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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR A MATHEMATICAL MODEL ON CHEMICAL INTERFACIAL REACTIONS

Dedicated to Professor Hiroki Tanabe on his sixtieth birthday

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1. Introduction

In the present paper we investigate the asymptotic behavior of solutions for parabolic systems closely related with a chemical interfacial reaction model which is considered in Yamada and Yotsutani [7]. Let I and $\bar{I}$ denote the intervals $(0, 1)$ and $[0, 1]$ respectively. Consider an initial boundary value problem for $u_i=u_i(x, z)$ ($i=1, 2, 3$ and $(x, z)\in\bar{I}\times[0, \infty)$):

$$\text{(P)}\qquad \begin{cases} a_i(x)\dfrac{\partial u_i}{\partial z} = \dfrac{\partial^2 u_i}{\partial x^2}\,, & (x, z)\in I\times(0, \infty)\,, \\ \dfrac{\partial u_i}{\partial x}(0, z) = R_i(u_1(0, z), u_2(0, z), u_3(0, z))\,, & z\in(0, \infty)\,, \\ \dfrac{\partial u_i}{\partial x}(1, z) = 0\,, & z\in(0, \infty)\,, \\ u_i(x, 0) = \phi_i(x)\geq 0\,, & x\in I\,, \end{cases}$$

where $a_i(x)$ ($i=1, 2, 3$) are given functions, $\phi_i(x)$ ($i=1, 2, 3$) are given nonnegative initial data and

$$\begin{aligned} R_1(u_1, u_2, u_3) &= k_1 R_0(u_1, u_2, u_3)\,, \\ R_2(u_1, u_2, u_3) &= k_2 R_0(u_1, u_2, u_3)\,, \\ R_3(u_1, u_2, u_3) &= -k_3 R_0(u_1, u_2, u_3)\,, \\ R_0(u_1, u_2, u_3) &= (u_1^{n_1} u_2^{n_2} - u_3^{n_3})\,\beta(u_1, u_2, u_3) \end{aligned}$$

with positive constants $k_i (i=1, 2, 3)$, positive integers $n_i (i=1, 2, 3)$ and a suitable positive function β.

The initial boundary value problem (P) models chemical reactions on interfaces. Such a model has been proposed by Kawano et al. [3]. They put

$$a_i(x) = c_i(1-x^2) \quad (i = 1, 2, 3)\,,$$

$$R_0(u_1, u_2, u_3) = \frac{u_1 u_2 - u_3}{1+u_2+u_1 u_2},$$

$$\text{i.e., } \beta(u_1, u_2, u_3) = \frac{1}{1+u_2+u_1 u_2},$$

where $c_i (i=1, 2, 3)$ are positive constants. As to the derivation of (P), see also Appendix of [7].

Taking account of the chemical background, we impose the following conditions on a_i and $R_i (i=1, 2, 3)$:

(A) $a_i \in C^\infty(\bar{I})$, $a_i \geq 0$ on $\bar{I}$ and $a_i > 0$ on $[0, 1)$.

(R.1) There exist an open subset U of $\boldsymbol{R}^3$ and a positive constant δ_R such that

$$U \supset [-\delta_R, \delta_R]^3 \cup [0, \infty)^3,$$
$$\beta(u_1, u_2, u_3) \in C^\infty(U) \quad \text{and} \quad \beta(u_1, u_2, u_3) > 0 \text{ on } U.$$

(R.2) There exists a positive constant C_R such that

$$-\sum_{i=1}^{3} R_i(u_1, u_2, u_3)\, u_i^{2p-1} \leq C_R \sum_{i=1}^{3} u_i^{2p}$$

for all $(u_1, u_2, u_3) \in [0, \infty)^3$ and $p \in [1, \infty)$.

These conditions assure the existence and uniqueness of nonnegative global solutions for (P) by the results of [7].

The purpose of the present paper is to obtain the uniform convergence of solutions for (P), together with all their derivatives, as $z \to \infty$. Recently, a related problem has been discussed by [2], [4] and [6] in a simpler interfacial reaction model with two components ($R_1(u_1, u_2) = k_1 u_1^{n_1} u_2^{n_2}$, $R_2(u_1, u_2) = k_2 u_1^{n_1} u_2^{n_2}$). We have established a method of constructing infinitely many Lyapunov functions systematically in [2]. We will develop such an energy method. A new ingredient is to introduce a Lyapunov function of special form fitting in with the nonlinearity peculiar to chemical reactions.

