Hang, C.W., Miyanishi, M., Nishida K. and Zhang, D.Q.
Osaka J. Math.
29 (1992), 393–404

ON ALGEBRAS WHICH RESEMBLE THE LOCAL WEYL ALGEBRA

Chang Woo HANG, Masayoshi MIYANISHI, Kenji NISHIDA and De-Qi ZHANG

(Received April 11, 1991)

1. Introduction

Let K be an algebraically closed field of characteristic zero and let $\hat{\mathcal{O}}_n(K)=K[[x_1, \cdots, x_n]]$ be the formal power series ring over K in n variables. According to Björk [1], we denote by $\hat{D}_n(K)$ the subring of $\mathrm{End}_K(\hat{\mathcal{O}}_n(K))$ generated over K by the left multiplications by elements of $\hat{\mathcal{O}}_n(K)$ and partial differentials $\partial_i=\partial/\partial x_i$,

$$\hat{D}_n(K)=\hat{\mathcal{O}}_n(K)\langle\partial_1, \cdots, \partial_n\rangle$$

where $\partial_i x_j-x_j \partial_i=\delta_{ij}$ (Kronecker's delta) and $\partial_i \partial_j=\partial_j \partial_i$. The ring $\hat{D}_n(K)$, called the *local Weyl algebra*, has the Σ-filtration $\{\Sigma_v\}_{v\geq 0}$ such that $\Sigma_0=\hat{\mathcal{O}}_n(K)$ and $\Sigma_v=\{\Sigma_\alpha f_\alpha \partial^\alpha; f_\alpha\in\hat{\mathcal{O}}_n(K)$ and $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ with $|\alpha|=\alpha_1+\cdots+\alpha_n\leq v\}$ and that the associated graded ring $\mathrm{gr}_\Gamma(\hat{D}_n(K))$ is a polynomial ring over $\hat{\mathcal{O}}_n(K)$ in n variables. Moreover, $\hat{D}_n(K)$ has weak global dimension n, i.e., $\mathrm{w.gl.dim}(\hat{D}_n(K))=n$.

These are ring-theoretic, algebraic properties which the local Weyl algebra $\hat{D}_n(K)$ has. In the present article, we consider whether or not these properties are sufficient to characterize the ring $\hat{D}_n(K)$. For this purpose, we introduce the notion of pre-W-algebra and W-algebra (see below for the definition) and show that a W-algebra, which satisfies the above-listed properties $\hat{D}_n(K)$ has and one additional condition, i.e., $L=\Sigma_1/\Sigma_0$ is essentially abelian, is realized as a subalgebra of some $\hat{D}_n(K)$. After all, we are successful only in the case $n=1$. We are, however, convinced that our approach of computing the weak global dimension of a W-algebra will be useful to study locally a vector field at a smooth point on an algebraic variety.

We employ the terminology and notation in [1].

2. Structure theorems

To simplify the notation, we denote $\hat{\mathcal{O}}_n(K)$ by R. Let A be a (not necessarily commutative) K-algebra containing R generated by finitely many elements over R. Consider the following three conditions on A:

(i) A has a Σ-filtration $\{\Sigma_v\}_{v\geq 0}$ such that $\Sigma_v(v\geq 0)$ is a two-sided R-submodule of A, $\Sigma_0=R$, Σ_1 generates A over R, $\Sigma_v\cdot\Sigma_w\subset\Sigma_{v+w}$ for any $v, w\geq 0$ and $A=\bigcup_{v\geq 0}\Sigma_v$;

(ii) The associated graded ring $\mathrm{gr}_\Sigma(A):=\bigoplus_{v\geq 0}\Sigma_v/\Sigma_{v-1}$ is a polynomial ring $R[y_1, \cdots, y_m]$ in m variables;

(iii) w.gl.dim $(A)=n$.

If A satisfies the above conditions (i) and (ii), we call it a *pre-W-algebra over R*. We denote by L the free R-module $\Sigma_1/\Sigma_0=\bigoplus_{i=1}^m Ry_i$.

Lemma 2.1. *Let A be a pre-W-algebra over R. Then we have the following:*

(1) *Let* $Y_1, \cdots, Y_m$ *be elements of* Σ_1 *such that* $y_i\equiv Y_i \pmod{\Sigma_0}$ *for any i. Then A is generated by* $Y_1, \cdots, Y_m$ *over R, which we write as* $A=R\langle Y_1, \cdots, Y_m\rangle$.

(2) *For any* $y\in L$ *and* $a\in R$, *define* $y[a]$ *by*

$$y[a] = Ya-aY$$

for $Y\in\Sigma_1$ *with* $y\equiv Y\pmod{\Sigma_0}$. *Then* $y[a]$ *is independent of the choice of Y, and y is considered as a K-derivation on R. So, we have an R-linear map* $\rho: L\to \mathrm{Der}_K(R)$; *we write* $y[a]$ *as* $\rho(y)(a)$ *as well and we use this map* ρ *in the subsequent discussions without referring explicitly to this lemma.*

(3) *Define a bracket product* $[y, z]$ *on L by*

$$[y, z] \equiv YZ-ZY \pmod{\Sigma_0}$$

for $Y, Z\in\Sigma_1$ *with* $y\equiv Y \pmod{\Sigma_0}$ *and* $z\equiv Z \pmod{\Sigma_0}$. *Then* $[y, z]$ *is well-defined and* ρ *is a Lie-algebra homomorphism, i.e.,* $\rho([y, z])=[\rho(y), \rho(z)]$.

