Harada, M.
Osaka J. Math.
29 (1992), 435–446

NOTE ON ALMOST RELATIVE PROJECTIVES AND ALMOST RELATIVE INJECTIVES

MANABU HARADA

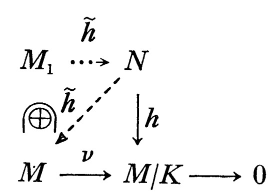
(Received April 19, 1990)

This paper is supplemental to [4], [6] and [7]. We shall show, under assumption of finite length, that when we study almost relative projectives, we may restrict ourselves only to certain special homomorphisms h in the definition of almost relative projectives [6] (see §1). In the similar manner to proof of the above fact, we shall give a criterion for an R-module M_0 to be almost M_1-projective, where R is a perfect ring and M_1 is an indecomposable R-module. We shall obtain, in §3, a generalization of [6], Theorem 1, where direct sums of local modules were studied. In this section we shall show the same property on direct sum of indecomposalbe modules. §§2 and 4 are the dual versions of §§1 and 3.

1. Almost relative simple-projectives

In this paper we always assume that R is a ring with identity and that every module is a unitary right R-module. Let M be an R-module. We denote *the socle, the Jacobson radical*, and *the length of* M by Soc(M), J(M) and $|M|$, respectively. If End$_R(M)$ is a local ring, we say M is *an* LE *module*. We recall here the definition of almost relative projectives [6]. Let M and N be R-modules. For any diagram with row exact:

(1)



if there exists $\tilde{h}$: $N \to M$ with $v\tilde{h}=h$ or there exist a non-zero direct summand M_1 of M and $\tilde{h}$: $M_1 \to N$ with $h\tilde{h}=v|M_1$, then N is called *almost M-projective*. (if we obtain only the first case, we say that N is *M-projective* [2]).

Here we shall introduce a little weaker condition than the above. In the diagram (1) we take only the h': $N \to M/K$ whose image is simple. If for any h' above there exists $\tilde{h}$ in the definition, then we say N is *almost M-simple-*

projective. We can similarly define *M-simple-projective* (resp. *simple-projective*). As an application of [10], Theorem*, we shall show in this section that the above weaker conditon coincides with original one when R is a semi-perfect ring and M, N are R-modules of finite length.

REMARK 1. If we restrict ν to have a simple image, i.e., K is maxiaml, instead of h in (1), then this is nothing but the lifting property of simple modules (see the definition before Theorem 1 below).

First we note that many arguments in [6] and [8] are valid for almost relative simple-projectives. We shall use those facts without proofs, and refer [6], [8] and [9] for definitions of local (hollow) modules and uniform modules.

Lemma 1. *Let R be any ring and let M_0, M be R-modules with $|M_0|<\infty$ or $|M|<\infty$. Then M_0 is M-projective if and only if M_0 is M-simple-projective.*

Proof. Take a diagram with row exact:

$$(2)\qquad \begin{array}{ccccc} & & M_0 & & \\ & & \downarrow h & & \\ M & \xrightarrow{\nu} & H & \longrightarrow & 0 \end{array}$$

Since $|M_0|<\infty$ or $|M|<\infty$, we can find a maximal submodule T in $h(M_0)$. Then we obtain a new diagram

$$\begin{array}{ccccccc} & & & & M_0 & & \\ & & & & \downarrow \nu' h & & \\ M & \xrightarrow{\nu} & H & \xrightarrow{\nu'} & H/T & \longrightarrow & 0 \end{array}$$

where ν': $H\to H/T$ is the canonical epimorphism.
Since $\nu' h(M_0)$ is simple, there exists $\tilde{h}_1$: $M_0\to M$ with $\nu'\nu\tilde{h}_1=\nu' h$. Hence $(\nu\tilde{h}_1-h)(M_0)\subset T\subsetneqq h(M_0)$. Replacing h by $\nu\tilde{h}_1-h$, we obtain $\tilde{h}_2$: $M_0\to M$ such that $(\nu\tilde{h}_2-(\nu\tilde{h}_1-h))(M_0)\subsetneqq(\nu\tilde{h}_1-h)(M_0)\subsetneqq h(M_0)$. $|h(M_0)|<\infty$ implies $h=\nu(\sum_{i=1}^{n}(-1)^{i+1}\tilde{h}_i)$ for some n.

