalence of (i), (ii) and (iii) are immediate because $R[X]/A_{[\alpha]}\simeq R[\alpha]$, and implications (ii)$\Rightarrow$(v)$\Rightarrow$(iv) are obvious.
(iv)$\Rightarrow$(ii): Suppose that $I_{[\alpha]}\subset p$ for some $p\in Dp_1(R)$. Since $J_{[\alpha]}=I_{[\alpha]}C(\varphi_\alpha(X))\not\subset p$ by Theorem 1.8, $(I_{[\alpha]})_p$ is an invertible ideal of R_p and hence $(I_{[\alpha]})_p$ is a principal ideal bR_p of R_p for some b. So $(A_{[\alpha]})_p=(I_{[\alpha]})_p\varphi_\alpha(X)R_p[X]=(b\varphi_\alpha(X))R_p[X]$. Since $I_{[\alpha]}\subset p$, $b\varphi_\alpha(X)\in R_p[X]$ is not monic. Hence either $R[\alpha]\otimes_R k(p)\simeq k(p)[T]$, a polynomial ring or $\mathrm{rank}_{k(p)}R[\alpha]\otimes_R k(p)<d$, a contradiction. Q.E.D.

By the above theorem, we see that the obstrutcion of integrality of anti-integral extensions is given by $I_{[\alpha]}$. Namely, we obtain the following.

Corollay 2.3. *Assume that α is an anti-integral element over R. Then* $V(I_{[\alpha]})=\{p\in\mathrm{Spec}(R)\mid R[\alpha]_p$ *is not integral over* $R_p\}$.

Proof. The integrality is a local-global property. So our conclusion follows from Theorem 2.2. Q.E.D.

REMARK 2.4. Let R be a Noetherian normal domain and let α be an element in a field L containing R. If α is integral over R, then it is a super-primitive element over R. Indeed, when $\varphi_\alpha(X)\in K[X]$ denotes the minimal polynomial of α over R, it is known that α is integral over R if and only if $\varphi_\alpha(X)$ belongs to $R[X]$ ([4, (9.2)]. Since R is normal, $p\in Dp_1(R)\Rightarrow ht(p)=1\Rightarrow R_p$ is a DVR. As $R[\alpha]$ is a finite R-module, $R[\alpha]_p$ is free over R_p for any $p\in Dp_1(R)$. By Theorem 1.10, α is a super-primitive element over R. Moreover $R[\alpha]$ is flat over R by Theorems 1.8 and 3.2 because $R[\alpha]/R$ is super-primitive, integral and flat.

Summing up the results in the preceding argument, we obtain the following:

Assume that α is an anti-integral elmeent of degree d. Let p be a prime ideal of R. Then
(1) $R[\alpha]_p$ is flat over R_p if and only if $\text{rank}_{k(p)} R[\alpha]\otimes_R k(p)\leq d$,
(2) $R[\alpha]_p$ is integral over R_p if and only if $\text{rank}_{k(p)} R[\alpha]\otimes_R k(p)=d$.

In particular, we conclude:

Corollary 2.5. *Assume that α is an anti-integral element of degree d. If $R[\alpha]$ is integral over R, then $R[\alpha]$ is flat over R.*

In view of Proposition 1.11, we extend Theorem 1.8 to the following.

Proposition 2.6. *Assume that α is an anti-integral element of degree d over R. Then the following are equivalent:*
(i) *$R[\alpha]$ is flat over R,*
(ii) $J_{[\alpha]}=R$,
(iii) $\text{rank}_{k(p)} R[\alpha]\otimes_R k(p)<\infty$ *for any* $p\in\text{Spec}(R)$,
(iv) $\text{rank}_{k(p)} R[\alpha]\otimes_R k(p)\leq d$ *for any* $p\in\text{Spec}(R)$,
(v) *$R[\alpha]$ is not a blowing-up at any point in* $\text{Spec}(R)$,
(vi) *$R[\alpha]$ is quasi-finite over R,*
(vii) *$A_{[\alpha]}$ contains a polynomial $f(X)$ with $C(f(X))=R$.*

Proof. The proof follows from Theorem 1.8 and Proposition 1.11 (b).