Proof. (1) For any $f\in A$, we define $\nu(f)$ as the smallest integer r with $f\in\Sigma_r$. If $\nu(f)=r$, there exists $F_r(y_1, \cdots, y_m)\in R[y_1, \cdots, y_m]_r$=the r-th homogeneous part of $\mathrm{gr}_\Sigma(A)$ such that $f-F_r(Y_1, \cdots, Y_m)\in\Sigma_{r-1}$. By induction on $\nu(f)$, we can verify the assertion straightforwardly.

(2) Replace Y be $Y+b$ with $b\in R$. Then we have

$$(Y+b)a-a(Y+b) = Ya-aY,$$

whence $y[a]$ is independent of the choice of Y. Furthermore, we have

$$\begin{aligned} y[ab] &= Y(ab)-(ab)Y = (aY+y[a])b-abY \\ &= a(Yb-bY)+y[a]b = ay[b]+y[a]b\,. \end{aligned}$$

So, $y[\ \ ]$ is a K-derivation on R.

(3) The assertion can be verified by a straightforward computation.

Q.E.D.

The structure of a pre-W-algebra over R is given in the following:

Theorem 2.2. (1) *Let A be a pre-W-algebra over R. Let $Y_1, \cdots, Y_m$ be elements of Σ_1 as chosen in the previous lemma. Write*

$$(2.0)\qquad Y_i Y_j - Y_j Y_i = \sum_{k=1}^{m} \rho_{ij,k} Y_k + \sigma_{ij}, \quad 1 \leq i, j \leq m,$$

where $\rho_{ij,k}, \sigma_{ij} \in R$. Then we have the following equalities:

$$(2.1)\qquad \sum_{l=1}^{m} (\rho_{ij,l}\,\rho_{lk,s} + \rho_{jk,l}\,\rho_{li,s} + \rho_{ki,l}\,\rho_{lj,s}) = y_i[\rho_{jk,s}] + y_j[\rho_{ki,s}] + y_k[\rho_{ij,s}], \quad 1 \leq i, j, k, s \leq m$$

$$(2.2)\qquad \sum_{l=1}^{m} (\rho_{ij,l}\,\sigma_{lk} + \rho_{jk,l}\,\sigma_{li} + \rho_{ki,l}\,\sigma_{lj}) = y_i[\sigma_{jk}] + y_j[\sigma_{ki}] + y_k[\sigma_{ij}], \quad 1 \leq i, j, k \leq m$$

$$(2.3)\qquad \rho_{ij,k} = -\rho_{ji,k}, \ \sigma_{ij} = -\sigma_{ji}, \quad 1 \leq i, j, k \leq m.$$

The elements $\{\rho_{ij,k}; 1 \leq i, j, k \leq m\}$ are determined uniquely by the Lie algebra L and the choice of R-free basis $\{y_1, \cdots, y_m\}$ of L.

(2) *Suppose we are given as in Lemma 2.1 the Lie algebra L and an R-linear map $\rho: L \to \mathrm{Der}_K R$ which is a Lie-algebra homomorphism. For an R-free basis $\{y_1, \cdots, y_m\}$ of L, suppose we are given elements $\{\sigma_{ij}; 1 \leq i, j \leq m\}$ satisfying the conditions* (2.2) *and* (2.3) *above. Then there exists a K-algebra A with a Σ-filtration $\{\Sigma_v\}_{v \geq 0}$ such that*

(i) *A is generated over R by elements $Y_1, \cdots, Y_m$;*
(ii) *The equalities* (2.0)-(2.3) *hold;*
(iii) $\Sigma_v = \{\Sigma_\alpha f_\alpha Y^\alpha; f_\alpha \in R, Y^\alpha = Y_1^{\alpha_1} \cdots Y_m^{\alpha_m}, |\alpha| \leq v\}$ *for any $v \geq 0$;*
(iv) $\mathrm{gr}_\Sigma(A) \cong R[y_1, \cdots, y_m] :=$ *the symmetric algebra of L over R.*

Proof. (1) By the definition of $[y_i, y_j]$ in Lemma 2.1, $\{\rho_{ij,k}; 1 \leq i, j, k \leq m\}$ are the multiplication constants of the Lie algebra L. Hence they are uniquely determined by the choice of the R-free basis $\{y_1, \cdots, y_m\}$ of L. If one chooses $\{Y_1, \cdots, Y_m\}$ as in Lemma 2.1, then $\{1, Y_1, \cdots, Y_m\}$ is an R-free basis of Σ_1. Then the equalities (2.1) and (2.2) follow from the Jacobi identity:

$$[[Y_i, Y_j], Y_k] + [[Y_j, Y_k], Y_i] + [[Y_k, Y_i], Y_j] = 0,$$

where $[Y_i, Y_j] = Y_i Y_j - Y_j Y_i$.

(2) Let $\{Y_1, \cdots, Y_m\}$ be indeterminates and let A be the free K-algebra generated by $Y_1, \cdots, Y_m$ over R modulo the two-sided ideal I generated by

$$\{Y_i Y_j - Y_j Y_i - \sum_{k=1}^{m} \rho_{ij,k} Y_k - \sigma_{ij}; 1 \leq i, j, k \leq m\}$$

and

$$\{Y_i f - f Y_i - \rho(y_i)(f); 1 \leq i \leq m, \forall f \in R\}.$$

We write $y_i[f]=\rho(y_i)(f)$ by identifying Y_i's with y_i's in L, We can employ the proof of the Poincaré-Birkoff-Witt theorem (cf. Jacobson [2]) without major changes in the present situation to show that every element of A is written uniquely as a linear combination of standard monomials in $Y_1, \cdots, Y_m$ with coefficients in R. In particular, the equalities (2.1) and (2.2) imply that Σ_1 (with the notation in (iii)) is a free R-module generated by 1, $Y_1, \cdots, Y_m$. Note that there is a surjective homomorphism $\theta: R[y_1, \cdots, y_m] \to \mathrm{gr}_\Sigma(A)$. Its kernel is generated by the relations $y_i y_j - y_j y_i$ and $y_i f - f y_i$, $1 \leq i, j \leq m$. But these elements are already zero in $R[y_1, \cdots, y_m]$. Hence $\mathrm{gr}_\Sigma(A) \cong R[y_1, \cdots, y_m]$.