Corollary. *Let M_0 be an R-module of finite length. Then M_0 is projective if and only if M_0 is simple-projective.*

From the above proof and the definition, we obtain

Lemma 2. *Let M_0 and M be as above and M an indecomposable R-module. Assume that M_0 is almost M-simple-projective. If the h in the diagram* (2) *is not an epimorphism, h is liftable to $\tilde{h}$: $M_0\to M$.*

From [7], Theorem 1 we note the following fact:

Lemma 3. *Let R be any ring and N, M R-modules. Further we assume that M is a non-hollow and indecomposable module. If, for any non-epic homomorphism h in* (1), *there exists $\tilde{h}$: $N\to M$ with $\nu\tilde{h}=h$, then N is M-projective.*

Proof. From the assumption and the technique in the proof of [7], Theorem 1, we can reduce the h in (1) to non-epimorphism by relpacing K with suitable submodule of K.

Corollary. *Let R be any ring and let M_0, M be as in Lemma 2. If M_0 is almost M-simple-projective and M is not a hollow module, then M_0 is M-projective.*

Proof. This is clear from Lemmas 2 and 3.

From the above corollary we study in a case where M is a local module.

Lemma 4. *Let R be a semi-perfect ring and let M_0 and $M=eR/A$ be R-modules with $|M_0|<\infty$ or $|eR/A|<\infty$, where e is a primitive idempotent. Assume that M_0 is almost eR/A-simple-projective. Then M_0 is almost eR/A-projective.*

Proof. Take a diagram for any right ideal $B\supset A$:

$$\text{(3)}\qquad \begin{array}{ccccc} & & M_0 & & \\ & & \downarrow h & & \\ eR/A & \xrightarrow{\nu} & eR/B & \longrightarrow & 0 \end{array}$$

By Lemma 2 we may assume that h is an epimorphism. Then from (3) we obtain the derived diagram:

$$\begin{array}{ccccccc} & & & & M_0 & & \\ & & & & \downarrow \nu' h & & \\ eR/A & \xrightarrow{\nu} & eR/B & \xrightarrow{\nu'} & eR/eJ & \longrightarrow & 0 \end{array}$$

By assumption and Lemma 2, if there exists $\tilde{h}'$: $M_0\to eR/A$ with $\nu'\nu\tilde{h}'=\nu'h$, then we can find $\tilde{h}$: $M_0\to eR/A$ with $\nu\tilde{h}=h$ (cf. the proof of Lemma 1). Hence we assume that there exists $\tilde{h}'$: $eR/A\to M_0$ with $\nu'h\tilde{h}'=\nu'\nu$. Put $\tilde{h}'(\tilde{e})=m_0(=m_0e)$, where $\tilde{e}=e+A$ in eR/A. Since $\nu'h\tilde{h}'=\nu'\nu$, $h(m_0)=\nu(\tilde{e}(e+j))$ for some $j\in eJe$. Therefore putting $(e+j)^{-1}=e+j'$: $j'\in eJe$, $\nu(\tilde{e})=h(m_0(e+j'))$. We note that [8], Lemma 3 was obtained from [8], Lemma 2, where we used the property of almost simple-projectives and the fact: $h(M_0)J^n=0$ for some n. Hence there exists f: $M_0\to M_0$ with $f(m_0)=m_0+m_0j'$. Put $\tilde{h}=f\tilde{h}'$, and $h\tilde{h}(\tilde{e})=hf\tilde{h}'(\tilde{e})=hf(m_0)=h(m_0(e+j'))=\nu(\tilde{e})$. Hence $h\tilde{h}=\nu$. Therefore M_0 is almost eR/A-projective.

We recall here the definition of LPSM. Assume that $M/\mathrm{J}(M)$ is semisimple. If for any simple submodule S in $M/\mathrm{J}(M)$ there exists a direct decomposition

$M=M_1\oplus M_2$ such that $(M_1+\mathrm{J}(M))/\mathrm{J}(M)=S$, we say that M has *the lifting property of simple modules modulo radical,* briefly LPSM [5] and [8].

The following theorem is useful when we want to check almost relative projectivity.

Theorem 1. *Let R be a semi-perfect ring. Then the concept of almost relative projectivity coincides with one of almost relative simple-projectivity on R-modules of finite length.*

Proof. First we note that every module of finite length has a projective cover. Let M_0 and M be R-modules of finite length. Assume that M_0 is almost M-simple-projective. We take a direct decomposition $M=\Sigma_i\oplus T_i\oplus\Sigma_k\oplus N_k$ into indecomposable modules such that M_0 is almost T_i-simple-projective (but not T_i-projective) and M_0 is N_k-projective (cf. Lemma 1). Then T_i is a local module by Corollary to Lemma 3, and hence M_0 is almost T_i-projective from Lemma 4. It is clear that M_0 is almost $T_i\oplus T_j$-simple-projective for $i\neq j$. As the remark given before Lemma 1, we used only a property of almost relative simple-projectives in the proof of [7], Proposition 5. Hence $T_i\oplus T_j$ has LPSM. As a consequence T_i and T_j are mutually almost relative projective by [10], Lemma 3# and the dual result to Corollary to Lemma 2 in [10] (cf. [10], the proof of Lemma 4#). Therefore M_0 is almost M-projective by [10], Theorem#.