REMARK 2.7. Let A be over-ring of R (i.e., $R\subset A$ and $K(A)=K$). If A is integral and flat over R on $Dp_1(R)$, then $A=R$. Indeed, it is known that $R=\bigcap_{p\in Dp_1(R)} R_p$. For $p\in Dp_1(R)$, A_p is integral, flat over R_p by the assumption. So A_p is a free R_p-module of rank one. Thus $A_p=R_p$ and hence $R=\bigcap_{p\in Dp_1(R)} R_p\supset A$.

Relating to this remark, we have the following.

Theorem 2.8. *Let α be an algebraic element over R. If $R[\alpha]$ is integral and flat at any point in $Dp_1(R)$, then $R[\alpha]$ is a free R-module and α is a super-primitive element over R.*

Proof. First, we shall show that $I_{[\alpha]}=R$. Suppose that $I_{[\alpha]}\neq R$. Since $I_{[\alpha]}$ is a divisorial ideal of R, there exists $p\in Dp_1(R)$ such that $I_{[\alpha]}\subset p$. Since $R[\alpha]_p$ is integral over R_p by assumption, $R[\alpha]_p$ is a flat extension of R_p. As $R[\alpha]_p$ is flat over R_p, $R[\alpha]_p$ is a free R_p-module of rank d. We want to show that $R[\alpha]_p=R_p+R_p\alpha+\cdots+R_p\alpha^{d-1}$. For this purpose, we have only to show that $1', \alpha', \cdots, \alpha'^{d-1}\in R[\alpha]_p/pR[\alpha]_p$ are linearly independent over $k(p)$, where α' denotes its residue class in $R[\alpha]_p/pR[\alpha]_p$. Suppose the contrary. Then $R[\alpha]_p/pR[\alpha]_p=k(p)[\alpha']=k(p)+k(p)\alpha'+\cdots+k(p)\alpha'^s$ for some $s<d$. But $R[\alpha]_p$ is a free R_p-module of rank d, which asserts that $\text{rank}_{k(p)} R[\alpha]_p/pR[\alpha]=d$,

a contradiction. Thus we have shown that $R[\alpha]_p = R_p + R_p\,\alpha + \cdots + R_p\,\alpha^{d-1}$. So we have a relation: $\alpha^d = \lambda_0 + \lambda_1\,\alpha + \cdots + \lambda_{d-1}\,\alpha^{d-1}$ ($\lambda_i \in R_p$). Since the minimal polynomial $\varphi_\alpha(X)$ of α is unique, we have $\varphi_\alpha(X) = X^d - \lambda_{d-1}\,X^{d-1} - \cdots - \lambda_0$. So $I_{[\alpha]} \not\subset p$, a contradiction. Thus $\varphi_\alpha(X) \in R[X]$, which implies that $A_{[\alpha]} = \varphi_\alpha(X)\,R[X]$ and $R[\alpha]$ is a free R-module. Since $C(\varphi_\alpha(X)) = R$, we conclude that $J_{[\alpha]} = R$. By Theorem 1.12, α is a super-primitive element over R. Q.E.D.

Now we consider a certain over-ring of R which is seen in [5].

DEFINITION 2.9. Let J be a fractional ideal of R. Let $\mathcal{R}(J) := J :_K J$, which is an over-ring of R.

Lemma 2.10. *Let J be a divisorial ideal of R. Then $\mathcal{R}(J) = R$ if and only if $\mathcal{R}(J^{-1}) = R$.*

Proof. Since J is divisorial, $(J^{-1})^{-1} = J$. So we have only to prove one of the implications. Assm Assume that $\mathcal{R}(J) = R$. The implication $\mathcal{R}(J^{-1}) \supset R$ is obvious. Take $\lambda \in \mathcal{R}(J^{-1})$. Then $\lambda J^{-1} \subset J^{-1}$. Thus $R: \lambda J^{-1} \supset R: J^{-1} = (J^{-1})^{-1} = J$. On the other hand, we have $R: \lambda J^{-1} = \lambda^{-1} R: J^{-1} = \lambda^{-1}(R: J^{-1}) = \lambda^{-1}(J^{-1})^{-1} = \lambda^{-1} J$. Thus $\lambda^{-1} J \supset J$, which shows that $J \supset \lambda J$, and hence $\lambda \in \mathcal{R}(J) = R$. Q.E.D.

