Funaki, T.
Osaka J. Math.
28 (1991), 495–516

REGULARITY PROPERTIES FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

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(Received June 2, 1990)

1. Introduction

We consider the following semilinear stochastic partial differential equation (SPDE) of parabolic type:

$$(1.1)\qquad dS_t(x) = -\mathcal{A}S_t(x)dt+\sum_{j=1}^{J}\mathcal{B}_j\{B_j(x, S_t)\}dt+\sum_{j=1}^{J}C_j dw_t^j(x)\,, \qquad x\in\boldsymbol{R}^d,\quad t>0\,.$$

Here $\mathcal{A}=\sum_{|\alpha|\le 2m}a_\alpha D^\alpha$, $\mathcal{B}_j=\sum_{|\alpha|\le n}b_{j,\alpha}D^\alpha$ and $C_j=\sum_{|\alpha|\le l}c_{j,\alpha}D^\alpha$, $m\ge 1, n, l\ge 0$, are differential operators with coefficients a_α, $b_{j,\alpha}$, $c_{j,\alpha}\in C_b^\infty(\boldsymbol{R}^d)$, $1\le j\le J$, and $\{B_j(x, S)\}_{j=1}^J$ are certain functions of x and $S=\{S(x); x\in\boldsymbol{R}^d\}$. We denote $D^\alpha \equiv D_x^\alpha=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$ and $|\alpha|=\sum_{i=1}^d \alpha_i$ for $\alpha=(\alpha_1\cdots, \alpha_d)\in\boldsymbol{Z}_+^d=\{0, 1, 2, \cdots\}^d$, while $C_b^\infty(\boldsymbol{R}^d)$ stands for the class of all C^∞-functions on $\boldsymbol{R}^d$ possessing bounded derivatives of all orders. The system $\{w_t^j(x)\}_{j=1}^J$ consists of J independent $\{\mathcal{F}_t\}$-cylindrical Brownian motions (c.B.m.'s) ([6], [7]) on the space $L^2(\boldsymbol{R}^d)$ which are defined on an appropriate probability space $(\Omega, \mathcal{F}, P)$ equipped with a reference family $\{\mathcal{F}_t\}$.

The general theory for the SPDE's has been developed by several authors based mainly on two different approaches, namely, the semigroup method (e.g. Dawson [4]) and the variational one (e.g. Pardoux [14], Krylov and Rozovskii [12]). It is actually possible to establish the existence and uniqueness of solutions to (1.1) by employing these former results; see Remark 2.2 below. However, in order to continue further investigation of the behavior of solutions, the meaning of solutions due to their theory happens not to be sufficiently strong. In other words, as a rule, they sometimes require too large space for solutions. The main purpose of this article is to fill this gap up by showing that the solutions live on *nice* spaces. This will be accomplished by studying the regularity properties, strong and weak differentiability, of solutions of (1.1).

Let us now introduce the state spaces for the solutions S_t of (1.1). A positive function $\chi\in C^\infty(\boldsymbol{R}^d)$ satisfying $\chi(x)=|x|$ for x; $|x|\ge 1$ and $\chi(-x)=\chi(x)$

will be fixed. We set $\boldsymbol{L}_r^2=L^2(\boldsymbol{R}^d, e^{-2r\chi(x)}dx)$, $r\in\boldsymbol{R}$, the Hilbert spaces having norms defined by $|S|_r=\{\int_{\boldsymbol{R}^d}S(x)^2e^{-2r\chi(x)}dx\}^{1/2}$, $S\in\boldsymbol{L}_r^2$, and $\boldsymbol{L}_e^2=\bigcap_{r>0}\boldsymbol{L}_r^2$, a countably Hilbertian space. Let $\boldsymbol{B}_r$, $r\in\boldsymbol{R}$, be the space of all Borel measurable functions S on $\boldsymbol{R}^d$ satisfying $|||S|||_r=\text{esssup}_{x\in\boldsymbol{R}^d}|S(x)|e^{-r\chi(x)}<\infty$. Set $\boldsymbol{C}_r=\boldsymbol{B}_r\cap C(\boldsymbol{R}^d)$ and $\boldsymbol{C}_e=\bigcap_{r>0}\boldsymbol{C}_r$, a countably normed space. We also introduce Banach spaces $\hat{\boldsymbol{C}}_r$, $r\in\boldsymbol{R}$, consisting of all $S\in\boldsymbol{C}_r$ such that $\lim_{|x|\to\infty}|S(x)|e^{-r\chi(x)}=0$. The spaces with parameter $r>0$ will play the role of the state spaces for the SPDE (1.1).

Let $L_b(\boldsymbol{L}_r^2)$ be the class of all bounded and Lipschitz continuous functions $B(\cdot, S)$ of $\boldsymbol{L}_r^2\to\boldsymbol{L}_r^2$, i.e.

$$|B(\cdot, S)|_{r,(\infty)}=\sup_{S\in\boldsymbol{L}_r}|B(\cdot, S)|_r<\infty$$

and

$$|B(\cdot, S)-B(\cdot, S')|_r\leq\text{const }|S-S'|_r\,,\quad S, S'\in\boldsymbol{L}_r^2\,.$$

The space $L_b(\boldsymbol{B}_r)$ of all bounded and Lipschitz continuous functions $B(\cdot, S)$ of $\boldsymbol{B}_r\to\boldsymbol{B}_r$ and the norm $|||B(\cdot, S)|||_{r,(\infty)}$ are defined similarly. We mention the assumptions imposed on the SPDE (1.1); $\bar{r}>0$ is arbitrary but fixed.