Using the above argument we shall give a criterion for M_0 to be almost M_1-projective (cf. [7], Proposition 2).

Theorem 2. *Assume that R is a perfect ring. Let M_1 be an indecomposable R-module and M_0 an R-module. Then M_0 is almost M_1-projective if and only if the following conditions are satisfied. Let $P\xrightarrow{\theta}M_0$ be a projective cover of M_0.*

1) $\mathrm{Hom}_R(P, N_1)=\mathrm{Hom}_R(M_0, N_1)$, *where N_1 is any maximal submodule of M_1* (cf. [1], *p*.22, Exercise 4).

2) *Any element in* $\mathrm{Hom}_R(M_0/N_0, M_1/N_1)$ *is liftable to an element in* $\mathrm{Hom}_R(M_0, M_1)$ *or in* $\mathrm{Hom}_R(M_1, M_0)$, *where N_0, N_1 are any maximal submodules of M_0 and M_1, respectively.*

Proof. We have $\mathrm{Hom}_R(P, N_1)\supset\mathrm{Hom}_R(M_0, N_1)$ for any maximal submodule N_1 of M_1. First we assume that M_0 is almost M_1-projective. Let f be in $\mathrm{Hom}_R(P, N_1)$. Then f is not an epimorphism onto M_1. Hence f induces an element in $\mathrm{Hom}_R(M_0, N_1)$ by [8], Lemma 1. 2) is clear from definition. Conversely we assume 1) and 2). Consider a diagram with K a submodule of M_1:

$$\begin{array}{ccccc} & & M_0 & & \\ & & \big\downarrow h & & \\ M_1 & \xrightarrow{\ \nu\ } & M_1/K & \longrightarrow & 0 \end{array}$$

If h is not an epimorphism, then $h(M_0)\subset N_1/K$, where N_1 is a maximal submodule of M_1. Since P is projective, there exists $\tilde{h}'\colon P\to M_1$ with $\nu\tilde{h}'=h\theta$, and $\tilde{h}'(P)\subset N_1$. Hence from 1) there exists $\tilde{h}\colon M_0\to M_1$ with $\nu\tilde{h}=h$. Therefore if M_1 is not local, then M_0 is M_1-projective by the remark in the proof of Lemma 3. Finally suppose that M_1 is local. If h is not an epimorphism, we obtain $\tilde{h}$ above. We assume that h is an epimorphism. We reproduce the same argument in the proof of Lemma 2. From the above diagram we can derive the following one:

$$\begin{array}{ccccccl}
 & & & & M_0 & & \\
 & & & & \downarrow \nu_0 & & \\
 & & & & M_0/N_0 & & \nu'h \\
 & \nu & & \nu' & \downarrow \nu'h & & \\
M_1 & \longrightarrow & M_1/K & \longrightarrow & M_1/\mathrm{J}(M_1) & \longrightarrow & 0
\end{array}$$

where $N_0=(\nu'h)^{-1}(0)$.
From 2) we assume first that $\nu'h$ is liftable to an element $\tilde{h}_1\colon M_0\to M_1$. Then $\nu\tilde{h}_1-h\colon M_0\to \mathrm{J}(M_1)/K$ is not an epimorphism onto M_1/K_1. Hence by the initial argument there exists $\tilde{h}_2\colon M_0\to M_1$ such that $\nu\tilde{h}_2=\nu\tilde{h}_1-h$. Therefore $h=\nu(\tilde{h}_1-\tilde{h}_2)$. We assume next that $\nu'h$ is liftable to an element $\tilde{h}'\colon M_1\to M_0$. Then in the manner given in the proof of Lemma 4, we can find $\tilde{h}\colon M_1\to M_0$ with $h\tilde{h}=\nu$. Hence M_0 is almost M_1-projective.

REMARKS 2. In the above, if M_1 is not indecomposable, then the situation is very much different. If M_1 is a finite direct sum of indecomposable modules, then we can use [8], Theorem.

3. Let Z be the ring of integers and p prime. Put $R=Z_p$. Then Q, the module of rationals, is a hollow and infinitely generated R-module. Hence Q is trivially almost $\sum_1^\infty\oplus Q$-simple projective, but Q is not almost $\Sigma\oplus Q$-projective by [6], Theorem 1 and the remark on p. 450.