The organization of this paper is as follows. In §2 we state main results (Theorems 1, 2 and 3). In §3 we prepare a fundamental lemma which will be used throughout the paper. In §4 we show the existence and uniqueness of solutions for (P) and the corresponding stationary problem (SP). Finally, §5 contains the proof of the uniform convergence of solutions and all their derivatives as $z \to \infty$.

NOTATION

We use the following notation throughout this paper. For any vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, we simply write

$$u \geq v \quad \text{if and only if} \quad u_1 \geq v_1,\, u_2 \geq v_2 \quad \text{and} \quad u_3 \geq v_3\,,$$
$$u > v \quad \text{if and only if} \quad u_1 > v_1,\, u_2 > v_2 \quad \text{and} \quad u_3 > v_3\,,$$

and abbreviate $R_i(u_1, u_2, u_3)$ as $R_i(u)$. For a vector-valued function $u=u(x, z)=(u_1(x, z), u_2(x, z), u_3(x, z))$, we denote its derivatives by

$$\begin{aligned} D_z^i D_x^j u &= (D_z^i D_x^j u_1, D_z^i D_x^j u_2, D_z^i D_x^j u_3) \\ &= \left(\frac{\partial^{i+j} u_1}{\partial z^i \partial x^j}, \frac{\partial^{i+j} u_2}{\partial z^i \partial x^j}, \frac{\partial^{i+j} u_3}{\partial z^i \partial x^j} \right). \end{aligned}$$

For any vector-valued functions $u=u(x)=(u_1(x), u_2(x), u_3(x))$ and $\rho=\rho(x)=(\rho_1(x), \rho_2(x), \rho_3(x))$ with $\rho \geq 0$,

$$\begin{aligned} &||u_i||_p = \{\int_I |u_i|^p\, dx\}^{1/p}\,, && ||u||_p = \{\sum_{i=1}^{3} ||u_i||_p^p\}^{1/p}\,, \\ &||u_i|| = ||u_i||_2\,, && ||u|| = ||u||_2\,, \\ &||u_i||_\infty = \operatorname*{ess\,sup}_{x\in I} |u_i(x)|\,, && ||u||_\infty = \operatorname*{ess\,sup}_{x\in I} |u(x)| \\ & && \quad = \operatorname*{ess\,sup}_{x\in I} \{\sum_{i=1}^{3} |u_i(x)|^2\}^{1/2}\,, \\ &||u_i||_{2\,;\,\rho_i} = \{\int_I u_i^2 \rho_i\, dx\}^{1/2}\,, && ||u||_{2\,;\,\rho} = \{\sum_{i=1}^{3} ||u_i||_{2\,;\,\rho_i}^2\}^{1/2}\,. \end{aligned}$$

2. Main results

Our first theorem is concerned with the existence, uniqueness, regularity and positivity of solutions for (P).

Theorem 1. *In addition to* (A), (R.1) *and* (R.2), *assume that* $\phi=(\phi_1, \phi_2, \phi_3)$ *satisfies*

$$\phi \in L^\infty(I)^3 \quad \textit{and} \quad \phi \geq 0 \textit{ on } I\,.$$

Then the initial boundary value problem (P) *has a unique solution* $u=(u_1, u_2, u_3) \in C^\infty(\bar{I} \times (0, \infty))^3$ *satisfying the initial condition in* L^2*-sense, i.e.,*

$$\lim_{z\to 0} ||u(\cdot, z) - \phi(\cdot)|| = 0\,.$$

Moreover u has the following properties:

(i) *u is nonnegative and bounded on* $\bar{I} \times [0, \infty)$,

(ii) *u satisfies the law of "mass conservation", i.e.,*

$$\frac{1}{k_i} ||u_i(\cdot, z)\, a_i||_1 + \frac{1}{k_3} ||u_3(\cdot, z)\, a_3||_1 = E_i,\; z \in [0, \infty) \quad (i=1, 2)\,, \tag{2.1}$$

where

$$E_i = \frac{1}{k_i} ||\phi_i\, a_i||_1 + \frac{1}{k_3} ||\phi_3\, a_3||_1 \quad (i=1, 2)\,,$$