(A.1) The operator $\dfrac{\partial}{\partial t}+\mathcal{A}$ is uniformly parabolic in the sense of Petrovskiĭ, i.e.,

$$\inf_{x,\sigma\in\boldsymbol{R}^d;\,|\sigma|=1}\sum_{|\alpha|=2m}(-1)^m a_\alpha(x)\sigma^\alpha>0\,,$$

where $\sigma^\alpha=\sigma_1^{\alpha_1}\cdots\sigma_d^{\alpha_d}$ for $\sigma\in\boldsymbol{R}^d$ and $\alpha\in\boldsymbol{Z}_+^d$.

(A.2) $\quad 2m>2l+d\,.$

(A.3.1) $\quad B_j\in L_b(\boldsymbol{L}_{\bar{r}}^2),\quad 1\leq j\leq J\,.$

(A.3.2) $\quad B_j\in L_b(\boldsymbol{B}_{\bar{r}}),\quad 1\leq j\leq J\,.$

(A.4.1) $\quad 2m>n\,.$

(A.4.2) $\quad 2m>n+\dfrac{d}{2}\,.$

The following three cases (I), (I)′ and (II) will be considered: The conditions (A.1) and (A.2) are supposed in all cases; in addition, we assume (A.3.1) and (A.4.1) in the case (I), (A.3.1) and (A.4.2) in the case (I)′ and (A.3.2) and (A.4.1) in the case (II).

The contents of this paper is now summarized briefly. The strong regularity, i.e., the sample-path continuity as $\boldsymbol{L}_{\bar{r}}^2$ (or $\boldsymbol{C}_{\bar{r}}$)-valued processes or the differentiability and the Hölder continuity of derivatives in x in a.s.-sense, for solutions to the SPDE (1.1) is discussed in Sect. 2. The existence and uniqueness of solutions are also shown. In Sect. 3 we introduce and investigate the notion of the weak differentiability for the solutions to the SPDE (1.1). Sect. 4

has a slightly different character: The martingale problem associated with this equation is introduced and its well-posedness is established.

Important examples of the SPDE (1.1) are the time-dependent Ginzburg-Landau equation (TDGL eq.) of non-conservative tpye (1.2) and of conservative type (1.3):

$$dS_t(x) = -\mathcal{A}S_t(x)dt-\frac{1}{2}V'(x, S_t(x))dt+dw_t(x)\,, \tag{1.2}$$

$$dS_t(x) = \Delta\mathcal{A}S_t(x)dt+\frac{1}{2}\Delta\{V'(x, S_t(x))\}dt+\mathrm{div}\{d\boldsymbol{w}_t(x)\}\,, \quad x\in\boldsymbol{R}^d,\quad t>0\,, \tag{1.3}$$

where $\mathcal{A}$ is a differential operator of order $2m$ with $m>d/2$ satisfying (A.1), $\Delta=\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\boldsymbol{R}^d$, $V=V(x, s)$ is a function on $\boldsymbol{R}^d\times\boldsymbol{R}$ such that $V'(\cdot, S)=\frac{\partial V}{\partial s}(\cdot, S(\cdot))\in L_b(\boldsymbol{L}_{\bar r}^2)$, $\bar r>0$, and $w_t(x)$ and $\boldsymbol{w}_t(x)=\{w_t^j(x)\}_{j=1}^d$ are c.B.m.'s on $L^2(\boldsymbol{R}^d)$ and $L^2(\boldsymbol{R}^d, \boldsymbol{R}^d)$, respectively. Note that both equations satisfy (A.1), (A.2), (A.3.1) and (A.4.2). This paper is originally organized as a preparatory part of the exploration of the Ginzburg-Landau type equations (1.2) and (1.3). The results are applied in the collaborative papers [8], [9] that investigate the class of stationary measures (equilibrium states) of these equations and the hydrodynamic behavior of (1.3).

2. Strong differentiability of solutions

Under the assumption (A.1) the fundamental solution $q(t, x, y)$, $t>0, x, y\in \boldsymbol{R}^d$, of the parabolic operator $\frac{\partial}{\partial t}+\mathcal{A}$ exists and the following estimate holds:

$$\begin{gathered}|D_t^j D_x^\alpha D_y^\beta q(t, x, y)| \leq t^{-(|\alpha|+|\beta|)/2m-j}\bar q(t, x, y)\,, \\ 0<t\leq T,\ x, y\in\boldsymbol{R}^d, j\in\boldsymbol{Z}_+,\ \alpha, \beta\in\boldsymbol{Z}_+^d;\ D_t=\frac{\partial}{\partial t}\,,\end{gathered} \tag{2.1}$$

where

$$\bar q(t, x, y) \equiv \bar q_T(t, x, y) = K_1 t^{-d/2m} \exp\{-K_2(\frac{|x-y|^{2m}}{t})^{1/(2m-1)}\}\,.$$

The positive constants K_1 and K_2 depend on T, j, α and β but they can be taken uniformly in (j, α, β) such that $0\leq j, |\alpha|, |\beta|\leq c$ for arbitrary $c\in\boldsymbol{Z}_+$ (see Eidel'man [5]).