2. Almost relative simple-injectives

We shall study dual properties to ones in §1. We recall here the definition of almost relative injectives [3]. Let $U\supset V$ and U_0 be R-modules. Consider the following diagram with i the inclusion and two conditions 1) and 2):

$$(4)\qquad \begin{array}{ccccc}
U & \overset{i}{\longleftarrow} & V & \longleftarrow & 0 \\
 & \overset{\tilde{h}}{\searrow} & \downarrow h & & \\
\oplus & & & & \\
U' & \underset{\tilde{h}}{\longleftarrow} & U_0 & &
\end{array}$$

1) There exists $\tilde{h}\colon U\to U_0$ such that $\tilde{h}i=h$ or

2) There exist a non-zero direct summand U' of U and $\tilde{h}$: $U_0 \to U'$ such that $\tilde{h}h = \pi i$, where π: $U \to U'$ is the projection of U onto U'.

Then U_0 is called *almost U-injective* if the above 1) or 2) holds true on the above diagram with any V and any h (U_0 is called *U-injective* if we have only 1) [2]).

In the above definition we consider only h': $V \to U_0$, whose image is simple. Then we shall call this restricted property *almost relative simple-injective*. Similarly we can define *relative simple-injective*. We shall show that almost relative injectivity coincides with one of almost relative simple-injectivity under some assumptions.

We assume that every module in this section contains a non-zero socle. The following three lemmas are dual to ones in §1, and their proofs are categroical. Hence we skip them.

Lemma 1$^{\#}$ (dual to Lemma 1). *Let U_0 and U_1 be R-modules and either $|U_1| < \infty$ or $|U_0| < \infty$. If U_0 is U_1-simple-injective, then U_0 is U_1-injective.*

Lemma 2$^{\#}$. *Let U_0 and U_1 be as above and U_1 indecomposable. Assume that U_0 is almost U_1-simple-injective, and that the h in the diagram* (4) *is not monic. Then there exists $\tilde{h}$: $U_1 \to U_0$ such that $\tilde{h}i = h$.*

Lemma 3$^{\#}$. *Let U_0 and U_1 be as in Lemma 2$^{\#}$. If U_1 is an indecomposable, non-uniform module, and U_0 is almost U_1-simple-injective, then U_0 is U_1-injective.*

Let R be a semiperfect ring and U_0, U_1 R-modules. Assume that U_1 is a uniform module with $\mathrm{Soc}(U_1) = S_1$. We consider the following situation: there exist submodules $T_1 \supset T\,(\neq 0)$ in U_1 such that $T_1/T \approx S_1$ ($\approx eR/eJ$), and $\mathrm{Soc}(U_0)$ contains a simple component S_1' isomorphic to S_1; e is a primitive idempotent. Take any element t in T_1 with $t = te$ and $T_1 = tR + T$.

Lemma 5. *Let R be semi-perfect, and let U_0, $U_1 \supset T_1(\ni t) \supset T$, S_1 and S_1' be as above. If U_0 is almost U_1-simple-injective, then for any element x in S_1' with $xe = x$ there exists $\tilde{h}$: $U_1 \to U_0$ such that $\tilde{h}(t) = x$ and $\tilde{h}(T) = 0$.*

Proof. Since $t = te$ and $x = xe$, we obtain an isomorphirm h': $T_1/T \approx S_1'$ ($\approx eR/eJ$) with $h'(t+T) = x$. Then we have a homomorphism h: $T_1 \xrightarrow{\nu} T_1/T \to S_1' \subset U_0$, where ν is the natural epimorphism. Hence by assumption there exists a homomorphism $\tilde{h}$: $U_1 \to U_0$ with $\tilde{h}(t) = x$ and $\tilde{h}(T) = 0$.

The following lemma is very useful when we examine almost injectivity for modules.

Lemma 4$^{\#}$. *Let R be semi-perfect and U_0, U_1 R-modules. Assume that either U_0 or U_1 is of finite length and that U_1 is indecomposable. If U_0 is almost U_1-simple-injective, then U_0 is almost U_1-injective.*

Proof. From Lemmas 1[#] and 3[#] we may assume that U_1 is uniform (and U_0 is not U_1-injective). Take a diagram with V a submodule of U_1:

$$\begin{array}{ccl} U_1 \xleftarrow{\ i\ } & V \supset (S_1 = \mathrm{Soc}(U_1)) \leftarrow 0 \\ & \downarrow h & \\ & U_0 & \end{array}$$