(iii) *the positivity of* u_1, u_2 *and* u_3 *is assured by that of* E_1 *and* E_2, *i.e.*,

$$\begin{array}{llll} u_1>0,\, u_2>0 \text{ and } u_3>0 & \text{on } \bar{I}\times(0,\infty) & \text{if} & E_1>0 \text{ and } E_2>0\,, \\ u_1>0 \text{ and } u_2\equiv u_3\equiv 0 & \text{on } \bar{I}\times(0,\infty) & \text{if} & E_1>0 \text{ and } E_2=0\,, \\ u_2>0 \text{ and } u_1\equiv u_3\equiv 0 & \text{on } \bar{I}\times(0,\infty) & \text{if} & E_1=0 \text{ and } E_2>0\,, \\ u_1\equiv u_2\equiv u_3\equiv 0 & \text{on } \bar{I}\times(0,\infty) & \text{if} & E_1=0 \text{ and } E_2=0\,. \end{array}$$

In the study of asymptotic properties of (P), we should note (2.1). Therefore, the stationary problem associated with (P) will be written in the form of algebraic equations for $u^\infty=(u_1^\infty, u_2^\infty, u_3^\infty)\in \boldsymbol{R}^3$:

$$\text{(SP)} \qquad \begin{cases} u^\infty \geq 0\,, \\ R_0(u^\infty)=0\,, \\ \dfrac{||a_i||_1}{k_i}u_i^\infty+\dfrac{||a_3||_1}{k_3}u_3^\infty=E_i \quad (i=1,2)\,. \end{cases}$$

Concerning (SP), we get the following theorem.

Theorem 2. *Suppose that* (A) *and* (R.1) *hold. Then there exists a unique solution* $u^\infty=(u_1^\infty, u_2^\infty, u_3^\infty)$ *of* (SP). *More precisely,*

$$\begin{array}{lll} u_1^\infty>0,\, u_2^\infty>0 \text{ and } u_3^\infty>0 & \text{if} & E_1>0 \text{ and } E_2>0\,, \\ u_1^\infty=\dfrac{||\phi_1 a_1||_1}{||a_1||_1}>0 \text{ and } u_2^\infty=u_3^\infty=0 & \text{if} & E_1>0 \text{ and } E_2=0\,, \\ u_2^\infty=\dfrac{||\phi_2 a_2||_1}{||a_2||_1}>0 \text{ and } u_1^\infty=u_3^\infty=0 & \text{if} & E_1=0 \text{ and } E_2>0\,, \\ u_1^\infty=u_2^\infty=u_3^\infty=0 & \text{if} & E_1=0 \text{ and } E_2=0\,. \end{array}$$

Now we are ready to state results on the asymptotic behavior of solutions for (P) as $z\to\infty$.

Theorem 3. *Suppose that* (A), (R.1) *and* (R.2) *hold. It follows that*

$$\lim_{z\to\infty}||u(\cdot,z)-u^\infty||_\infty=0\,, \tag{2.2}$$

and

$$\lim_{z\to\infty}||D_z^i D_x^j u||_\infty=0 \tag{2.3}$$

for all nonnegative integers i, j *with* $(i,j)\neq(0,0)$.

REMARK 2.1. It is possible to derive the rates of the convergence stated in Theorem 3. Indeed, in the case when $E_1>0$ and $E_2>0$,

$$\begin{aligned} ||u(\cdot,z)-u^\infty||_\infty &= O(\exp(-\lambda_+ z)) \quad \text{as} \quad z\to\infty\,, \\ ||D_z^i D_x^j u||_\infty &= O(\exp(-\lambda_+ z)) \quad \text{as} \quad z\to\infty\,, \end{aligned}$$

where λ_+ is the least positive eigenvalue for the linearized operator around u^∞ associated with (P). We can also obtain analogous rates of convergence in the remaining cases. We will prove this fact in the forthcoming paper.

3. Preliminary lemma

The following lemma is very useful to derive several uniform convergence properties from various estimates of solutions.