Let $\mathcal{S}(\boldsymbol{L}_r^2)$ (or $\mathcal{S}(\boldsymbol{B}_r)$), $r>0$, be the class of all $\{\mathcal{F}_t\}$-adapted stochastic processes $S_t=\{S_t(x; \omega);\ x\in\boldsymbol{R}^d\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ such that the mappings $(t, \omega)\in[0, \infty)\times\Omega\mapsto S_t(\cdot\,; \omega)\in\boldsymbol{L}_r^2$ (or $\boldsymbol{B}_r$, resp.) are measurable. For given $S_0 \in \boldsymbol{L}_r^2$ (or $\hat{\boldsymbol{C}}_r$), $r>0$, and $S. \in \mathcal{S}(\boldsymbol{L}_r^2)$ (or $\mathcal{S}(\boldsymbol{B}_r)$), we set

$$S_{t,1}(x) \equiv S_{t,1}(x; S_0) = \int_{\boldsymbol{R}^d} q(t, x, y) S_0(y) dy$$

$$S_{t,2}(x) = \sum_{j=1}^{J} \int_0^t \int_{\boldsymbol{R}^d} C_{j,y}^* q(t-u, x, y)\, dw_u^j(y) dy\,,$$

and

$$S_{t,3}(x) \equiv S_{t,3}(x; S.) = \sum_{j=1}^{J} \int_0^t \int_{\boldsymbol{R}^d} \mathcal{B}_{j,y}^* q(t-u, x, y) B_j(y, S_u)\, du dy$$

where $\mathcal{B}_j^*$ and C_j^* denote the formal adjoint of the differential operators $\mathcal{B}_j$ and C_j, respectively, and the subscript y to these operators means that they act on the variable y. We call $S_t \in \mathcal{S}(\boldsymbol{L}_r^2)$ (or $\mathcal{S}(\boldsymbol{B}_r)$) a solution of the SPDE (1.1) with initial data $S_0 \in \boldsymbol{L}_r^2$ (or $\hat{\boldsymbol{C}}_r$) if it satisfies the following equation which is formally obtained by rewriting (1.1) into integral form:

$$S_t(x) = S_{t,1}(x; S_0) + S_{t,2}(x) + S_{t,3}(x; S.), \quad \text{a.e.-}(t, x, \omega)\,. \tag{2.2}$$

The purpose of this section is to study the differentiability and the Hölder continuity of derivatives of the solutions $S_t(x)$ of (1.1) (in the a.s.-ω's sense). The existence and uniqueness problem is also discussed.

Let us begin with the investigation of the first term $S_{t,1}(x)$. We set $T_t S(x) = \int_{\boldsymbol{R}^d} q(t, x, y) S(y) dy (= S_{t,1}(x; S))$ and $\bar{T}_t S(x) = \int_{\boldsymbol{R}^d} \bar{q}(t, x, y) S(y) dy$. Some properties of the operators $\{T_t\}_{t \geq 0}$ and $\{\bar{T}_t\}_{t \geq 0}$ are summarized in the next two lemmas. The following estimate which is shown easily will be useful:

$$\int_{\boldsymbol{R}^d} \bar{q}(t, x, y) e^{-r\chi(y)}\, dy \leq K e^{-r\chi(x)}, \quad 0 < t \leq T,\ x \in \boldsymbol{R}^d, \tag{2.3}$$

for every $r \in \boldsymbol{R}$ with some $K = K(r, T) > 0$.

Lemma 2.1. *(i) $\{\bar{T}_t\}$ has the following properties for every $r \in \boldsymbol{R}$ and $T>0$:*

$$\bar{T}_t\colon \boldsymbol{L}_r^2 \to \boldsymbol{L}_r^2,\ t \geq 0, \qquad \sup_{0 \leq t \leq T} \|\bar{T}_t\|_{\boldsymbol{L}_r^2 \to \boldsymbol{L}_r^2} < \infty \tag{2.4}$$

$$\bar{T}_t\colon \boldsymbol{L}_r^2 \to \boldsymbol{B}_r,\ t > 0, \qquad \sup_{0 < t \leq T} t^{d/4m} \|\bar{T}_t\|_{\boldsymbol{L}_r^2 \to \boldsymbol{B}_r} < \infty \tag{2.5}$$

$$\bar{T}_t\colon \boldsymbol{B}_r \to \boldsymbol{B}_r,\ t \geq 0, \qquad \sup_{0 \leq t \leq T} \|\bar{T}_t\|_{\boldsymbol{B}_r \to \boldsymbol{B}_r} < \infty \tag{2.6}$$

where $\|T\|_{\boldsymbol{E} \to \boldsymbol{E}'}$ denotes the operator norm of $T\colon \boldsymbol{E} \to \boldsymbol{E}'$ for two normed spaces $\boldsymbol{E}$ and $\boldsymbol{E}'$.

(ii) $\{T_t\}$ has the same properties (2.4)–(2.6) as $\{\bar{T}_t\}$. It satisfies the followings for every $r \in \boldsymbol{R}$ as well

$$T_t\colon \boldsymbol{L}_r^2 \to C^\infty(\boldsymbol{R}^d),\ t > 0, \tag{2.7}$$

(2.8) $$T_tS(x)\in C^\infty((0,\infty)\times\boldsymbol{R}^d)\ \textit{for}\ S\in\boldsymbol{L}_r^2$$

(2.9) $$T_t\colon \hat{\boldsymbol{C}}_r\to\hat{\boldsymbol{C}}_r,\ t\geq 0$$

Proof. The property (2.4) is shown from (2.3) since Schwarz's inequality implies

$$|\bar{T}_tS(x)|^2\leq\int_{\boldsymbol{R}^d}\bar{q}(t,x,y)S^2(y)dy\int_{\boldsymbol{R}^d}\bar{q}(t,x,y)\,dy\,,$$

while (2.5) is verified again by using Schwarz's inequality and (2.3) as follows:

$$\begin{aligned}|\bar{T}_tS(x)|&\leq|S|_r\{\int_{\boldsymbol{R}^d}\bar{q}^2(t,x,y)e^{2r\chi(y)}dy\}^{1/2}\\&\leq|S|_r\{K(-2r,T)e^{2r\chi(x)}\sup_{y\in\boldsymbol{R}^d}\bar{q}(t,x,y)\}^{1/2},\ 0<t\leq T\,.\end{aligned}$$