We may suppose from Lemma 2[#] that h is a monomorphism.
a): Assume that there exists $\tilde{h}$: $U_0 \to U_1$ with $\tilde{h}(h|S_1)=i|S_1$. Put $f=(\tilde{h}h-i)|V$: $V\to U_1$ and $T=\ker f \supset S_1$. Then

$$\tilde{h}h|T = i|T\,. \tag{5}$$

Suppose $T\neq V$ and put $T_1=f^{-1}(S_1)$. Then $T_1/T\approx S_1\,(\approx eR/eJ)$, since U_1 is uniform. Take t in T_1 such that $t=te$ and $tR+T=T_1$. Put $t_0=\tilde{h}h(t)\in U_1$, and $t_0 e=t_0$. V being uniform, $\tilde{h}h$: $V\to U_1$ is a monomorphism. Further since U_1 is uniform, $xR\supset S_1$ for any non-zero x in U_1. Hence there exists j in R such that

$$t_0-t = (\tilde{h}h-i)\,(t) = f(t) = t_0 j = t_0 je, \text{ i.e., } t=t_0(1-j)\;(j=je)\,.$$

From the fact that $t_0 j\in S_1$ and $\tilde{h}h$ is a monomorphism, we have $tj\in S_1$, and $h(tj)=h(tj)\,e$ is in a simple submodule S_1' of U_0. Further $S_1\approx T_1/T\approx(t_0R+\tilde{h}h(T))/\tilde{h}h(T)$. Hence there exists $\tilde{h}'$: $U_1\to U_0$ such that

$$\tilde{h}'(t_0) = h(tj) \quad \text{and} \quad \tilde{h}'(\tilde{h}h(T)) = 0$$

by Lemma 5. Put $h^*=1_{U_0}-\tilde{h}'\,\tilde{h}$: $U_0\to U_0$, and

$$(\tilde{h}h^*)\,h(t) = \tilde{h}h(t)-\tilde{h}\tilde{h}'\,\tilde{h}h(t) = t_0-\tilde{h}\tilde{h}'(t_0) = t_0-\tilde{h}h(tj) = t_0(1-j) = t, \text{ i.e.}$$

$$(\tilde{h}h^*)\,(h(t)) = t \quad \text{and} \tag{6}$$

$$(\tilde{h}h^*)\,h\,|\,T = (1_T-\tilde{h}\tilde{h}'\,\tilde{h}h)\,|\,T = 1_T \quad \text{by} \quad (5)\,, \tag{7}$$

since $\tilde{h}'\,\tilde{h}h(T)=0$. Hence we obtain $\tilde{h}_1:=\tilde{h}h^*$: $U_0\to U_1$ with

$$\tilde{h}_1\,h\,|\,T_1 = 1_{T_1}$$

by (6), (7) and the fact: $T_1=tR+T$, and further $T_1\supsetneqq T\supset S_1$. Repeating this argument, we finally obtain $\tilde{h}_n$: $U_0\to U_1$ with $\tilde{h}_n\,h=i$ since $|V|<\infty$.
b): Assume that there exists $\tilde{h}$: $U_1\to U_0$ with $\tilde{h}i|S_1=h|S_1$.
Put $h_1=h-\tilde{h}i$: $V\to U_0$. Then ker $h_1\supset S_1\neq 0$. Hence there exists $\tilde{h}'$: $U_1\to U_0$ with $\tilde{h}'\,i=h_1=h-\tilde{h}\,i$ from Lemma 2[#]. Therefore $h=(\tilde{h}+\tilde{h}')\,i$ and $\tilde{h}+\tilde{h}'$: $U_1\to U_0$.

Theorem 1* (dual to Theorem 1). *Let R be a semi-perfect ring. Then the concept of almost relative injectivity coincides with that of almost relative simple-injectivity on R-modules of finite length.*

Theorem 2*. *Assume that R is a right semi-artinian ring. Let U_1 be an indecomposable R-module and U_0 an R-module. Then U_0 is almost U_1-injective if and only if the following conditions are satisfied. Let E be an injective hull of U_0.*

1) $\mathrm{Hom}_R(U_1/S_1, U_0)=\mathrm{Hom}_R(U_1/S_1, E)$, *where S_1 is any smiple submodule of U_1.*

2) *Any element in* $\mathrm{Hom}_R(S_1, S_0)$ *is extendible to an element in* $\mathrm{Hom}_R(U_1, U_0)$ *or in* $\mathrm{Hom}_R(U_0, U_1)$, *where S_0, S_1 are simple submodules of U_0 and U_1, respectively.*

REMARK 4. Let R be a local commutative and non-valuation domain. Then R is not almost R-injective as R-modules. However R is semi-perfect and trivially R is almost R-simple-injective. Hence we need the assumption on length in Lemma 4*.

3. Condition (D)

We shall give a supplemental reslut of [6]. We recall the condition (D) in [6]. Let $\{M_i\}_I$ be a set of R-modules and $M=\Sigma_I \oplus M_i$. By π_i we denote the projection of M onto M_i. Concerning to this decomposition we consider the following condition:

(D) *any submodule N of M with $\pi_i(N)=M_i$ for some i contains a non-zero direct summand of M.*

If all the M_i are hollow and I is a finite set, then (D) is equivalent to M being a lifting module by [6], Theorem 1.