Q.E.D.

Let A be a pre-W-algebra over R. We are interested in the existence of an algebra homomorphism from A to the local Weyl algebra $\hat{D}_n(K)$, which is the identity homomorphism when restricted on the subalgebra R. We call it a *K-algebra homomorphism over R.*

Theorem 2.3. *Let A be a pre-W-algebra over R. Then the following conditions on A are equivalent:*

(1) *There is a K-algebra homomorphism $\tilde{\rho}: A \to \hat{D}_n(K)$ over R such that $\tilde{\rho}(\Sigma_v) \subset \Sigma_v$ for all $v \geq 0$ and $\tilde{\rho}|_{\Sigma_1}$ induces the Lie-algebra homomorphism $\rho: L := \Sigma_1/\Sigma_0 \to \mathrm{Der}_K(R)$ (cf. Lemma 2.1).*

(2) *There exists a lifting $\{Y_1, \cdots, Y_m\}$ of the R-free basis $\{y_1, \cdots, y_m\}$ in Σ_1 for which $\sigma_{ij}=0$, $1 \leq i, j \leq m$.*

(3) *There exist $\{a_i\}_{1 \leq i \leq m}$ in R such that*

$$\sigma_{ij} = \sum_{l=1}^{m} \rho_{ij,l}\, a_l + y_j[a_i] - y_i[a_j]\,, \quad 1 \leq i, j \leq m\,. \tag{2.4}$$

(4) *There exists an R-free submodule $\tilde{L}$ of Σ_1 such that $\tilde{L}$ is closed under the bracket product $[Y, Z] = YZ - ZY$ and the natural residue homomorphism $\pi: \Sigma_1 \to L$ induces a Lie-algebra isomorphism $\pi|_{\tilde{L}}: \tilde{L} \to L$.*

Proof.

(1)$\Rightarrow$(2). Note that $\hat{D}_n(K)$ acts on R in the natural fashion. So, A acts on R via the homomorphism $\tilde{\rho}$. For $Y \in \Sigma_1$, let $a = \tilde{\rho}(Y) \cdot 1$ and let $Y' = Y - a$. Then, since $\tilde{\rho}(Y) \in \Sigma_1 := \oplus_{i=1}^{n} R\partial/\partial x_i + R$, we know that $\tilde{\rho}(Y') \in \mathrm{Der}_K(R)$. In particular, $\tilde{\rho}(Y') \cdot 1 = 0$. Now, for the given lifting $\{Y_1, \cdots, Y_m\}$, we set $Y'_i = Y_i - \tilde{\rho}(Y_i) \cdot 1$, $1 \leq i \leq m$. Then $\{Y'_1, \cdots, Y'_m\}$ is a lifting of $\{y_1, \cdots, y_m\}$ in Σ_1. We assume from the beginning that $Y'_i = Y_i$, $1 \leq i \leq m$. Then the equality (2.0) implies $\sigma_{ij} = 0$ $(1 \leq i, j \leq m)$ because $\tilde{\rho}(Y_i) \in \mathrm{Der}_K(R)$.

(2)$\Rightarrow$(3). Suppose $\{Y_1, \cdots, Y_m\}$ is the given lifting of $\{y_1, \cdots, y_m\}$ and $\{Y'_1, \cdots, Y'_m\}$ is a lifting for which $\sigma'_{ij} = 0$ when we write

$$Y'_i Y'_j - Y'_j Y'_i = \sum_{k=1}^{m} \rho_{ij,k} Y'_k + \sigma'_{ij}\,, \quad 1 \leq i, j \leq m\,. \tag{2.0$'$}$$

Then $Y_i' = Y_i + a_i$ with $a_i \in R$. Replacing Y_i' in (2.0)′ by this expression, we obtain the equality (2.4).

(3)⇒(2). Conversely, if we are given $\{a_i\}_{1 \le i \le m}$ satisfying (2.4), set $Y_i' = Y_i + a_i$. Then $\{Y_1', \cdots, Y_m'\}$ is a lifting of $\{y_1, \cdots, y_m\}$ for which $\sigma_{ij}' = 0$.

(2)⇒(4). Let $\{Y_1, \cdots, Y_m\}$ be as in (2) above. Let $\tilde{L}$ be the R-submodule of Σ_1 generated by $Y_1, \cdots, Y_m$. Then $\tilde{L}$ is a free R-module. Since $\sigma_{ij} = 0$, we readily verify that $[Y, Z] \in \tilde{L}$ for any $Y, Z \in \tilde{L}$. Clearly, π induces an isomorphism between $\tilde{L}$ and L.