The property (2.6) is an easy consequence of (2.3). Since $|T_tS(x)|\leq(\bar{T}_t|S|)(x)$, the same statements (2.4)–(2.6) hold also for $\{T_t\}_{t\geq 0}$. The properties (2.7) and (2.8) of $\{T_t\}$ are shown without difficulty by using (2.1). Finally for the proof of (2.9), we notice that $T_t\colon C_0(\boldsymbol{R}^d)\to\hat{\boldsymbol{C}}_r$. Indeed, this follows from (2.6) and (2.7) since $C_0(\boldsymbol{R}^d)\subset\boldsymbol{B}_{\tilde{r}}$ for all $\tilde{r}\in\boldsymbol{R}$. Here $C_0(\boldsymbol{R}^d)$ denotes the space of all continuous functions on $\boldsymbol{R}^d$ having compact supports. Then, (2.9) is verified by using (2.6); note that $C_0(\boldsymbol{R}^d)$ is dense in the Banach space $\hat{\boldsymbol{C}}_r$. □

Lemma 2.2. *A family of the operators $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup on the spaces $\boldsymbol{L}_r^2$ and $\hat{\boldsymbol{C}}_r$ for every $r\in\boldsymbol{R}$.*

Proof. Since the semigroup property ($T_tT_s=T_{t+s}$, $t,s\geq 0$, and T_0=identity) is automatic, the proof is completed only by showing the strong-continuity:

(2.10) $$\lim_{t\downarrow 0}|T_tS-S|_r=0,\ S\in\boldsymbol{L}_r^2\quad\text{and}\quad\lim_{t\downarrow 0}|||T_tS-S|||_r=0,\ S\in\hat{\boldsymbol{C}}_r.$$

Since $C_0(\boldsymbol{R}^d)$ is dense in both $\boldsymbol{L}_r^2$ and $\hat{\boldsymbol{C}}_r$, it suffices to verify (2.10) only for $S\in C_0(\boldsymbol{R}^d)$; use (2.4) and (2.6) . For such S, however, it is known that $T_tS(x)\to S(x)$ as $t\downarrow 0$ uniformly in x on each compact set of $\boldsymbol{R}^d$ ([2, p241]). Hence (2.10) is proved by noticing that (2.6) gives a uniform decay estimate: $|T_tS(x)|\leq$const $|||S|||_{-r'}e^{-r'\chi(x)}$, $0<t<1$, $S\in C_0(\boldsymbol{R}^d)$, for arbitrary $r'>0$. □

The next task is to establish the Hölder property of the second term $S_{t,2}(x)$ appearing in the RHS of (2.2).

Lemma 2.3. *For every $T>0$, $|\alpha|<m-l-\dfrac{d}{2}$ and $0<\delta<2m-2l-d-2|\alpha|$,*

(2.11) $$\begin{aligned}E[|D^\alpha S_{t,2}(x)-D^\alpha S_{t',2}(x')|^2]\leq\text{const}\ \{&|t-t'|^{(2m-2l-d-2|\alpha|)/2m}\\&+|x-x'|^{(2m-2l-d-2|\alpha|-\delta)\wedge 2}\},\quad 0\leq t,t'\leq T,\ x,x'\in\boldsymbol{R}^d.\end{aligned}$$

Proof. First we consider the case of $t=t'$. In this case

$$\text{LHS of (2.11)} = \sum_{j=1}^{J} \int_0^t du \int_{R^d} \{I_j\}^2 \, dy ,$$

where

$$I_j \equiv I_j(u, x, x', y) = C^*_{j,y}\{D^\alpha_x q(u, x, y) - D^\alpha_x q(u, x', y)\} .$$

Since the estimate (2.1) verifies two kinds of bounds on I_j:

$$|I_j| = |\int_0^1 \nabla_x C^*_{j,y} D^\alpha_x q(u, x(\xi), y)\cdot(x'-x)\, d\xi|$$
$$\leq \text{const}\, |x'-x| u^{-(l+|\alpha|+1)/2m} \int_0^1 \bar{q}(u, x(\xi), y)\, d\xi , \quad x(\xi) = x+\xi(x'-x) ,$$

and

$$|I_j| \leq \text{const}\, u^{-(l+|\alpha|)/2m}\{\bar{q}(u, x, y)+\bar{q}(u, x', y)\} ,$$

we get

$$\text{LHS of (2.11)} \leq \text{const}\, |x'-x|^a \int_0^t u^{-(a+2l+2|\alpha|)/2m} \cdot II\, du ,$$

for arbitrary $a\in[0, 2]$, where

$$II \equiv II(u, x, x'; a)$$
$$= \int_{R^d} \{\int_0^1 \bar{q}(u, x(\xi), y)\, d\xi\}^a \{\bar{q}(u, x, y)+\bar{q}(u, x', y)\}^{2-a}\, dy .$$