In the above let I be a finite set $\{1, 2, \cdots, n\}$, and M_0, M_i R-modules, where the M_i are indecomposable. Suppose that M_0 is almost M_i-projective for all i. Consider a diagram with K a submodule of M:

$$\text{(8)} \qquad \begin{array}{ccccc} & & M_0 & & \\ & & \downarrow h & & \\ M & \xrightarrow{\nu} & M/K & \longrightarrow & 0 \end{array}$$

We can derive the following diagram from (8):

$$(8\text{–}j) \qquad \begin{array}{ccccc} & & M_0 & & \\ & & \downarrow \nu_j' h & & \\ M_j & \xrightarrow{\nu_j' \nu i_j} & M_j/K^j & \longrightarrow & 0\,, \end{array}$$

where i_j is the inclusion of M_j into M, $K^i=\pi_i(K)$ and ν_j': $M/K \to M/(\Sigma \oplus K^i)=\Sigma \oplus M_i/K^i \to M_j/K^j$ (see [8]).

Lemma 6. *In the above we assume that $\nu_j' h(M_0) \neq M_j/K^j$ for all j. Then there exists $\tilde{h}$: $M_0 \to M$ with $\nu\tilde{h}=h$.*

Proof. We can prove the lemma by induction on n in a manner similar to the proof of [4], Lemma 1.

The following theorem is given in [4] and [6], when the M_i are hollow. In general we obtain

Theorem 3. *Let $\{M_i\}_I$ be a set of* LE *modules and $M=\Sigma_I \oplus M_i$. Then (D) holds true for the decomposition $M=\Sigma_I \oplus M_i$ if and only if $\Sigma_J \oplus M_i$ is almost $\Sigma_{I-J} \oplus M_k$-projective for any subset J of I. If I is finite, then (D) holds true for any direct decomposition of M if and only if M_i is almost M_j-projective whenever $i \neq j$.*

Proof. The first part was given in [11]. Hence we shall show the second half. Assume that I is finite and M_i is almost M_j-projective for any pair i and j $(i \neq j)$. By making use of an argument similar to the proof of [4], Theorem 1, we shall show that (D) holds true for the decomposition $M=\Sigma_I \oplus M_i$. If $I=\{1, 2\}$, i.e. $M=M_1 \oplus M_2$, then M satisfies (D) by definition. We shall show (D) by induction on n; $I=\{1, 2, \cdots, n\}$. Let N be the submodule of M given in (D). We may assume $\pi_1(N)=M_1$. Put $\pi^*=1-\pi_1$, $M^*=M_2 \oplus \cdots \oplus M_n$ and $N^*=\pi^*(N)$. Further putting $N_1=N \cap M_1$ and $N_*=N \cap M^*$, we obtain an isomorphism h: $M_1/N_1 \approx N^*/N_*$, i.e., $N=M_1(h)\, N^*$ (see [6]). From those data we have the diagram:

$$\begin{array}{ccccc} & & M_1 & & \\ & & \downarrow \nu_1 & & \\ & & M_1/N_1 & & \\ & & \downarrow h & & \\ M^* & \xrightarrow{\nu^*} & M^*/N_* & \longrightarrow & 0 \end{array}$$

We can derive the diagram similar to $(8-j)$ from the above. Considering the diagram

$$\begin{array}{ccccc} & & M_1/N_1 & & \\ & & \downarrow h & & \\ N^* & \xrightarrow{\nu^* \mid N^*} & N^*/N_* & \xrightarrow{\nu_j'} & \pi_j(N^*)/\pi_j(N_*) \\ \cap & & \cap & & \cap \\ M^* & \xrightarrow{\nu^*} & M^*/N_* & \xrightarrow{\nu_j'} & M_j/\pi_j(N_*) \end{array}$$

we have $\nu_j' h \nu_1(M_1)=\nu_j' \nu^* \pi_j(N_*)=\nu_j' \nu^* \pi_j(N)$. If $\pi_j(N) \neq M_j$ for all $j \geqslant 2$, $\nu_j' h \nu_1$ is not an epimorphism for all $j \geqslant 2$. Hence there exists $\tilde{h}_1$: $M_1 \to M^*$ with $\nu^* \tilde{h}_1=h \nu_1$ by Lemma 6. Therefore $N=M_1(h)\, N^* \supset M_1(\tilde{h}_1)$ (cf. the proof of [6], Theorem 1). As a consequence we may assume $\pi_i(N)=M_i$ for some $i \geqslant 2$, say