(4)⇒(1). Define $\tilde{\rho}\colon \tilde{L} \to \mathrm{Der}_K(R)$ by $\tilde{\rho}(Y) = \rho(\pi(Y))$. Extend this to Σ_1 in a natural fashion by putting $\tilde{\rho}|_{\Sigma_0} = \mathrm{id}_R$. Furthermore, we extend $\tilde{\rho}$ to the free K-algebra F generated over R by $Y_1, \cdots, Y_m$ as follows. For an element $Y_{i_1} f_{i_1} \cdots Y_{i_r} f_{i_r}$ of F with $Y_{i_j} \in \{Y_1, \cdots, Y_m\}$ and $f_{i_j} \in R$, define

$$Y_{i_1} f_{i_1} \cdots Y_{i_r} f_{i_r} \cdot (a) = y_{i_1}[f_{i_1}[y_{i_2}[\cdots [f_{i_r} a] \cdots]]],$$

where $y_{i_j} = \pi(Y_{i_j})$ and $f[b] := fb \in R$. In view of (2) of Theorem 2.2, A is identified with the residue ring of F by the two-sided ideal I considered in Theorem 2.2. So, in order to have $\tilde{\rho}$ as above, we have only to show that

$$y_i[y_j[a]] - y_j[y_i[a]] = \sum_{k=1}^{m} \rho_{ij,k}\, y_k[a] \quad \text{and} \quad y_i[fa] = f y_i[a] + y_i[f]\, a$$

for $a \in R$. These equations hold, in fact, because $\rho\colon L \to \mathrm{Der}_K(R)$ being a Lie-algebra homomorphism implies

$$y_i[y_j[a]] - y_j[y_i[a]] = [y_i, y_j]\,[a] = \sum_{k=1}^{m} \rho_{ij,k}\, y_k[a]$$

and the second equality above. Q.E.D.

If a pre-W-algebra A over R satisfies one of the equivalent conditions in Theorem 2.3, we call *A a W-algebra over R.*

REMARK 2.4. (1) Suppose that $\rho\colon L \to \mathrm{Der}_K(R)$ is an isomorphism. Then, as an R-free basis $\{y_1, \cdots, y_m\}$ of L, we can take $y_i = \rho^{-1}(\partial/\partial x_i)$. Then $\rho_{ij,k} = 0$ for all $1 \le i, j, k \le m$. So the case with all $\rho_{ij,k} = 0$ can take place. We then say that L is *essentially abelian.*

(2) Suppose L is essentially abelian. Let $\{y_1, \cdots, y_m\}$ be an R-free basis of L such that $[y_i, y_j] = 0$, $1 \le i, j \le m$ and let $\{Y_1, \cdots, Y_m\}$ be such that $y_i \equiv Y_i$ (mod Σ_0) and $Y_i\, Y_j - Y_j\, Y_i = \sigma_{ij} \in R$. Suppose we can take $\sigma_{ij} = c_{ij} \in K^* = K - (0)$ for $1 \le i, j \le m$ and $i \ne j$ and that $\rho(y_i)(\mathcal{M}) \subset \mathcal{M}$, where $\mathcal{M}$ is the maximal ideal of R. Then we cannot find $\{a_i\}_{1 \le i \le m}$ so that the equality (2.4) holds. There exists a K-algebra A over R satisfying these conditions. In fact, we take $m = n$, $\rho\colon L \to \mathrm{Der}_K(R)$ to be a homomorphism such that $\rho(y_i) = \partial/\partial x_i$, $1 \le i \le n$, and A to be the residue ring of a free K-algebra F over R generated by $Y_1, \cdots, Y_n$ modulo the two-sided ideal I as considered in Theorem 2.2, (2). Then ρ cannot

be extended to a K-algebra homomorphism $\tilde{\rho}$: $A \to \tilde{D}_n(K)$ over R as considered in Theorem 2.3.

3. Case L is essentially abelian

We begin with the following:

Lemma 3.1. *Let A be a W-algebra over R with a K-algebra homomorphism $\tilde{\rho}$: $A \to \hat{D}_n(K)$ over R which is an extension of the Lie-algebra homomorphism ρ: $L \to \mathrm{Der}_K(R)$. Then we have* w.gl.dim $(A) \geq n$.

Proof. Note that any element ξ of A can be expressed as $\xi = \Sigma_\alpha f_\alpha Y^\alpha$, where $f_\alpha \in R$ and $Y^\alpha = Y_1^{\alpha_1} \cdots Y_m^{\alpha_m}$ (cf. the equality $Ya - aY = y[a]$ in Lemma 2.1). Furthermore, this expression is unique. Indeed, if we have a nontrivial expression $\Sigma_\alpha f_\alpha Y^\alpha = 0$ then this yields a homogeneous nontrivial relation

$$\sum_{|\alpha| = v} f_\alpha y^\alpha = 0, \quad y^\alpha = y_1^{\alpha_1} \cdots y_m^{\alpha_m}$$

where $v = \max\{|\alpha|; f_\alpha \neq 0\}$. This contradicts the hypothesis that $\mathrm{gr}_\Sigma(A)$ is a polynomial ring in $y_1, \cdots, y_m$ over R. Hence A is a free R-module, whence A is R-flat as a left R-module. Similarly, ξ can be expressed uniquely as $\xi = \Sigma_\beta Y^\beta g_\beta$. So, A is R-flat as a right R-module. Hence A is R-flat as a ring. In view of Björk [1, Cor.2.9, p.42], we have

$$(*) \quad \mathrm{w.dim}_R(A \otimes_R M) \leq \mathrm{w.dim}_A(A \otimes_R M)$$

for any left R-module M. Take an R-module $K = R/\mathcal{M}$ with $\mathcal{M} = (x_1, \cdots, x_n) R$. Then, by the theory of syzyzy, we know that $\mathrm{w.dim}_R(K) = n$; in fact, $\mathrm{Tor}_n^R(K, K) = K \neq (0)$. Then the above inequality $(*)$ implies that $\mathrm{w.dim}_A(A \otimes_R K) \geq n$. Hence w.gl.dim $(A) \geq n$. Q.E.D.