By these arguments, we extend Theorem 1.12 to the following.

Theorem 2.11. *The following conditions are equivalent*:

(i) *α is a super-primitive element over R,*

(ii) *for each $p \in Dp_1(R)$, there exists $f(X) \in A_{[\alpha]}$ with $(A_{[\alpha]})_p = f(X)\,R_p[X]$,*

(iii) *for each $p \in Dp_1(R)$, there exists $a \in I_{[\alpha]}$ with $(I_{[\alpha]})_p = aR_p$,*

(iv) $\mathcal{R}(I_{[\alpha]}) = R$.

Proof. Denote the degree of α by d.

(i)$\Rightarrow$(ii): Since $J_{[\alpha]} = I_{[\alpha]}\,C(\varphi_\alpha(X)) \not\subset p$ for any $p \in Dp_1(R)$, there exists $a \in I_{[\alpha]}$ with $f(X) := a\varphi_\alpha(X) \in pR[X]$. Note that $(A_{[\alpha]})_K \cap R_p[X] = (A_{[\alpha]})_p$ and $f(X) \in (A_{[\alpha]})_p$. By Proposition 1.2, $f(X)$ is a Sharma polynomial of degree d in $R_p[X]$. So $(A_{[\alpha]})_p = f(X)\,R_p[X]$.

(ii)$\Rightarrow$(iii): Suppose that $(A_{[\alpha]})_p = f(X)\,R_p[X]$. Then $\deg f(X) = d$. Let a be the leading coefficient of $f(X)$. Then $\varphi_\alpha(X) = (1/a)f(X)$ by the uniqueness of the minimal polynomial of α. So $f(X) = a\varphi_\alpha(X)\,R[X]$, and hence $a \in I_{[\alpha]}$. Since $(A_{[\alpha]})_p = f(X)\,R_p[X]$, $(I_{[\alpha]})_p = aR_p$.

(iii)$\Leftrightarrow$(iv): We know that $\mathcal{R}(I_{[\alpha]}) = R$ if and only if $\mathcal{R}(I_{[\alpha]}^{-1}) = R$ by Lemma 2.10. So apply a result of [5, (3.2)] and we conclude that (iii) and (iv) are equivalent.

(iii)$\Rightarrow$(i): Since $(I_{[\alpha]})_p$ is a principal ideal of R_p for any $p \in Dp_1(R)$, there exists $f(X) \in A_{[\alpha]}$ such that $\deg f(X) = d$ and $(A_{[\alpha]})_p = f(X)\,R_p[X]$. Since $f(X)$ is a

Sharma polynomial in $R_p[X]$ by Proposition 1.2 and depth $R_p=1$, $C(f(X))\not\subset p$. Thus $J_{[\alpha]}\not\subset p$ for any $p\in Dp_1(R)$ and hence α is a super-primitive element over R by definition. Q.E.D.

3. Vanishing Points and Blowing-Up Points

Assume that α is an anti-integral element over R. For $p\in\mathrm{Spec}(R)$, $\mathrm{rank}_{k(p)} R[\alpha]\otimes_R k(p)<\infty$ if and only if $R[\alpha]_p$ is flat over R_p by Theorem 2.2. So it may be natural to ask when $\mathrm{rank}_{k(p)} R[\alpha]\otimes_R k(p)$ is infinite or zero.

Let α be an element which is algebraic over R. Recall that $\varphi_\alpha(X)=X^d+\eta_1 X^{d-1}+\cdots+\eta_d$ is the minimal polynomial of α over K, where $d=[K(\alpha):K]$ and $J_{[\alpha]}:=I_{[\alpha]}C(\varphi_\alpha(X))=I_{[\alpha]}+I_{[\alpha]}\eta_1+\cdots+I_{[\alpha]}\eta_d$. Define $B_{[\alpha]}:=I_{[\alpha]}+I_{[\alpha]}\eta_1+\cdots+I_{[\alpha]}\eta_{d-1}$, which is an ideal of R.

We use this notation throughout §3.