However, it is easily shown from (2.1) that $II \leq \text{const}\, u^{-d/2m}$, $0<u\leq T$, $a\in[0, 2]$ and therefore we obtain the desired estimate (2.11) with $t=t'$ by choosing $a=(2m-2l-d-2|\alpha|-\delta)\wedge 2$. Now assume $x=x'$, $0\leq t\leq t'\leq T$ and set $\tau=t'-t$. Then

$$\text{LHS of (2.11)} = III+IV$$
$$= \sum_{j=1}^{J} \int_t^{t'} du \int_{R^d} \{D^\alpha_x C^*_{j,y} q(t'-u, x, y)\}^2\, dy$$
$$+ \sum_{j=1}^{J} \int_0^t du \int_{R^d} \{D^\alpha_x C^*_{j,y} q(t'-u, x, y) - D^\alpha_x C^*_{j,y} q(t-u, x, y)\}^2\, dy .$$

The first term III in this equality has a bound which follows from (2.1):

$$III \leq \text{const}\, \tau^{(2m-2l-d-2|\alpha|)/2m} .$$

The estimate (2.11) with $x=x'$ is a consequence of this bound and the following one on the second term IV:

$$\begin{aligned}|IV| &= |\sum_{j=1}^{J}\int_0^t du\int_{R^d}\{\int_u^{u+\tau}\frac{\partial}{\partial v}D_x^\alpha C_{j,y}^* q(v,x,y)\,dv\}^2\,dy| \\ &\leq \text{const}\int_0^t du\int_{R^d}dy\{\int_u^{u+\tau}v^{-(l+|\alpha|)/m-2}\bar{q}(v,x,y)\,dv\}\times\{\int_u^{u+\tau}\bar{q}(v,x,y)\,dv\} \\ &\leq \text{const}\,\tau\times\int_0^t du\int_u^{u+\tau}v^{-(2l+d+2|\alpha|)/2m-2}\,dv \\ &\leq \text{const}\,\tau^{(2m-2l-d-2|\alpha|)/2m},\end{aligned}$$

where we have used (2.1) and then Schwarz's inequality for deriving the second line. □

Let $C^\lambda(\boldsymbol{R}^d)$, for non-integer $\lambda>0$, be the class of all $f\in C^{[\lambda]}(\boldsymbol{R}^d)$ having locally Hölder continuous derivatives $\{D^\alpha f;\ |\alpha|=[\lambda]\}$ of order $\lambda-[\lambda]$, where $[\lambda]$ is an integral part of λ.

Corollary 2.1. (*i*) *The process* $S_{t,2}\in C([0,\infty),\boldsymbol{C}_e)$ (a.s.) *and consequently* $S_{t,2}\in C([0,\infty),\boldsymbol{L}_e^2)$ (a.s.).
(*ii*) *For every* $t\geq 0$, $S_{t,2}\in\bigcap_{\delta>0}C^{m-l-d/2-\delta}(\boldsymbol{R}^d)$ (a.s.).

Proof. Since $\{D^\alpha S_{t,2}(x);\ t\geq 0,\ x\in\boldsymbol{R}^d\}$ is a Gaussian system, the $2p$-th moment $E[|D^\alpha S_{t,2}(x)-D^\alpha S_{t',2}(x')|^{2p}]$ is bounded by $\text{const}\{E[|D^\alpha S_{t,2}(x)-D^\alpha S_{t',2}(x')|^2]\}^p$, $p\geq 1$. Therefore, using Lemma 2.3, Kolmogorov-Totoki's regularization theorem (see [15] for example) verifies the conclusion. □

Let us give estimates on the third term $S_{t,3}$. We shall sometimes denote $S_{t,3}(x;S.)$ by $S_{t,3}(x;S.,B)$ in order to elucidate its dependence on the function $B=\{B_j\}_{j=1}^J$. Set $|B|_{r,(\infty)}=\sum_{j=1}^J|B_j|_{r,(\infty)}$ and $|||B|||_{r,(\infty)}=\sum_{j=1}^J|||B_j|||_{r,(\infty)}$. Recall the three cases (I), (I') and (II) introduced in Sect. 1 having differences in the assumptions.

Lemma 2.4. *For every* $T>0$ *and* $|\alpha|<N$,

$$\|D^\alpha S_{t,3}-D^\alpha S_{t',3}\|_1\leq\text{const}\,\|B\|_2|t-t'|^{(N-|\alpha|)/2m},\quad 0\leq t,t'\leq T, \tag{2.12}$$

where the triplet $(\|\cdot\|_1,\|\cdot\|_2,N)$ *is given by* $(|\cdot|_{\bar r},|\cdot|_{\bar r,(\infty)},2m-n)$ *in the case* (I), $(|||\cdot|||_{\bar r},|\cdot|_{\bar r,(\infty)},2m-n-\frac{d}{2})$ *in the case* $(I)'$ *and* $(|||\cdot|||_{\bar r},|||\cdot|||_{\bar r,(\infty)}\ 2m-n)$ *in the case* (II). *This especially implies that* $S_{t,3}\in C([0,\infty),\boldsymbol{E})$ *with* $\boldsymbol{E}=\boldsymbol{L}_{\bar r}^2$ *in the case* (I) *and* $\boldsymbol{E}=\boldsymbol{C}_{\bar r}$ *in the cases* $(I)'$ *and* (II), *respectively.*

Proof. Assume $0\leq t\leq t'\leq T$. Then, using (2.1), we have

$$\begin{aligned}&|D^\alpha S_{t,3}(x)-D^\alpha S_{t',3}(x)| \\ &\quad\leq\text{const}\sum_{j=1}^J\int_t^{t'}du(t'-u)^{-(n+|\alpha|)/2m}\{\bar T_{t'-u}|B_j(\cdot,S_u)|\}(x) \\ &\qquad+\text{const}\sum_{j=1}^J\int_0^t du\int_{t-u}^{t'-u}dv\,v^{-(2m+n+|\alpha|)/2m}\{\bar T_v|B_j(\cdot,S_u)|\}(x).\end{aligned}$$