We shall be concerned with the condition w.gl.dim $(A) = n$ for a W-algebra over R.

Theorem 3.2. *Let A be a W-algebra over R with a K-algebra homomorphism $\tilde{\rho}$: $A \to \hat{D}_n(K)$ over R. Suppose that L is essentially abelian and A has* w.gl.dim $(A) = n$. *Then $\tilde{\rho}$ is an injection.*

Proof. Let $\tilde{\rho}_1 := \tilde{\rho} |_{\tilde{L}}$, where $\tilde{L}$ is an R-free submodule of Σ_1 isomorphic to L as a Lie algebra (cf. Theorem 2.3). Then there exists an R-free basis $\{Y_1, \cdots, Y_m\}$ of $\tilde{L}$ such that $Y_i Y_j = Y_j Y_i$ for $1 \leq i, j \leq m$. Let $\tilde{L}_0 = \oplus_{i=1}^m KY_i$ and let $Q = \mathrm{Ker}(\tilde{\rho}_1 |_{\tilde{L}_0})$. Then $\tilde{L}_0 \cong Q \oplus \tilde{\rho}_1(\tilde{L}_0)$ is a direct sum as Lie algebras and Q is contained in the center of A. Let B be the R-subalgebra of $\hat{D}_n(K)$ generated by $\tilde{\rho}_1(\tilde{L}_0)$ and let J be the two-sided ideal of A generated by Q. Then $B \cong A/J$ and B is a W-algebra over R. Indeed, we may take $\{Y_1, \cdots, Y_m\}$ so that $\{Y_{r+1}, \cdots, Y_m\}$ is a K-basis of Q. Let $\bar{Y}_i = \tilde{\rho}_1(Y_i)$, $1 \leq i \leq r$. Then B is generated by $\bar{Y}_1, \cdots, \bar{Y}_r$ over R which act on R via the derivations $\delta_i = y_i[\ \ ]$, $1 \leq i \leq r$. Note that $\{\bar{Y}_1, \cdots, \bar{Y}_r\}$ are linearly independent over R. So, $r \leq n$. We claim:

Lemma 3.3. *$\{\delta_1, \cdots, \delta_r\}$ are algebraically independent over R. Namely, if $\Sigma_\gamma f_\gamma \delta^\gamma = 0$ with $f_\gamma \in R$ and $\delta^\gamma = \delta_1^{\gamma_1} \cdots \delta_r^{\gamma_r}$ then $f_\gamma = 0$ for all γ.*

Proof. Denote by $Q(R)$ the quotient field of R. We can find $\Delta_1, \cdots, \Delta_r \in \oplus_{i=1}^r Q(R) \delta_i$ satisfying the following conditions:
(1) $\oplus_{i=1}^r Q(R) \delta_i = \oplus_{i=1}^r Q(R) \Delta_i$;
(2) We can express $\Delta_i = \sum_{j=1}^n a_{ij} \partial_j$ with $a_{ij} \in R$ and $\partial_j = \partial / \partial x_j$, and if we define s_i as $\min\{j;\ a_{ij} \neq 0\}$ then $s_1 < s_2 < \cdots < s_r$.

Suppose we have a nontrivial relation $\Sigma_\gamma f_\gamma \delta^\gamma = 0$. Let $v = \max\{|\gamma|\,; f_\gamma \neq 0\}$. Expressing δ_i as a $Q(R)$-linear combination of Δ_j's and substituting it for δ_i in $\Sigma_\gamma f_\gamma \delta^\gamma = 0$, we obtain a nontrivial relation $\Sigma_\gamma g_\gamma \Delta^\gamma = 0$ with $\max\{|\gamma|\,; g_\gamma \neq 0\} = v$. Expressing then Δ^γ in terms of $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, we obtain

$$(*) \qquad \sum_{|\gamma| = v} (g_\gamma \prod_{i=1}^r (a_{is_i})^{\gamma_i})\, \partial^{\tilde{\gamma}} + \cdots = 0\,,$$

where $\tilde{\gamma}$, as an n-tuple, has γ_i at the s_i-th entry for $1 \leq i \leq r$ and 0 elsewhere if $\gamma = (\gamma_1, \cdots, \gamma_r)$. Among g_γ's with $|\gamma| = v$ and $g_\gamma \neq 0$, let $(\alpha_1, \cdots, \alpha_r)$ be the smallest with respect to the lexicographic relation: $(\gamma_1, \cdots, \gamma_r) \leq (\gamma_1', \cdots, \gamma_r')$ if and only if $\gamma_1 = \gamma_1', \cdots, \gamma_{t-1} = \gamma_{t-1}', \gamma_t \leq \gamma_t'$. Then $(g_\alpha \prod_{i=1}^r (a_{is_i})^{\alpha_i})\, \partial^{\tilde{\alpha}}$ has no other terms in $(*)$ to cancel with. This is a contradiction. Q.E.D.

Proof of Theorem 3.2 resumed. The above lemma implies that B is isomorphic to a W-algebra over R generated by $Y_1, \cdots, Y_r$. Since any element ξ of A is expressed uniquely in the form

$$(**) \qquad \xi = \sum_\gamma f_\gamma Y^\gamma + \eta\,, \quad f_\gamma \in R \quad \text{and} \quad \eta \in J\,,$$

where $Y^\gamma = Y_1^{\gamma_1} \cdots Y_r^{\gamma_r}$, we know that A/J is isomorphic to B.