Lemma 3.1. *For a positive integer m, let* $\{p_k(z)\}_{0\le k\le m}$, $\{\bar{p}_k(z)\}_{1\le k\le m}$, $\{q_k(z)\}_{0\le k\le m}$ *and* $\{\bar{q}_k(z)\}_{0\le k\le m-1}$ *be sequences of nonnegative functions of class* $C^1([1,\infty))$, *and let* $s_0(z)$ *be a nonnegative function of class* $C([1,\infty))$. *Suppose that there exists a positive constant* η *such that*

$$\begin{cases} \dfrac{dp_0}{dz}+q_0+s_0\le 0\,, \\ \dfrac{d\bar{q}_0}{dz}+\bar{p}_1\le \dfrac{1}{\eta}\, q_0\,, \\ \dfrac{dp_1}{dz}+q_1\le \dfrac{1}{\eta}\,(\bar{p}_1+q_0)\,, \\ \dfrac{d\bar{q}_{k-1}}{dz}+\bar{p}_k\le \dfrac{1}{\eta}\,(\sum_{j=1}^{k-1}\bar{p}_j+\sum_{j=0}^{k-1}q_j) \quad (k=2,\cdots,m)\,, \\ \dfrac{dp_k}{dz}+q_k\le \dfrac{1}{\eta}\,(\sum_{j=1}^{k}\bar{p}_j+\sum_{j=0}^{k-1}q_j) \quad (k=2,\cdots,m)\,, \\ \left|\dfrac{dq_{k-1}}{dz}\right|\le \dfrac{1}{\eta}\,(\sum_{j=1}^{k}\bar{p}_j+\sum_{j=0}^{k}q_j) \quad (k=1,2,\cdots,m)\,, \\ p_k\le \dfrac{1}{\eta}\,(\sum_{j=1}^{k}\bar{p}_j+\sum_{j=0}^{k}q_j) \quad (k=1,2,\cdots,m)\,. \end{cases}$$

Then it holds that

$$\int_1^\infty (\sum_{k=1}^{m}\bar{p}_k+\sum_{k=0}^{m}q_k+s_0)\,dz<\infty\,,$$

$$\int_1^\infty (\sum_{k=0}^{m}\left|\frac{dp_k}{dz}\right|+\sum_{k=0}^{m-1}\left|\frac{d\bar{q}_k}{dz}\right|)\,dz<\infty\,,$$

$$\lim_{z\to\infty} p_k(z)=0 \quad (k=1,2,\cdots,m)\,,$$

$$\lim_{z\to\infty} q_k(z)=0 \quad (k=0,1,\cdots,m-1)\,.$$

Proof. See Lemma 3.2 in Iida, Yamada and Yotsutani [2]. ■

4. Existence and uniqueness of solutions for (P) and (SP)

In this section we give the proof of Theorems 1 and 2.

Proof of Theorem 1. It is easy to see that the boundary functions $R_i(u_1, u_2, u_3)$ $(i=1, 2, 3)$ satisfy the assumption in Theorem 2 of [7] (see also Appendix of [7]). Thus the existence of a unique solution and the facts (i), (ii) are assured by [7] and [8]. Among them, we can show the fact $u \in C^\infty(\bar{I}\times(0, \infty))^3$ by proving $u \in \cap_{i=0}^m H^{m-i}(\delta, \gamma; H^{2i+1}(I)^3)$ for any $\gamma>\delta>0$ and all integers m (see, e.g., [8]).

Now we will prove (iii). It follows from Friedman's lemma (a parabolic version of Hopf's lemma) on the maximum principle, see [1], that

$$(4.1) \qquad u_i(x, z)>0 \text{ on } \bar{I}\times(0, \infty) \quad \text{or} \quad u_i(x, z)\equiv 0 \text{ on } \bar{I}\times(0, \infty)$$

for each $i=1, 2, 3$. In fact, by (R.1) and the nonnegativity of solutions, $u_i(0, z)=0$ implies $D_x u_i(0, z)=R_i(u(0, z))\leq 0$.

Consider the case where $E_1>0$ and $E_2>0$. We will show $u(x, z)>0$ for all $(x, z)\in\bar{I}\times(0, \infty)$ by contradiction. Suppose that $u_1\equiv 0$ on $\bar{I}\times(0, \infty)$. Then it follows that

$$D_x u_1(0, z) = 0 \quad \text{for all} \quad z\in(0, \infty),$$

which, together with the boundary condition at $x=0$, implies

$$u_3(0, z) = 0 \quad \text{for all} \quad z\in(0, \infty).$$

Therefore, by (4.1), we get

$$u_3(x, z) = 0 \quad \text{for all} \quad (x, z)\in\bar{I}\times(0, \infty),$$

which contradicts $E_1>0$; so that (4.1) assures

$$u_1(x, z)>0 \quad \text{on} \quad \bar{I}\times(0, \infty).$$

Similarly we see that

$$u_2(x, z)>0 \quad \text{and} \quad u_3(x, z)>0 \quad \text{on} \quad \bar{I}\times(0, \infty).$$