$i=2$. Now $M^*=M_2\oplus\cdots\oplus M_n$ and let π_2' be the projection of M^* onto M_2. Then $\pi_2'(N^*)=\pi_2'\pi^*(N)=\pi_2(N)=M_2$. Hence by induction hypothesis, there exists a non-zero direct summand M_2' of M^* contained in N^*; $M^*=M_2'\oplus M_*'$. Put $N'=\pi^{*-1}(M_2')\cap N$. Then $N'\subset N$ and $N'\subset M_1\oplus M_2'$ with $\pi_2''(N')=M_2'$, where $\pi_2''(=\pi^*|M_2')$ is the projection of $M_1\oplus M_2'$ onto M_2'. Here we note that M_2' is isomorphic to some $M_j(j>1)$. Hence N' contains a non-zero direct summand of $M_1\oplus M_2'$ from the initial and hence of M. Since every direct summand of M is a direct sum of indecomposable modules isomorphic to $\{M_i\}$, (D) holds true for any direct decomposition of M. The converse is clear from the first equivalence.

REMARK 5. If I is infinite in the second part, Theorem 3 is not true (see [4]). We shall give a module which satisfies the conditions in Theorem 3, but which is not a lifting module. Let K be a field and put

$$R=\begin{pmatrix} K & 0 & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$$

Then $\mathrm{Soc}(e_{11}R)\approx\mathrm{Soc}(e_{22}R)$ via f, where e_{ii} is a matrix unit. Put $M_1=e_{22}R$ and $M_2=(e_{11}R\oplus e_{22}R)/\{x+f(x)\,|\,x\in\mathrm{Soc}(e_{11}R)\}$. Then M_2 is an indecomposable and non-local module. Hence $M=M_1\oplus M_2$ is not a lifting module, but M satisfies (D) by Theorems 2 and 3.

4. Condition ($D^\sharp$)

We shall give the property dual to one in the previous section. Let $\{V_i\}_{i=1}^n$ be a set of R-modules and $V=\sum_{i=1}^n\oplus V_i$. We define the dual condition to (D).

($D^\sharp$) *For a submodule N of V, if $N\cap V_i=0$ for some i, then N is contained in a proper direct summand of V.*

Clearly from [4], Theorem 4, ($D^\sharp$) is equivalent to V being an extending module, provided the V_i are uniform and LE modules.

Lemma 6$^\sharp$. *Let $\{U_i\}_{i=0}^n$ be a set of indecomposable R-modules. Assume that U_0 is almost U_i-injective for all $i\geqslant 1$ and take any diagram with V a submodule of $U_1\oplus\cdots\oplus U_n$:*

$$\begin{array}{ccccc} U_1\oplus\cdots\oplus U_n & \overset{i}{\leftarrow} & V & \leftarrow & 0 \\ & & \downarrow h & & \\ & & U_0 & & \end{array}$$

Put $V'=\ker h$. If $V'\cap U_i\neq 0$ for all $i\geqslant 1$, then there exists $\tilde{h}\colon U_1\oplus\cdots\oplus U_n\to U_0$ with $\tilde{h}i=h$.

Lemma 7. *Let $X=X_1\oplus X_2\oplus X_3$ be an R-module. If X satisfies $(D^\sharp)$ for the above decomposition, then so does $X_1\oplus X_2$.*

Proof. Put $Y=X_1\oplus X_2$. Let N be a submodule of Y with $X_1\cap N=0$. Setting $W=N\oplus X_3\subset X$, we know $W\cap X_1=0$. Hence there exists a proper direct summand V of X such that $V\supset W$, i.e. $X=V\oplus V'$ and $V'\neq 0$. Since $X_3\subset W\subset V$, $V=X_3\oplus(Y\cap V)$ and $X=V\oplus V'=X_3\oplus(Y\cap V)\oplus V'$. Hence $Y=(Y\cap V)\oplus(Y\cap(X_3\oplus V'))$ and $Y\cap V\supset N$. If $Y=Y\cap V$, $X=X_3\oplus(Y\cap V)\oplus V'=X_3\oplus Y\oplus V'$ and hence $V'=0$, a contradiction. Therefore Y satisfies $(D^\sharp)$.

The following theorem is given in [4], when the U_i are uniform. We obtain in general

Theorem $3^\sharp$. *Let $\{U_i\}_{i=1}^n$ be a set of* LE *R-modules and $U=\sum_{i=1}^n\oplus U_i$. Then the following are equivalent:*

1) *U satisfies $(D^\sharp)$ for any direct decomposition of U.*
2) *U_i is almost U_j-injective for all $i\neq j$.*