Now we can easliy show that $A \cong B[Y_{r+1}, \cdots, Y_m]$, a polynomial ring in $Y_{r+1}, \cdots, Y_m$ over B (cf. the above expression $(**)$ of ξ). By Björk [1, Th. 3.4, p.43], we have $\text{w.gl.dim}(A) = \text{w.gl.dim}(B) + (m-r) \geq n + m - r$ (cf. Lemma 3.1). By the hypothesis $\text{w.gl.dim}(A) = n$, we have $m = r$. This implies $J = (0)$. Hence $A \cong B$. Q.E.D.

A W-algebra A over R is called a *W-subalgebra of $\hat{D}_n(K)$* provided $\tilde{\rho}$ is injective.

Theorem 3.4. *There is a one-to-one correspondence between the set of W-subalgebras of $\hat{D}_n(K)$ and the set of R-submodules $\tilde{L}$ of $\text{Der}_K(R)$ satisfying the conditions:*

(L-1) *$\tilde{L}$ is a free R-submodule of* $\mathrm{Der}_K(R)$;
(L-2) *$\tilde{L}$ is closed under the bracket product of* $\mathrm{Der}_K(R)$.

Proof. Let A be a W-subalgebra of $\hat{D}_n(K)$. Then we can find an R-free submodule $\tilde{L}$ of Σ_1 which is isomorphic to $L:=\Sigma_1/\Sigma_0$. Since $\tilde{\rho}$ is injective, so is $\rho: L\to\mathrm{Der}_K(R)$. Hence $\tilde{L}$ is an R-free submodule of $\mathrm{Der}_K(R)$. Since $\rho\cdot(\pi\,|\,\tilde{L})$ is a Lie-algebra homomorphism, $\tilde{L}$ is closed under the bracket product of $\mathrm{Der}_K(R)$ (cf. Theorem 2.3). Conversely, let $\tilde{L}$ be an R-submodule of $\mathrm{Der}_K(R)$ satisfying the conditions (L-1) and (L-2). Let $\{Y_1, \cdots, Y_m\}$ be an R-free basis of $\tilde{L}$. Then we have:
(1) $Y_i Y_j - Y_j Y_i = \sum_{k=1}^m \rho_{i j,k} Y_k$, $1\leq i,j\leq m$,
(2) $Y_i f - f Y_i = Y_i[f]$ for $f\in R$ and $1\leq i\leq m$.

Construct a K-algebra A over R as in Theorem 2.2, (2). Then the natural K-algebra homomorphism $A\to\hat{D}_n(K)$ over R is injective (cf. the proof of Lemma 3.3). Q.E.D.

A W-subalgebra A of $\hat{D}_n(K)$ is said to be *of maximal rank if* $\mathrm{rank}\,\tilde{L}=n$. We shall consider the case $n=1$. Then L is essentially abelian. Hence there exists a K-algebra homomorphism $\tilde{\rho}$: $A\to\hat{D}_1(K)$ over R which must be injective by virtue of Theorem 3.4. We set $Y=Y_1$, a free generator of the R-module $\tilde{L}$ (cf. Theorem 2.3). Then we have $Yx-xY=f$, where $f=x^r u$ with $u\in R^*$. Replacing Y by $u^{-1}Y$, we may assume that $f=x^r$. We shall show:

Lemma 3.5. $\mathrm{Tor}_2^A(K,K)=K$ *if* $r\geq 2$, *while it is zero if* $r=1$. $\mathrm{Tor}_1^A(K,K)=K$ *if* $r=1$.

Proof. Suppose $r>0$. Then K is a two-sided A-module. As a right A-module K has the following free A-module resolution:

$$0\to e_2 A \xrightarrow{\varphi_1} e_1 A\oplus e_1' A \xrightarrow{\varphi_0} e_0 A \xrightarrow{\varepsilon} K\to 0\,,$$

where ε is the natural residue homomorphism and $\varphi_i (i=0, 1)$ is given as:

$$\varphi_0(e_1)=e_0 Y,\ \ \varphi_0(e_1')=e_0 x \quad\text{and}\quad \varphi_1(e_2)=e_1 x - e_1'(Y+x^{r-1})\,.$$

Take the tensor product of this sequence with a left A-module $K=Av$ to obtain the complex:

$$0\to e_2 A\otimes_A Av \xrightarrow{\bar{\varphi}_1} (e_1 A\otimes_A Av)\oplus(e_1' A\otimes_A Av) \xrightarrow{\bar{\varphi}_0} e_0 A\otimes_A Av\to 0\,,$$

where we can identify $e_i A\otimes_A Av$ with $e_i\otimes Kv$ for $e_i=e_0, e_1, e_1'$ and e_2. Then it is clear that $\bar{\varphi}_1=\bar{\varphi}_0=0$ if $r\geq 2$. Hence $\mathrm{Tor}_2^A(K,K)=K$ if $r\geq 2$. If $r=1$, then $\bar{\varphi}_1(e_2\otimes v)=-e_1'\otimes v$, whence $\bar{\varphi}_1$ is injective. So, $\mathrm{Tor}_2^A(K,K)=0$ if $r=1$. If $r=1$, $\mathrm{Tor}_1^A(K,K)=K$ because $\bar{\varphi}_0=0$. Q.E.D.