The estimate (2.12) in each case (I), (I)′ and (II) follows without difficulty from (2.4), (2.5) and (2.6), respectively. □

REMARK 2.1. (i) Let $H_r^s(\boldsymbol{R}^d)$, $s, r\in\boldsymbol{R}$, be the Hilbert space consisting of all generalized functions S on $\boldsymbol{R}^d$ such that the products $e^{-r\chi}S$ belong to the Sobolev space $H^s(\boldsymbol{R}^d)$ of order s (see [13]). The norm of this space is naturally introduced by $\|S\|_{H_r^s(\boldsymbol{R}^d)}=\|e^{-r\chi}S\|_{H^s(\boldsymbol{R}^d)}$. This norm is equivalent to another one defined by $|S|_{r,s}=\{\sum_{|\alpha|\le s}|D^\alpha S|_r^2\}^{1/2}$ if $s\in\boldsymbol{Z}_+$. In fact, it is easy to see $\|S\|_{H_r^s(\boldsymbol{R}^d)}\le\text{const }|S|_{r,s}$ and therefore the equivalence of two norms follows from the open mapping principle.
(ii) Assume "$2m>n+1$". Then, the family of functions $\{S_{t,3}(\cdot\,;B);\ |B|_{r,(\infty)}\le M\}$, $M, r>0$, is relatively compact in the space $C([0,T],\boldsymbol{L}_{\tilde r}^2)$ if $\tilde r>r$. Indeed, Lemma 2.4-case (I) (by replacing $\tilde r$ with r) verifies the equicontinuity of this family in $C([0,T],\boldsymbol{L}_r^2)$ and therefore in $C([0,T],\boldsymbol{L}_{\tilde r}^2)$. On the other hand, Lemma 2.4-case (I) with $t'=0$ and $|\alpha|=0,1$ proves that $\sup\{|S_{t,3}(\cdot\,;B)|_{r,1};\ 0\le t\le T, |B|_{r,(\infty)}\le M\}<\infty$. Therefore the conclusion follows from Ascoli-Arzelà's theorem, since the imbedding map of $H_r^s(\boldsymbol{R}^d)\to\boldsymbol{L}_{\tilde r}^2$ is compact if $s>0$ and $\tilde r>r$ (use Rellich's theorem [13, p99]). This remark has been useful in [8].

The Hölder continuity of $S_{t,3}(x)$ (especially in the variable x) is given by the following lemma in the cases (I)′ and (II).

Lemma 2.5. *Let $T>0$ and $\boldsymbol{K}$ be a compact set of $\boldsymbol{R}^d$. Then the following estimate holds for every $0\le t, t'\le T$, $x, x'\in\boldsymbol{K}$, $|\alpha|<N$ and $0<\delta<N-|\alpha|$:*

$$|D^\alpha S_{t,3}(x)-D^\alpha S_{t',3}(x')|\\ \le\text{const }\|B\|_2\{|t-t'|^{(N-|\alpha|)/2m}+|x-x'|^{(N-|\alpha|-\delta)\wedge 1}\},$$

where $(\|\cdot\|_2, N)$ is given by $(|\cdot|_{\tilde r,(\infty)}, 2m-n-\frac{d}{2})$ in the case (I)′ and $(|||\cdot|||_{\tilde r,(\infty)}, 2m-n)$ in the case (II), respectively.

Proof. First assume $t=t'$. Then we have

$$|D^\alpha S_{t,3}(x)-D^\alpha S_{t,3}(x')|=|\sum_{j=1}^J\int_0^t I_j\,du|,$$

where

$$I_j\equiv I_j(u,t,x,x')=\int_{\boldsymbol{R}^d}\mathcal{B}_{j,y}^*\{D_x^\alpha q(t-u,x,y)-D_x^\alpha q(t-u,x',y)\}B_j(y,S_u)\,dy\,.$$

This term is bounded similarly to I_j appearing in the proof of Lemma 2.3 and we obtain for arbitrary $a\in[0,1]$:

$$|I_j|\le\text{const }(t-u)^{-(a+n+|\alpha|)/2m}|x'-x|^a\{\int_0^1(\bar T_{t-u}|B_j(\cdot,S_u)|)(x(\xi))\,d\xi\}^a\\ \times\{(\bar T_{t-u}|B_j(\cdot,S_u)|)(x)+(\bar T_{t-u}|B_j(\cdot,S_u)|)(x')\}^{1-a}.$$

However, by using (2.5) and (2.6), $(\bar{T}_{t-u}|B_j(\cdot, S_u)|)(x)$ is bounded by const$\times$ $|B_j|_{\bar{r},(\infty)}e^{\bar{r}\chi(x)}(t-u)^{-d/4m}$ in the case (I)′ and by const$|||B_j|||_{\bar{r},(\infty)}e^{\bar{r}\chi(x)}$ in the case (II), respectively. Therefore, taking $a=(2m-n-\frac{d}{2}-|\alpha|-\delta)\wedge 1$ in the case (I)′ and $a=(2m-n-|\alpha|-\delta)\wedge 1$ in the case (II), we obtain the conclusion when $t=t'$. If $x=x'$, the conclusion follows from Lemma 2.4-cases (I)′ and (II), respectively. □