Lemma 3.1. *Assume that α is an anti-integral element over R and let $A=R[\alpha]$. For $q\in\mathrm{Spec}(R)$, the following are equivalent*:
i) $qA_q=A_q$,
ii) $qA\cap R\not\subset q$,
iii) $q\supset B_{[\alpha]}$ *and* $q\not\supset I_{[\alpha]}\eta_d$.

Proof. (i)⇒(ii): Since $qA_q=A_q$, there exist $a_i\in q$, $\beta_i\in A$ and $s_i\in R\backslash q$ such that $1=\sum a_i\beta_i/s_i$. Put $s=\prod s_i$. Then $s=\sum a_i\beta_i b_i\in qA\cap R$ with $s\notin q$, where $s\beta_i/s_i=b_i\in A$. Thus $qA\cap R\not\subset q$.
(ii)⇒(i): Take $s\in qA\cap R$ with $s\notin q$. Then $s\in qA_q$ and s is invertible in A_q. Thus $qA_q=A_q$.
(iii)⇒(ii): Take $a\in I_{[\alpha]}$ with $a\eta_d\notin q$. Put $f(X)=a\varphi_\alpha(X)$ and $a\eta_i=b_i$, $a=b_j$, so that $f(X)=b_0X^d+b_1X^{d-1}+\cdots+b_d$. Since $f(\alpha)=0$, $b_0\alpha^d+b_1\alpha^{d-1}+\cdots+b_d=0$. Noting that $b_d\notin q$, b_d is a unit in A_q. Since $b_0,\cdots,b_{d-1}\in q$, $b_d\in qA\subset qA_q$. Thus $qA_q=A_q$.
(ii)⇒(iii): Sinse $qA_q=A_q$, $1=b_0+b_1\alpha+\cdots+b_n\alpha^n$ for some $b_i\in qR_q$. Put $f(x)=b_nX^n+\cdots+b_1X+b_0-1$. Then $f(\alpha)=0$ and b_0-1 is a unit in R_q. The kernel of $R_q[X]\to R[\alpha]_q$ is $(I_{[\alpha]})_q\varphi_\alpha(X)R_q[X]$. So $f(X)\in(I_{[\alpha]})_q\varphi_\alpha(X)R_q[X]$ and $C(f(X))_q=R_q$. Thus it follows that $(J_{[\alpha]})_q=(I_{[\alpha]})_q C(\varphi_\alpha(X))_q=R_q$, which means that $R[\alpha]_q$ is flat over R_q by Theorem 1.8. So $(I_{[\alpha]})_q\varphi_\alpha(X)R_q[X]$ is an invertible ideal of $R_q[X]$. Hence $(I_{[\alpha]})_q$ is a principal ideal of R_q. Let $(I_{[\alpha]})_q=aR_q$. We shall show that all of $a, a\eta_1,\cdots,a\eta_{d-1}$ belong to qR_q. Note that $f(X)\in a\varphi_\alpha(X)R_q[X]$ because $f(\alpha)=0$. So there exists $h(X)\in R_q[X]$ such that $f(X)=a\varphi_\alpha(X)h(X)$. We have $-1\equiv a\varphi_\alpha(X)h(X)\pmod{qR_q[X]}$. Thus $a\eta_i, a\in qR_q$, for $1\le i\le d-1$ and $a\eta_d\notin qR_q$. Therefore $I_{[\alpha]}, I_{[\alpha]}\eta_1,\cdots,I_{[\alpha]}\eta_{d-1}\subset q$ and $I_{[\alpha]}\eta_d\not\subset q$. Q.E.D.

DEFINITION 3.2. Let A be an extension of R and let $p\in\mathrm{Spec}(R)$. We say that p is a *vanishing point* of A/R if $pA_p=A_p$.

Recall that A is a *blowing-up at* p or p is a *blowing-up point* of A/R if the following two conditions are satisfied:
i) $pA_p \cap R_p=pR_p$ (equivalently $pA \cap R=p$, cf. Lemma 3.1),
ii) A_p/pA_p is isomorphic to a polynomial ring $(R_p/pR_p)[T]$.

By Lemma 3.1, we obtain the following theorem.