Corollary 3.6. *Let A be a W-subalgebra of $\hat{D}_1(K)$ with* w.gl.dim $(A)=1$. *Then $A=\hat{D}_1(K)$.*

Proof. With the same notations as in Lemma 3.5, it suffices to show that w.gl.dim$(A)=2$ if $r=1$. Suppose $r=1$ and consider the following exact sequence

$$0 \to e_2 A \xrightarrow{\varphi_1} e_1 A \oplus e_1' A \xrightarrow{\varphi_0} \operatorname{Im} \varphi_0 \to 0 .$$

Suppose that w.gl.dim$(A)=1$. Then $\operatorname{Im}\varphi_0$ is a projective A-module in view of the free A-module resolution of K given in the proof of Lemma 3.5. So, the above sequence must split. Hence there exists an A-homomorphism $\psi: e_1 A\oplus e_1' A \to e_2 A$ such that $\psi\varphi_1=id_{e_2A}$, Write $\psi(e_1)=e_2a$ and $\psi(e_1')=e_2b$ for some a, b of A. Then we have $ax-b(Y+1)=1$. We claim, however, that $Ax+A(Y+1)$ is a proper left ideal of A. Indeed, $Ax=xA$ (cf. Lemma 3.7 below) and A/Ax is isomorphic to a polynomial ring $K[Y]$. Hence $A/Ax+A(Y+1)=K$ and our claim is proved. This is a contradiction. Consequently, we have w.gl.dim $(A)=2$. Q.E.D.

We still remain in the case $n=r=1$. A simple right or left A-module M is said to be *unfaithful* if $\operatorname{ann}_A(M)\neq 0$. For $\alpha\in K$, define $K_\alpha=A/xA+(Y-\alpha)A$. Then we have the following:

Lemma 3.7. *The following assertions hold true:*

(1) *K_α is a simple right A-module as well as a simple left A-module.*

(2) *$K_\alpha\cong K_\beta$ if and only if $\alpha=\beta$.*

(3) *Every unfaithful simple right or left A-module is isomorphic to K_α for some $\alpha\in K$.*

(4) *Let S_A and ${}_AT$ be unfaithful simple right and left A-modules, respectively. Then* $\operatorname{Tor}_1^A(S, T)=0$.

Proof. The first three assertions can be proved as in the case of a skew polynomial ring or in the case of the universal enveloping algebra of a two-dimensional Lie algebra over K. For the convenience of the readers, we shall sketch the proof.

(1) By the relation $Yx-xY=x$, we have

$$(Y-\alpha)x-x(Y-\alpha)=x \quad \text{for} \quad \alpha\in K$$

This implies that

$$xA=Ax \quad \text{and} \quad xA+(Y-\alpha)A=Ax+A(Y-\alpha)$$

Since $K_\alpha\cong K[Y]/(Y-\alpha)$, K_α is simple as right and left A-modules.

(2) This easily follows from the first assertion.

(3) Since $xA\subset\operatorname{ann}_A(K_\alpha)$, K_α is unfaithful. Let I be a nonzero two-sided

ideal of A. Then $x^n \in I$ for some n. Indeed, let ξ be a nonzero element of I and write it as

$$\xi = \sum_{i=0}^{r} f_i Y^i \quad \text{with} \quad f_i \in R \quad \text{and} \quad f_r \neq 0 .$$

Then $\xi x - x\xi = rx f_r Y^{r-1} +$(terms of lower degree) is an element of I. Since $rx f_r \neq 0$, we can continue this step of finding an element of I with lower degree in Y. After the r-steps repeated, we find an element $x^r f_r$ of I. Multiplying to this element a unit in R, we find $x^r \in I$. Let S be an unfaithful simple right A-module. Set $I = \mathrm{ann}_A(S) \neq 0$. Then $x^n \in I$ and $x^{n-1} \notin I$ for some n. Since $Sx^{n-1} \neq 0$, there exists $s \in S$ such that $sx^{n-1} \neq 0$. Since S is simple, we have $S = sx^{n-1}A = sAx^{n-1}$, whence $Sx = sAx^n = 0$. Hence $x \in I$. So, $xA \subset I$. It is clear that I is a prime ideal of A in the sense that $J_1 J_2 \subset I$ for two-sided ideals J_1, J_2 of A implies $J_1 \subset I$ or $J_2 \subset I$. Let $\bar{A} = A/xA \cong K[Y]$ and $\bar{I}$ the image of I in $\bar{A}$. Since $\bar{I}$ is a prime ideal of $K[Y]$, we have $\bar{I} = (Y-\alpha)K$ for some $\alpha \in K$. Hence $I = xA + (Y-\alpha)A$ and $S \cong A/I = K_\alpha$. A similar argument applies to a simple left A-module.

(4) In order to prove the assertion, we have to show

$$\mathrm{Tor}_1^A(K_\alpha, K_\beta) = 0 \quad \text{for} \quad \alpha, \beta \in K .$$

We can easily show this result by replacing Y by $Y-\alpha$ in the proof of Lemma 3.5. Q.E.D.

If $n \geq 2$, we know little on W-subalgebras of $\hat{D}_n(K)$ even if it is of maximal rank. We shall give two partial results.

Proposition 3.8. *Let A be a W-subalgebra of maximal rank of $\hat{D}_n(K)$ corresponding to a Lie subalgebra $\tilde{L} = \oplus_{i=1}^n RY_i$ wiih $Y_i = x_i^{r_i} \partial/\partial x_i$ and $r_i \geq 1$. Then we have*

$$\mu := \max\{v;\ \mathrm{Tor}_v^A(K, K) \neq 0\} = 2\#\{i;\ r_i \geq 2\} + \#\{i;\ r_i = 1\} .$$

Hence $r_i = 1$ for all i provided $\text{w.gl.dim}(A) = n$.