The Lipschitz condition (in the assumptions (A.3.1) or (A.3.2)) of the functions $\{B_j\}_{j=1}^J$ is required only for the proof of the next lemma:

Lemma 2.6. *We consider two cases (I) and (II). For every* $p>\dfrac{2m}{2m-n}$ *and* $T>0$,

$$||S_{t,3}(\cdot\,;S.)-S_{t,3}(\cdot\,;\bar{S}.)||^p\leq \text{const}\int_0^t ||S_u-\bar{S}_u||^p\,du\,,$$
$$0\leq t\leq T,\ S.,\bar{S}.\in C([0,\infty),\boldsymbol{E})\,,$$

where $(||\cdot||,\boldsymbol{E})$ *is taken to be* $(|\cdot|_{\bar{r}},\boldsymbol{L}^2_{\bar{r}})$ *in the case (I) and* $(|||\cdot|||_{\bar{r}},\boldsymbol{C}_{\bar{r}})$ *in the case (II), respectively.*

Proof. Under the assumption (A.3.1) (or (A.3.2)), by using (2.4) (or (2.6), resp.), we obtain

$$||S_{t,3}(\cdot\,;S.)-S_{t,3}(\cdot\,;\bar{S}.)||\leq \text{const}\int_0^t (t-u)^{-n/2m}||S_u-\bar{S}_u||\,du,\ 0\leq t\leq T\,.$$

Therefore the concluding estimate follows by using Hölder's inequality. □

Let us summarize the result.

Theorem 2.1. *The solution of the SPDE (1.1) with initial data* $S_0\in\hat{\boldsymbol{E}}$ *exists uniquely and satisfies* $S_t\in C([0,\infty),\boldsymbol{E})$ a.s., *where* $\hat{\boldsymbol{E}}=\boldsymbol{E}=\boldsymbol{L}^2_{\bar{r}}$ *in the cases (I), (I)′ or* $\hat{\boldsymbol{E}}=\hat{\boldsymbol{C}}_{\bar{r}}$, $\boldsymbol{E}=\boldsymbol{C}_{\bar{r}}$ *in the cases (I)′, (II). Moreover, we have* $S_t\in C((0,\infty),\boldsymbol{C}_{\tilde{r}})$ a.s. *for every* $\tilde{r}>\bar{r}$ *even if* $S_0\in\boldsymbol{L}^2_{\bar{r}}$ *in the case (I)′. For* $t>0$, $S_t\in\bigcap_{\delta>0}C^{\tilde{N}-\delta}(\boldsymbol{R}^d)$ a.s., $\tilde{N}=(m-l-\frac{d}{2})\wedge N$, *where* $N=2m-n-\frac{d}{2}$ *in the case (I)′ and* $N=2m-n$ *in the case (II).*

Proof. Lemmas 2.2, 2.4 and Corollary 2.1 show that every solution S_t of the SPDE (1.1), if exists, has the property $S_t\in C([0,\infty),\boldsymbol{E})$, or more precisely saying, has such modification. Therefore the uniqueness of solutions may be discussed in this class. However, this is an immediate consequence of Lemma 2.6. The usual method of successive approximation can be used for the construction of solutions. The other properties of S_t follow from Lemmas 2.1, 2.2, 2.4, 2.5 and Corollary 2.1. □

REMARK 2.2. Suppose the conditions (A.2), (A.3.1) and "$m \geq n$" in place of (A.4.1). In addition, instead of (A.1), we assume that $\mathcal{A}$ satisfies the Gårding type inequality

$$(2.13) \qquad {}_V\langle S, \mathcal{A}S\rangle_{V^*} \geq c_1 |S|_V^2 - c_2 |S|_H^2, \quad c_1, c_2 > 0.$$

Here $(\boldsymbol{V}, \boldsymbol{H}, \boldsymbol{V}^*) = (H_{\bar{r}}^0(\boldsymbol{R}^d), H_{\bar{r}}^{-m}(\boldsymbol{R}^d), H_{\bar{r}}^{-2m}(\boldsymbol{R}^d))$ is a Gelfand triple; recall the definition of the spaces $H_{\bar{r}}^s(\boldsymbol{R}^d)$ given in Remark 2.1-(i) and notice that $\boldsymbol{V} = \boldsymbol{L}_{\bar{r}}^2$. Then, the result of Krylov and Rozovskii [12] based on this triple verifies that the solution of (1.1) exists uniquely and satisfies $S_t \in \boldsymbol{V}$(a.e.-(t, ω)) and $\in C([0, \infty), \boldsymbol{H})$ (a.s.-ω) if $E[|S_0|_H^2] < \infty$. In fact, the conditions listed in [12, p1252], especially the monotonicity and coercivity conditions, for the pair $(A(S), B(S)) = (-\mathcal{A}S + \sum_{j=1}^J \mathcal{B}_j\{B_j(\cdot, S)\}, \{C_j\}_{j=1}^J)$ can be checked from (2.13) and the following two facts: (1) The imbedding map of $H^{-l}(\boldsymbol{R}^d) \to \boldsymbol{H} = H_{\bar{r}}^{-m}(\boldsymbol{R}^d)$ is Hilbert-Schmidt (HS) operator if $m > l + \frac{d}{2}$ since $\bar{r} > 0$ (modify the proof in [1, p176]) and therefore (A.2) implies that C_j is HS operator of $L^2(\boldsymbol{R}^d) \to \boldsymbol{H}$. (2) The conditions "$m \geq n$" and (A.3.1) show that

$$|{}_V\langle S_1 - S_2, \mathcal{B}_j\{B_j(\cdot, S_1) - B_j(\cdot, S_2)\}\rangle_{V^*}| \leq \varepsilon |S_1 - S_2|_V^2 + K|S_1 - S_2|_H^2$$

for arbitrary $\varepsilon > 0$ with some $K = K_\varepsilon > 0$. The final remark is that $\mathcal{A} = P(-\Delta)$ satisfies (2.13) if $P(\lambda) = \sum_{k=0}^m c_k \lambda^k$ is a polynomial such that $c_m > 0$ and $P(\lambda) > 0$ for $\lambda \geq 0$, although the theory of [14], [12] is powerful in the case when $\mathcal{A}$ is nonlinear.