Consequently $E_1>0$ and $E_2>0$ imply $u>0$ on $\bar{I}\times(0, \infty)$. The rest of (iii) is straightforward from (2.1) and (4.1). ■

Proof of Theorem 2. Let us consider the case where $E_1>0$ and $E_2>0$. We set $N(u_1, u_2, u_3)=u_1^{n_1}u_2^{n_2}-u_3^{n_3}$. Note that $N(u_1, u_2, u_3)$ satisfies

$$\begin{cases} N(u_1, 0, 0) = 0 & \text{for any} \quad u_1\geq 0, \\ N(0, u_2, 0) = 0 & \text{for any} \quad u_2\geq 0, \\ \dfrac{\partial N}{\partial u_1}>0, \dfrac{\partial N}{\partial u_2}>0 & \text{for any} \quad u_1>0, u_2>0, u_3\geq 0, \\ \dfrac{\partial N}{\partial u_3}<0 & \text{for any} \quad u_1\geq 0, u_2\geq 0, u_3>0. \end{cases}$$

We will show the existence and uniqueness of solutions of

$$f(\xi)=0 \quad \text{on} \quad [0, k_3\|a_3\|_1^{-1} \min_{i=1,2} E_i],$$

where

$$f(\xi)=N\Big(\frac{k_1}{\|a_1\|_1}\Big(E_1-\frac{\|a_3\|_1}{k_3}\xi\Big), \frac{k_2}{\|a_2\|_1}\Big(E_2-\frac{\|a_3\|_1}{k_3}\xi\Big), \xi\Big).$$

We see from the above properties of $N(u_1, u_2, u_3)$ that $f(\xi)$ satisfies

$$\begin{cases} f(0)>0, \\ f(k_3\|a_3\|_1^{-1} \min_{i=1\cdot 2} E_i)<0, \\ \dfrac{df}{d\xi}<0 \quad \text{for} \quad \xi\in(0, k_3\|a_3\|_1^{-1} \min_{i=1,2} E_i). \end{cases}$$

Hence there exists a unique real number ξ_0 such that

$$f(\xi_0)=0 \quad \text{and} \quad \xi_0\in(0, k_3\|a_3\|_1^{-1} \min_{i=1,2} E_i).$$

Consequently the solution of (SP) is given by

$$u_1^\infty=\frac{k_1}{\|a_1\|_1}\Big(E_1-\frac{\|a_3\|_1}{k_3}\xi_0\Big), \quad u_2^\infty=\frac{k_2}{\|a_2\|_1}\Big(E_2-\frac{\|a_3\|_1}{k_3}\xi_0\Big), \quad u_3^\infty=\xi_0,$$

which implies $u^\infty>0$.

If $E_1=0$ or $E_2=0$, then (SP) clearly has a unique solution

$$u_1^\infty=\frac{k_1}{\|a_1\|_1}E_1, \quad u_2^\infty=\frac{k_2}{\|a_2\|_1}E_2, \quad u_3^\infty=0.$$

Thus we complete the proof. ■

5. Proof of Theorem 3

The following Lyapunov functional plays a crucial role. For vector-valued functions $u=u(x)=(u_1(x), u_2(x), u_3(x))$ and $v=v(x)=(v_1(x), v_2(x), v_3(x))$ with $u>0$ and $v>0$, set

$$\Psi(u, v):=\sum_{i=1}^{3} k_i^{-1} n_i \int_I \psi(u_i, v_i)\, a_i\, dx,$$

where

$$\psi(\xi, \xi_0):=\int_{\xi_0}^{\xi} (\log t-\log \xi_0)\, dt=\xi(\log \xi-\log \xi_0)-(\xi-\xi_0).$$

We give some estimates of solutions for (P) which are essential for the proof of (2.2).