Theorem 3.3. *Assume that α is an anti-integral element over R and let $A=R[\alpha]$. Then the set of vanishing points (i.e., $\{q\in \operatorname{Spec}(R) \mid qA_q=A_q\}$) is given by $\bigcap_{i=0}^{d-1} V(I_{[\alpha]}\eta_i)\setminus V(I_{[\alpha]}\eta_d)$, where $\eta_0=1$.*

Proposition 3.4. *Assume that α is an anti-integral element of degree d over R and let $A=R[\alpha]$. Consider the following conditions:*
(i) *A is flat over R,*
(ii) *$J_{[\alpha]}=R$,*
(iii) *If $pA_p=A_p$ for $p\in \operatorname{Spec}(R)$, then $pA=A$.*
Then we have implications (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii). *If moreover R is a local ring and $\sqrt{B_{[\alpha]}} \not\supset I_{[\alpha]}\eta_d$, then* (i), (ii) *and* (iii) *are equivalent to each other.*

Proof. (i)$\Leftrightarrow$(ii) was proved in Proposition 2.6. (ii)$\Rightarrow$(iii): Take $p\in \operatorname{Spec}(R)$ and assume that $pA_p=A_p$. Then $p\supset B_{[\alpha]}=I_{[\alpha]}+I_{[\alpha]}\eta_1+\cdots+I_{[\alpha]}\eta_{d-1}$ and $p\not\supset I_{[\alpha]}\eta_d$ by Lemma 3.1. Take $a\in I_{[\alpha]}$ and put $f(X)=a\varphi_\alpha(X)=aX^d+a\eta_1X^{d-1}+\cdots+a\eta_d$. Since $f(\alpha)=0$, we get $a\eta_d\in pA$ and hence $I_{[\alpha]}\eta_d\subset pA$. So $J_{[\alpha]}=B_{[\alpha]}+I_{[\alpha]}\eta_d\subset pA$. Since $J_{[\alpha]}=R$, we conclude that $pA=A$. We will show the last part. Since $\sqrt{B_{[\alpha]}}\not\supset I_{[\alpha]}\eta_d$, there exists $q\in \operatorname{Spes}(R)$ such that $q\supset B_{[\alpha]}$ but $q\not\supset I_{[\alpha]}\eta_d$. Thus $qA_q=A_q$ and so $qA=A$. Let m denote the maximal ideal of R. Suppose that $m\supset J_{[\alpha]}$. Then we have $A/mA\simeq(R/m)[T]$, a polynomial ring (cf. Theorem 1.8). Hence $mA\neq A$. But $q\subset m$ implies that $mA=A$, a contradiction. Thus $J_{[\alpha]}=R$. Q.E.D.

REMARK 3.5. Let the notation be the same as in Proposition 3.4.
(i) When $d=1$ (i.e., α is an element of K), then (i), (ii) and (iii) of Proposition 3.4 are equivalent.
(2) $pA\cap R=p$ if and only if there exists $P\in \operatorname{Spec}(A)$ such that $P\cap R=p$.

REMARK 3.6. Let the notation be the same as in Lemma 3.1. If $B_{[\alpha]}\subset q$, then q is either a vanishing point (i.e., $I_{[\alpha]}\eta_d\not\subset q$) or a blowing-up point (i.e., $I_{[\alpha]}\eta_d\subset q$). So if $\sqrt{J_{[\alpha]}}$ contains $\sqrt{B_{[\alpha]}}$ properly, there exists a vanishing point. Thus $\operatorname{Spec}(A)\to\operatorname{Spec}(R)$ is not surjective.

Proposition 3.7. *Assume that α is an anti-integral element of degree d over R and let $A=R[\alpha]$. Then* $\operatorname{Spec}(A)\to\operatorname{Spec}(R)$ *is surjective if and only if* $\sqrt{J_{[\alpha]}}=\sqrt{B_{[\alpha]}}$.