Proof. 1)→2). $U_i\oplus U_j$ satisfies $(D^\sharp)$ by Lemma 7. Hence we obtain 2) from the proof of [4], Lemma 8.
2)→1) Since the U_i are LE, we may take a direct decomposition into indecomposable modules U_i. We shall show the implication by induction on n. If $n=1$, this is clear. Let N be a submodule of $U=\sum_{i=1}^n\oplus U_i$ with $N\cap U_i=0$ for some i, say $i=1$. Then $N=N^*(h)N^1$, where π_1: $U\to U_1$, π^*: $U\to U^*=\Sigma_{i\geqslant 2}\oplus U_i$ are the projections, $N^*=\pi^*(N)$, $N^1=\pi_1(N)$ and h: $N^*\to N^*/(N\cap U^*)\approx N^1$ (see the proof of Theorem 3 and note $N\cap U_1=0$). Since $N^*/(N\cap U^*)\approx N^1$, $N_*=N\cap U^*=h^{-1}(0)$. First assume that $U_j\cap N_*\neq 0$ for all $j\geqslant 2$. Then there exists $\tilde{h}$: $U^*\to U_1$ with $h\tilde{h}\,|\,N^*=h$ by Lemma $6^\sharp$. Hence $N\subset U^*(\tilde{h})\neq U$, which is a proper direct summand of U. Accordingly we may assume $N_*\cap U_2=0$. Then from the induction hypothesis, there exists a direct decomposition $U^*=U_2'\oplus U_3'\oplus\cdots\oplus U_n'$ such that U_i' is isomorphic to some in $\{U_i\}_{i=2}^n$ and $V'=U_3'\oplus\cdots\oplus U_n'\supset N_*(U^*=U_2'\oplus V')$. Consider $\bar{U}=U/N_*=U_1\oplus U_2'\oplus V'/N_*\supset N/N_*$, and take any element x_1 in N^1. Then there exists z in N such that $z=x_1+x_2+x'$, i.e. $h(x_2+\bar{x}')=x_1$; $x_1\in U_1$, $x_2\in U_2'$ and $x'\in V'$, $\bar{x}'\in V'/N_*$. Further if $z'=x_1+x_2'+x''\in N$; $x_2'\in U_2'$, $x''\in V'$, then $(x_2-x_2')+(x'-x'')\in N\cap U^*=N_*\subset V'$, namely $x_2=x_2'$. Hence the mapping g: $N^1\to U_2'$ given by $g(x_1)=x_2$ is a homomorphism, i.e. $z=x_1+g(x_1)+x'$. From this observation we obtain the following diagram:

$$\begin{array}{ccccc} U_1 & \xleftarrow{\ i\ } & N^1 & \longleftarrow & 0 \\ & & \big\downarrow{\scriptstyle g} & & \\ & & U_2' & & \end{array}$$

Since U_2' is almost U_1-injective, there exists either $\tilde{h}$: $U_1 \to U_2'$ with $i\tilde{h}=g$ or $\tilde{h}$: $U_2' \to U_1$ with $\tilde{h}g=i$. In the formar case $U=U_1(\tilde{h}) \oplus U_2' \oplus \cdots \oplus U_n'$ and $N \subset U_1(\tilde{h}) \oplus U_3' \oplus \cdots \oplus U_n'$ (in the later case g is a monomorphism and $N \subset U_2'(\tilde{h}) \oplus U_3' \oplus \cdots \oplus U_n'$).

References

[1] T. Albu and C. Nastasescu: Relative finiteness in module theory, Monographs Textbooks Pure Appl. Math. 84, Marcel Dekker Inc., New York and Basel.

[2] G. Azumaya, F. Mbuntum and K. Varadarajan: *On M-projective and M-injective modules*, Pacific J. Math. **59** (1975), 9–16.

[3] Y. Baba: *Note on almost M-injectives*, Osaka. J. Math. **26** (1989), 687–698.

[4] Y. Baba and M. Harada: *On almost M-projectives and almost M-injectives*, Tsukuba Math. J. **14** (1990), 53–69.

[5] M. Harada: *Uniserial rings and lifting properties*, Osaka J. Math. **19** (1982), 217–229.

[6] M. Harada and A. Tozaki: *Almost M-projectives and Nakayama rings*, J. Algebra **122** (1989), 447–474.

[7] M. Harada and T. Mabuchi: *On almost M-projectives*, Osaka J. Math. **26** (1989), 837–848.

[8] M. Harada: *On almost relative projectives ovre perfect rings*, Osaka J. Math. **27** (1990), 655–665.

[9] ————: *On almost relative injectives on artinian modules*, Osaka J. Math, **27** (1990), 963–971.

[10] ————: *Direct sums of almost relative injectives and ones of almost relative projectives*, to appear.

[11] A. Tozaki: *Almost M-projectives and one-sided serial rings*, Doctoral Dissertation at Osaka City Univ. 1988.

Department of Mathematics
Osaka City University
Sugimoto–3, Sumiyosi-ku
Osaka 558