Proposition 5.1. *Let u be the solution of* (P). *Suppose that* $E_1>0$ *and*

$E_2>0$. *Then*

(5.1)
$$\frac{d}{dz}\Psi(u, u^\infty)+\varepsilon\,||D_x u||^2+\varepsilon\{R_0(u(0, z))\}^2\leq 0\,,$$

(5.2)
$$\frac{1}{2}\frac{d}{dz}||D_x u||^2_{2\,;\,g}+||D_z u||^2_{2\,;\,h}=\frac{1}{2}||D_x u||^2\,,$$

$$\frac{d}{dz}||D_z u||^2_{2\,;\,a}+||D_z D_x u||^2\leq M_1\,||D_z u||^2_{2\,;\,h}\,,$$

$$\left|\frac{d}{dz}||D_x u||^2\right|\leq||D_x u||^2+||D_z D_x u||^2\,,$$

$$||D_z u||^2_{2\,;\,a}\leq M_1(||D_z u||^2_{2\,;\,h}+||D_z D_x u||^2)\,,$$

for $z\in(0, \infty)$, *where* ε *and* M_1 *are positive constants independent of* z *and*

$$g=g(x)=(g_1(x), g_2(x), g_3(x))\,,\quad h=h(x)=(h_1(x), h_2(x), h_3(x))$$

with

$$g_i(x)=\int_0^x a_i(\xi)\xi\,d\xi \quad and \quad h_i(x)=a_i(x)\,g_i(x) \quad (i=1, 2, 3)\,.$$

Proposition 5.2. *Let* u *be the solution of* (P). *Suppose that* $E_1>0$ *and* $E_2>0$. *Then it holds for* $z\in(0, \infty)$ *that*

$$K_1\,||u(\cdot, z)-u^\infty||^2_{2\,;\,a}\leq\Psi(u(\cdot, z), u^\infty)\leq K_2\,||u(\cdot, z)-u^\infty||^2_{2\,;\,a}\,,$$

where K_1 *and* K_2 *are positive constants independent of* z.

REMARK 5.1. This kind of Lyapunov functional Ψ was employed in Rothe [5]. He investigated the asymptotic behavior of solutions for the following reaction-diffusion system

$$\begin{cases}\dfrac{\partial u_1}{\partial t}=D_1\,\Delta\,u_1+u_3-u_1\,u_2\,,\\[2mm] \dfrac{\partial u_2}{\partial t}=D_2\,\Delta\,u_2+u_3-u_1\,u_2\,,\\[2mm] \dfrac{\partial u_3}{\partial t}=D_3\,\Delta\,u_3+u_1\,u_2-u_3\end{cases}$$

with the homogeneous Neumann boundary condition. This system models the situation where chemical substances are reacting in a bounded domain.

REMARK 5.2. Equality (5.2) is employed in Shinomiya [6]. The important point in (5.2) is that it does not include a term related to the boundary conditions $R_i(u(0, z))$ $(i=1, 2, 3)$.

The following proposition essentially shown in [2] plays a crucial role for the proof of (2.3).

Proposition 5.3. *Let u be the solution of* (P). *Then*

$$\frac{1}{2}\frac{d}{dz}\|D_z^{k-1}D_x u\|_{2;g}^2+\|D_z^k u\|_{2;h}^2=\frac{1}{2}\|D_z^{k-1}D_x u\|^2,$$

$$\frac{d}{dz}\|D_z^k u\|_{2;a}^2+\|D_z^k D_x u\|^2$$

$$\leq M_k\{\sum_{j=1}^{k}\|D_z^j u\|_{2;h}^2+\sum_{j=1}^{k-1}\|D_z^j D_x u\|^2\},$$

$$\left|\frac{d}{dz}\|D_z^{k-1}D_x u\|^2\right|\leq\|D_z^{k-1}D_x u\|^2+\|D_z^k D_x u\|^2,$$

$$\|D_z^k u\|_{2;a}^2\leq M_k(\|D_z^k u\|_{2;h}^2+\|D_z^k D_x u\|^2),$$

for $k=2, 3, \cdots$ *and* $z\in[1, \infty)$, *where* M_k *is a positive constant independent of* z *and* g, h *are the functions used in Proposition* 5.1.

Before proving the above propositions, we give the proof of Theorem 3 by using them.