Proof. ($\Rightarrow$): Since $J_{[\alpha[}\supset B_{[\alpha]}$, $\sqrt{J_{[\alpha]}}\supset\sqrt{B_{[\alpha]}}$. If $B_{[\alpha]}\subset q$ for some $q\in\mathrm{Spec}(R)$, there exists $Q\in\mathrm{Spec}(A)$ such that $Q\cap R=q$ because $\mathrm{Spec}(A)\to\mathrm{Spec}(R)$ is surjective. So $qA_q\neq A_q$, which means that q is not a vanishing point. Thus by Remark 3.6, q is a blowing-up point, that is, $q\supset J_{[\alpha]}$. Therefore $\sqrt{J_{[\alpha]}}=\sqrt{B_{[\alpha]}}$. ($\Leftarrow$): Suppose that $\mathrm{Spec}(A)\to\mathrm{Spec}(R)$ is not surjective. There exists $q\in\mathrm{Spec}(R)$ such that $qA_q=A_q$. So $q\supset\sqrt{B_{[\alpha]}}=\sqrt{J_{[\alpha]}}\supset J_{[\alpha]}\supset I_{[\alpha]}\eta_d$, a contradiction. Q.E.D.

Proposition 3.8. *Let the notation be the same as in Proposition* 3.7 *and let* $p\in\mathrm{Spec}(R)$ *satisfy* $pA_p=A_p$. *If* $q\supset pA\cap R$, *then* q *is a blowing-up point.*

Proof. Since $p\in\mathrm{Spec}(R)$ satisfies $pA_p=A_p$, we have $p\supset B_{[\alpha]}$. Thus $\eta_d I_{[\alpha]}\subset\alpha^d I_{[\alpha]}+\cdots+\eta_{d-1}\alpha I_{[\alpha]}\subset B_{[\alpha]}A\subset pA$. So $q\supset pA\cap R\supset B_{[\alpha]}+I_{[\alpha]}\eta_d=J_{[\alpha]}$, which means that q is a blowing-up point. Q.E.D.

REMARK 3.9. Let k be a field, a, b indeterminates and $R=k[a,b]$. Let α be a root of an equation $aX^2+bX+a=0$ and put $A=R[\alpha]$. Then $J_{[\alpha]}=(a,b)R$ and $\mathrm{grade}((a,b)R)=2$ so that α is a super-primitive element by Theorem 1.12. In this case, $J_{[\alpha]}=B_{[\alpha]}=(a,b)R$. Thus $\mathrm{Spec}(A)\to\mathrm{Spec}(R)$ is surjective, but not flat. Hence the implication (iii)$\Rightarrow$(i) in Proposition 3.4 does not necessarily hold.

Theorem 3.10. *Assume that* α *is an anti-integral element over* R *and let* $p\in\mathrm{Spec}(R)$. *If* $R[\alpha]$ *is not a blowing-up at* q, *then* $\mathrm{depth}\,R[\alpha]_Q=\mathrm{depth}\,R_q$ *for* $Q\in\mathrm{Spec}(R[\alpha])$ *with* $Q\cap R=q$.

Proof. Since α is an anti-integral element over R and q is not a blowing-up point, $R[\alpha]_q$ is flat over R_q by Theorem 1.8. Since $R[\alpha]_Q$ is obtained from $R[\alpha]_q$ by localizing at $QR[\alpha]_q$, $R[\alpha]_Q$ is flat over Rq. So we have $\mathrm{depth}\,R_q\leq\mathrm{depth}\,R[\alpha]_Q$. As q is not a blowing-up point, there exists $a\in I_{[\alpha]}$ such that $a\varphi_\alpha(X)R_q[X]=(A_{[\alpha]})_q$. Put $f(X):=a\varphi_\alpha(X)$. Since $Q\in\mathrm{Spec}(R[\alpha])$, there exists $P\in\mathrm{Spec}(R[X])$ such that $P\supset A_{[\alpha]}$ and $Q=P/A_{[\alpha]}$. Then $Q_q=P_q/(A_{[\alpha]})_q=P_q/f(X)R_q[X]$. So $QR[\alpha]_Q=PR[X]_P/f(X)R[X]_P$ implies that $\mathrm{depth}\,R[\alpha]_Q=\mathrm{depth}\,R[X]_P-1$. Now since $P\cap R=q$, we have $P\supset pR[X]$. Suppose that $P=qR[X]$. Then $qR[X]=P\supset A_{[\alpha]}$, which asserts that q is a blowing-up point. So we have $P\neq qR[X]$. Since $PR_q[X]/qR_q[X]$ $(\subset k(P)[X])\neq0$, we have $PR_q[X]=qR_q[X]+g(X)R_q[X]$ for some $g(X)\in R[X]\backslash qR[X]$. Hence $\mathrm{depth}\,R[X]_P\leq\mathrm{depth}\,R[X]_qR[X]+1$. We obtain that $\mathrm{depth}\,R[\alpha]_Q\leq\mathrm{depth}\,R_q$ because $\mathrm{depth}\,R[X]_qR[X]=\mathrm{depth}\,R_q$. Thus $\mathrm{depth}\,R_q=\mathrm{depth}\,R[\alpha]_Q$. Q.E.D.