Proof. Let S_i be the free algebra generated by Y_i over a one-dimensional polynomial ring $K[x_i]$ modulo the two-sided ideal generated by $Y_i x_i - x_i Y_i = x_i^{r_i}$. Since $Y_i Y_j = Y_j Y_i$ and $x_i Y_j = Y_j x_i$ if $i \neq j$, A is isomorphic to

$$(S_1 \otimes_K S_2 \otimes_K \cdots \otimes_K S_n) \otimes_{K[x_1, \cdots, x_n]} R ,$$

when $S_1 \otimes_K \cdots \otimes_K S_n$ is regarded as an algebra over $K[x_1, \cdots, x_n]$. Consider a complex

$$(\tilde{C}_i^\bullet):\ 0 \to e_2^{(i)} S_i \xrightarrow{\varphi_1} e_1^{(i)} S_i \oplus e_1'^{(i)} S_i \xrightarrow{\varphi_0} e_0^{(i)} S_i \xrightarrow{\varepsilon} K \to 0 ,$$

which is defined in the same fashion as in the proof of Lemma 3.5 with A replaced by S_i. It is a resolution of the two-sided S_i-module K by free right S_i-modules. The complex $\tilde{C}^\bullet := (\tilde{C}_1^\bullet \otimes_K \cdots \otimes_K \tilde{C}_n^\bullet) \otimes_{K[x_1,\dots,x_n]} R$ is a resolution of the two-sided A-module K by free right A-modules. Let $C_i^\bullet$ (resp. $C^\bullet$) be the complex obtained from $\tilde{C}_i^\bullet$ (resp. $\tilde{C}^\bullet$) by replacing K by 0. Then, taking the tensor products with the left A-module K, we obtain $\bar{C}^\bullet := C^\bullet \otimes_A K \cong \bar{C}_1^\bullet \otimes_K \cdots \otimes_K \bar{C}_n^\bullet$, where $\bar{C}_i^\bullet = C_i^\bullet \otimes_A K$. By the Künneth formula for homologies, we have

$$\mathrm{Tor}_v^A(K, K) \cong \oplus_{v_1+\cdots+v_n=v} \mathrm{Tor}_{v_1}^{S_1}(K, K) \otimes_K \cdots \otimes_K \mathrm{Tor}_{v_n}^{S_n}(K, K)\,.$$

Hence we obtain the stated formula in view of Lemma 3.5. Q.E.D.

Proposition 3.9. *Let A be a W-subalgebra of maximal rank of $\hat{D}_2(K)$ corresponding to a Lie subalgebra $\tilde{L} = RY_1 + RY_2$ with $Y_i = h\partial/\partial x_i$, where $h = x_1 f + x_2 g \in Rx_1 + Rx_2$. Suppose that h is a homogeneous polynomial in x_1 and x_2. Then* $\mathrm{Tor}_3^A(K, K) \neq 0$ *and* $\mathrm{Tor}_4^A(K, K) = 0$.

Proof. We have the following relations:

$$\begin{aligned} Y_1 Y_2 - Y_2 Y_1 &= -h_{x_2} Y_1 + h_{x_1} Y_2 \\ Y_1 x_1 - x_1 Y_1 &= h = Y_2 x_2 - x_2 Y_2 \\ Y_1 x_2 - x_2 Y_1 &= 0 = Y_2 x_1 - x_1 Y_2\,, \end{aligned}$$

where $h_{x_i} = \partial h/\partial x_i$. Construct a complex of right A-modules:

$$0 \to e_3 A \overset{\varphi_2}{\to} e_2 A \oplus e_2' A \oplus e_2'' A \oplus e_2''' A \overset{\varphi_1}{\to}$$
$$e_1 A \oplus e_1' A \oplus e_1'' A \oplus e_1''' A \overset{\varphi_0}{\to} e_0 A \overset{\varepsilon}{\to} K \to 0\,,$$

where:

(0) K is the two-sided A-module with $x_i \cdot 1 = Y_i \cdot 1 = 0$ for $i = 1, 2$;

(i) $\varepsilon(e_0) = 1$;

(ii) $\varphi_0(e_1) = e_0 Y_1$, $\varphi_0(e_1') = e_0 x_1$, $\varphi_0(e_1'') = e_0 Y_2$, $\varphi_0(e_1''') = e_0 x_2$;

(iii) $\varphi_1(e_2) = e_1 x_1 - e_1'(Y_1 + f) - e_1''' g$, $\varphi_1(e_2') = -e_1' f + e_1'' x_2 - e_1'''(Y_2 + g)$, $\varphi_1(e_2'') = e_1 x_2 - e_1''' Y_1$, $\varphi_1(e_2''') = -e_1' Y_2 + e_1'' x_1$;

(iv) $\varphi_2(e_3) = e_2 x_2(Y_2 + g + h_{x_2}) + e_2' x_1(Y_1 + f + h_{x_1}) - e_2'' x_1(Y_2 + g + h_{x_2}) - e_2''' x_2(Y_1 + f + h_{x_1})$.

It is straightforward to show that this complex is a resolution of K by right free A-modules. The stated result follows from this observation. Q.E.D.

References

[1] J.E. Björk: Rings of differential operators, North-Holland, Amsterdam, Oxford, New York, 1979.

[2] N. Jacobson: Lie algebras, Dover, New York, 1979.

Chang Woo Hang
Department of Mathematics
Dong-A University
Pusan, Korea

Masayoshi Miyanishi and De-Qi Zhang
Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560

Kenji Nishida
Department of Mathematics
College of General Education
Nagasaki University
Nagasaki 852