Proof of (2.2). Consider the case where $E_1>0$ and $E_2>0$. Observe that the estimates in Proposition 5.1 correspond to the inequalities (with $m=1$) in the assumption of Lemma 3.1. Therefore,

$$\int_1^\infty \{R_0(u(0, z))\}^2\,dz<\infty,\qquad \lim_{z\to\infty}\|D_x u(\cdot, z)\|=0; \tag{5.3}$$

in particular, $\{u(\cdot, z)\}_{z\geq 1}$ is uniformly bounded and equi-continuous. By virtue of (2.1) and (5.3), Ascoli-Arzelà's theorem assures that there exist a sequence $\{z_j\}_{j\geq 1}$ and a consatnt $\bar{u}=(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ such that

$$\begin{cases}\bar{u}\geq 0,\\ R_0(\bar{u})=0,\\ \dfrac{\|a_i\|_1}{k_i}\bar{u}_i+\dfrac{\|a_3\|_1}{k_3}\bar{u}_3=E_i\quad (i=1, 2),\end{cases}$$

and

$$\lim_{j\to\infty}\|u(\cdot, z_j)-\bar{u}\|_\infty=0.$$

It follows from Theorem 2 that $\bar{u}=u^\infty\in(0, \infty)^3$, which implies

$$\lim_{j\to\infty}\|u(\cdot, z_j)-u^\infty\|_\infty=0. \tag{5.4}$$

On the other hand, (5.1) in Proposition 5.1 yields

$$\frac{d}{dz}\Psi(u, u^{\infty})\leq 0\,,$$

which leads to

$$\lim_{z\to\infty}\Psi(u(\cdot, z), u^{\infty}) = \Psi(u^{\infty}, u^{\infty}) = 0$$

by (5.4). Hence it follows from Proposition 5.2 that

$$\lim_{z\to\infty}||u(\cdot, z)-u^{\infty}||_{2\,;\,a} = 0\,. \tag{5.5}$$

Since $\{u(\cdot, z)\}_{z\geq 1}$ is uniformly bounded and equi-continuous, we can derive

$$\lim_{z\to\infty}||u(\cdot, z)-u^{\infty}||_{\infty} = 0$$

by (5.5).

We consider the case where $E_1=0$ or $E_2=0$. Without loss of generality we may assume $E_1=0$. We see from (iii) of Theorem 1 that

$$u_1(x, z)\equiv 0\,,\quad u_3(x, z)\equiv 0\quad \text{on}\quad \bar{I}\times[0, \infty)$$

and u_2 satisfies

$$\begin{cases} a_2(x)\, D_z\, u_2 = D_x^2\, u_2\,, & (x, z)\in I\times(0, \infty)\,, \\ D_x\, u_2(0, z) = D_x\, u_2(1, z) = 0\,, & z\in(0, \infty)\,, \\ u_2(x, 0) = \phi_2(x)\,, & x\in I\,. \end{cases}$$

Thus we can show

$$\lim_{z\to\infty}\left\|u_2(\cdot, z)-\frac{||\phi_2\, a_2||_1}{||a_2||_1}\right\|_{\infty} = 0$$

in the standard manner for parabolic equations. In fact, we have only to observe that

$$\left\|u_2-\frac{||\phi_2\, a_2||_1}{||a_2||_1}\right\|_{\infty}\leq||D_x\, u_2||\,,$$

$$\frac{1}{2}\frac{d}{dz}\left\|u_2-\frac{||\phi_2\, a_2||_1}{||a_2||_1}\right\|_{2\,;\,a}^2+||D_x\, u_2||^2 = 0\,,$$

$$\frac{1}{2}\frac{d}{dz}||D_z\, u_2||_{2\,;\,a}^2+||D_z\, D_x\, u_2||^2 = 0$$

for $z\in(0, \infty)$. ■

Proof of (2.3). As in the proof of (2.2), Propositions 5.1 and 5.3 combined with Lemma 3.1 yield

$$\lim_{z\to\infty}||D_z^{k-1}\, D_x\, u|| = 0\quad (k = 1, 2, \cdots)\,,$$

$$\lim_{z\to\infty}||D_z^k\, u||_{2\,;\,a} = 0\qquad (k = 1, 2, \cdots)\,.$$

In the case $E_1=0$ or $E_2=0$, we can easily get the same results by slight modifications (e.g., use $||u-u^\infty||^2_{2;a}$ instead of $\Psi(u, u^\infty)$ in (5.1)). Thus we can derive the uniform convergence of any derivative $D^i_z D^j_x u$ by using imbedding theorems. Refer to the proof of Theorem 4 in [2] for the detail. ■

Now we will prove Propositions 5.1 and 5.2.