4. Unramifiedness and Etaleness of Super-Primitive Extensions

The following result can be proved by using [1, VI (6.8)] but we give a direct proof. If α is super-primitive and integral over R, $R[\alpha]$ is finite, flat over

R (cf. Proposition 1.11).

Proposition 4.1. *Assume that α is an anti-integral element which is integral over R. Then $R[\alpha]$ is unramified over R if and only if $R[\alpha]_p$ is unramified over R_p for any $p\in Dp_1(R)$.*

Proof. Since $A:=R[\alpha]$ is integral over R, $\varphi_\alpha(X)\in R[X]$ by Theorem 2.2. For a polynomial f, we denote the derivative of f by f'. Then $\varphi'_\alpha(\alpha)=d\alpha^{d-1}+(d-1)\eta_1\alpha^{d-2}+\cdots+\eta_{d-1}$ and let $p\in \mathrm{Spec}(R)$. Then $\varphi'_\alpha(\alpha)A\not\subset P$ for any $P\in \mathrm{Spec}(A)$ with $P\cap R=p$ if and only if A_p is unramified over R_p (cf. [1, VI (6.12)]). Suppose that $\varphi'_\alpha(\alpha)A\neq A$. Then there exists $P\in Ht_1(A)$ such that $\varphi'_\alpha(\alpha)\in P$. Put $p=P\cap R$. Then depth $A_P=1$ implies depth $R_p=1$ because A_p is flat over R_p. Thus A_q is unramified over R_P by the assumption. Hence A_p is unramified over R_p, which is a contradiction. So $\varphi'_\alpha(\alpha)A=A$, which means that A is unramified over R. Q.E.D.

REMARK 4.2. Let the notation be the same as in Proposition 4.1 and its proof. Let $B=A[1/\alpha]$. Then for $P\in \mathrm{Spec}(B)$, B_P is unramified over $R_{R\cap B}$ if and only if $P\not\ni\varphi'_\alpha(\alpha)B$. Indeed, let $P\subset B$ be a prime ideal and put $Q=P\cap A$ and $p=P\cap R$. When B_P/R_p is ramified, A_Q/R_p is ramified. So $\varphi'_\alpha(\alpha)\in Q\subset P$. Conversely, if $\varphi'_\alpha(\alpha)\in P$, then $Q=P\cap A\ni\varphi'_\alpha(\alpha)$. So $B_P=A_Q$ is ramified over R_p.

It is known that the purity of branch locus holds for a finite flat extension [1]. The following is a result similar to this fact.

Proposition 4.3. *Assume that α is a super-primitive element which is flat over R and that R contains an infinite field k. Then $R[\alpha]$ is unramified over R if and only if $R[\alpha]_p$ is unramified over R_p for any $p\in Dp_1(R)$.*

Proof. We have only to consider the case that R is a local ring. So we may assume that (R, m) is a local ring. If $A:=R[\alpha]$ is integral over R, we have shown this in Proposition 4.1. Assume that A is not integral over R. Since $J_{[\alpha]}=R$ by Theorem 2.2, replacing α by $\alpha-\lambda$ for some $\lambda\in k$, we may assume by Proposition 1.14, that α satisfies that

(a) $1/\alpha\in R[\alpha]$,

(b) $1/\alpha$ is a super-primitive element of degree d over R,

(c) $1/\alpha$ is integral over R.

Hence we have

$$R\subset R[1/\alpha]\subset R[\alpha, 1/\alpha]=R[\alpha]=A\,.$$

Apply Remark 4.2 to $B=R[1/\alpha][(1/\alpha)^{-1}]=A$. We conclude that for $P\in \mathrm{Spec}(A)$, A_P is unramified over $R_{P\cap R}$ if and only if $P\not\ni\varphi'_{1/\alpha}(1/\alpha)A$. In the same way as in the proof of Proposition 4.1, the assumption that A_p is unramified over R_p for any $p\in Dp_1(R)$ yields that $R[\alpha]$ is unramified over R. Q.E.D.