Proof of Proposition 5.1. We will show (5.1). The other estimates are obtained in the same way as the proof of Proposition 5.1 in [2]. Let $u=(u_1, u_2, u_3)$ be the solution of (P). Then we have

$$\begin{aligned}
&\frac{d}{dz}\Psi(u, u^\infty)\\
&=\sum_{i=1}^{3} k_i^{-1} n_i \frac{d}{dz}\int_I \psi(u_i, u_i^\infty)\, a_i\, dx\\
&=\sum_{i=1}^{3} k_i^{-1} n_i \int_I (\log u_i-\log u_i^\infty)\, a_i\, D_z u_i\, dx\\
&=\sum_{i=1}^{3} k_i^{-1} n_i \int_I (\log u_i-\log u_i^\infty)\, D_x^2 u_i\, dx\\
&=\sum_{i=1}^{3} k_i^{-1} n_i\, [(\log u_i-\log u_i^\infty)\, D_x u_i]_0^1-\sum_{i=1}^{3} k_i^{-1} n_i \int_I \frac{|D_x u_i|^2}{u_i}\, dx\\
&=\{\log (u_1^\infty)^{n_1} (u_2^\infty)^{n_2}-\log (u_3^\infty)^{n_3}\}\, R_0(u(0, z))\\
&\quad-\{\log u_1(0, z)^{n_1} u_2(0, z)^{n_2}-\log u_3(0, z)^{n_3}\}\, R_0(u(0, z))\\
&\quad-\sum_{i=1}^{3} k_i^{-1} n_i \int_I \frac{|D_x u_i|^2}{u_i}\, dx\,,
\end{aligned}$$

which, together with $(u_1^\infty)^{n_1} (u_2^\infty)^{n_2}=(u_3^\infty)^{n_3}$, implies

$$\begin{aligned}
&\frac{d}{dz}\Psi(u, u^\infty)+\sum_{i=1}^{3} k_i^{-1} n_i \int_I \frac{|D_x u_i|^2}{u_i}\, dx\\
&\quad+\{\log u_1(0, z)^{n_1} u_2(0, z)^{n_2}-\log u_3(0, z)^{n_3}\}\, R_0(u(0, z))=0\,.
\end{aligned}$$

We note that the following elementary inequality holds:

$$(\log \xi-\log \eta)\,(\xi-\eta)\geq\frac{(\xi-\eta)^2}{\bar{\xi}} \quad \text{for} \quad \xi, \eta\in(0, \bar{\xi}]\,.$$

Consequently we obtain (5.1) from (i) of Theorem 1. ■

Proof of Proposition 5.2. Noting (i) of Theorem 1, we can reduce this propopsition to the following fundamental lemma. ■

Lemma 5.1. *Let* ξ_0 *and* $\bar{\xi}$ *be constants with* $0<\xi_0<\bar{\xi}$. *Then*

$$\kappa_1|\xi-\xi_0|^2\leq\psi(\xi, \xi_0)\leq\kappa_2|\xi-\xi_0|^2 \quad \textit{for all} \quad \xi\in(0, \bar{\xi})\,,$$

where

$$\kappa_1 = \frac{\psi(\xi, \xi_0)}{|\xi - \xi_0|^2}, \quad \kappa_2 = \frac{1}{\xi_0}.$$

Proof. Consider a function

$$\zeta(\xi) := \begin{cases} \dfrac{\psi(\xi, \xi_0)}{|\xi - \xi_0|^2} & \text{for} \quad \xi \neq \xi_0, \\ \dfrac{1}{2\xi_0} & \text{for} \quad \xi = \xi_0. \end{cases}$$

Since $\zeta(\xi) \in C^1((0, \infty))$ and $(d\zeta)/(d\xi) < 0$ on $(0, \infty)$, we get

$$\frac{1}{\xi_0} = \lim_{t \to +0} \zeta(t) > \zeta(\xi) > \zeta(\bar{\xi}) = \frac{\psi(\bar{\xi}, \xi_0)}{|\bar{\xi} - \xi_0|^2} \quad \text{for} \quad \xi \in (0, \bar{\xi}). \quad \blacksquare$$

For the proof of Proposition 5.3, see §6 of [2].

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