As a consequence of Propositions 4.1 and 4.3, we obtain the following theorem.

Theorem 4.4. *Assume that α is a super-primitive element over R and that R contains an infinite field k. Then there exist $p_1, \cdots, p_t\in Dp_1(R)$ (t may be 0) such that the non-etale locus of $R[\alpha]$ is given by $V(J_{[\alpha]})\cup\cup_{i=1}^{t} V(p_i)$.*

EXAMPLE 4.5. Let k be a field, a, b indeterminates and $R=k[a, b]$. Let α be a root of an equation $aX^2+bX+a=0$ and put $A=R[\alpha]$. Then $J_{[\alpha]}=(a, b)R$. Assume that $p\in\mathrm{Spec}(R)$ and $p\not\supset J_{[\alpha]}$. When $a\notin p$, $(2\alpha+b/c)A_p$ is the ramification locus. When $a\in p$ and $b\notin p$, $(\alpha+1)A_p$ is the ramification locus.

DEFINITION 4.6. Let A be an extension of R with $[K(A): K]=d$. Define

$$\Delta(A):=\{q\in\mathrm{Sepc}(R)\,|\,\mathrm{rank}_{k(q)}\, A\otimes_R k(q)=d\}\ .$$

It is easy to see that when α is a super-primitive element of degree d over R, we have:

$$\begin{aligned}&\Delta(R[\alpha])\supset Dp_1(R)\\ &\Leftrightarrow R[\alpha] \text{ is integral over } R\\ &\Rightarrow R[\alpha] \text{ is flat over } R\,.\end{aligned}$$

When A is a finitely generated extension of R, define:

$$Ur(A):=\{p\in\mathrm{Spec}(R)\,|\,A_p \text{ is unramified over } R_p\}\ ,$$

which is an open set of $\mathrm{Spec}(R)$.

Under these preparations, we finally obtain the following.

Theorem 4.7. *Assume that $[L: k]=d$, and that $\alpha_1, \cdots, \alpha_n\in L$ are super-primitive elements of degree d, and let $A=R[\alpha_1, \cdots, \alpha_n]$. If $\Delta(R[\alpha_i])\supset Dp_1(R)$ $(1\leq i\leq n)$ and $Ur(R[\alpha_j])\supset Dp_1(R)$ for some j, then A is integral over R, and A_p is etale over R_p for any $p\in\Lambda(A)$. If $\Delta(A)=\mathrm{Spec}(R)$ in addition to the preceding assumptions, then A is integral and etale over R.*

Proof. The assumption $Dp_1(R)\subset\Delta(R[\alpha_i])$ implies that α_i is integral over R and $\Delta(R[\alpha_i])=\mathrm{Spec}(R)$ by Theorem 2.2, and hence A is integral gver R. Take $p\in\Delta(A)$. Then $p\in\Delta(R[\alpha_j])$ and $R[\alpha_j]$ is finite, flat over R as was shown in Theorem 1.8. Thus $R[\alpha_j]_q$ is an R_p-free module of rank d. Since $Ur(R[\alpha_j])\supset Dp_1(R)$, $R[\alpha_j]$ is unramified over R by Proposition 4.1. Hence $pR[\alpha_j]_p$ is a radical ideal. Noting that A is integral over $R[\alpha_j]$, we have $pA_p\cap R[\alpha_j]_p=$

$pR[\alpha_j]_p$. Thus $R[\alpha_j]_p/pR[\alpha_j]_p \subset A_p/pA_p$. As both of those sides have the same dimension d as vector spaces over $k(p)$, we have $R[\alpha_j]_p/pR[\alpha_j]_p = A_p/pA_p$, which means that $A_p = R[\alpha_j]_p + pA_p$. By Nakayama's lemma, we get $A_p = R[\alpha_j]_q$. Therefore A_p is unramified and flat (i.e., etale) over R_p for any $p \in \Delta(A)$.

Q.E.D.

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