3. Weak differentiability of solutions

We introduce the notion of weak differentiability for random fields (r.f.'s) on $\boldsymbol{R}^d$ and prove the solution $\{S_t(x); x \in \boldsymbol{R}^d\}$ of the SPDE (1.1) is actually differentiable in this sense for $t > 0$. The result of this section has been applied in [8] in order to characterize the class of reversible measures of the TDGL eq. More precisely saying, it has become necessary in [8] to construct a new r.f. Y from a given r.f. X in such a way that Y is distributed according to the Gibbs rule inside a bounded region G and coincides with X outside G (or has the same boundary data as X on ∂G). The weak differentiability plays a role to determine a sufficient number of boundary data of X.

Let $\mathcal{V}$ be the class of all bounded open sets in $\boldsymbol{R}^d$ having C^∞-boundaries. For a real valued r.f. $X = \{X(x), x \in \boldsymbol{R}^d\}$ and $\Gamma = \partial G$ with $G \in \mathcal{V}$, we set

$$F_X(h, \psi) \equiv F_X(h, \psi; \Gamma) = \int_\Gamma \psi(\underline{x})\, X(\underline{x} + h \cdot n(\underline{x}))\, d\sigma(\underline{x}),$$

for every $|h| < h_0$, $h_0 > 0$, and $\psi \in L^2(\Gamma) \equiv L^2(\Gamma, d\sigma)$, where $n(\underline{x}) = n_\Gamma(\underline{x})$ is the inner normal unit vector at $\underline{x}\in\Gamma$ and $d\sigma=d\sigma_\Gamma$ is the volume element on Γ. The Sobolev spaces of order $s\in\boldsymbol{R}$ on Γ and G are denoted as usual by $H^s(\Gamma)$ [13, p35] and $H^s(G)$ [13, p40, p70], respectively.

DEFINITION 3.1. (i) The r.f. X is called weakly C^p, $p\in\boldsymbol{Z}_+$, at Γ if there exists $h_0>0$ such that $F_X(\cdot, \psi; \Gamma)\in C^p((-h_0, h_0))$ a.s. for every $\psi\in L^2(\Gamma)$.
(ii) We say that the r.f. X satisfies the regularity condition $(RC)^s_G$, $s>0$, on G if

$$E[\langle X, \psi\rangle_G^2]\leq \text{const}\, \|\psi\|^2_{H^{-s}(G)},\quad \psi\in L^2(G)\,,$$

where $\langle X, \psi\rangle_G=\int_G X(x)\,\psi(x)\,dx$.
(iii) We say that a family of $p+1$ generalized random fields (g.r.f.'s) $Y=\{Y_i(\psi), \psi\in L^2(\Gamma)\}_{i=0}^p$ on Γ satisfies the regularity condition $(RC)^s_\Gamma$, $s>0$, on Γ if

$$E[Y_i(\psi)^2]\leq \text{const}\, \|\psi\|^2_{H^{-s+i+1/2}(\Gamma)},\quad \psi\in L^2(\Gamma),\ 0\leq i\leq p\,.$$

REMARK 3.1. We say that X is weakly C^p at Γ from inside G (or outside G) if $F_X(\cdot,\psi; \Gamma)\in C^p([0, h_0))$ (or $C^p((-h_0, 0])$, resp.) a.s. for every $\psi\in L^2(\Gamma)$.

For $G\in\mathcal{V}$ and a C^∞-diffeomorphism $f(x)=\{f_i(x)\}_{i=1}^d$ of $\boldsymbol{R}^d\to\boldsymbol{R}^d$, we set $G_f=\{f(x)\in\boldsymbol{R}^d;\ x\in G\}$, $\Gamma_f=\partial G_f$ and

$$\delta_\Gamma(f)=\sup_{\underline{x}\in\Gamma}\{\|\boldsymbol{I}-\boldsymbol{J}_f(\underline{x})\|+|\underline{x}-f(\underline{x})|\}$$

where $\boldsymbol{I}=(\delta_{ij})_{ij}$ is a unit matrix, $\boldsymbol{J}_f(x)=\left(\dfrac{\partial f_i}{\partial x_j}\right)_{ij}$ is a Jacobian matrix of f and $\|\cdot\|$ denotes the norm of $d\times d$ matrices.

DEFINITION 3.2. A family of r.v.'s $\{Y(\psi; \Gamma);\ \psi\in L^2(\Gamma),\ \Gamma=\partial G$ with $G\in\mathcal{V}\}$ indexed by ψ and Γ is called mean-square continuous in Γ if it satisfies

$$\lim_{\delta\downarrow 0}\sup_f E[|Y(\psi; \Gamma)-Y(\psi_f; \Gamma_f)|^2]=0\,,$$

for every $\Gamma=\partial G$, $G\in\mathcal{V}$, and $\psi\in L^2(\Gamma)$, where the supremum is taken over all C^∞-diffeomorphisms f satisfying $\delta_\Gamma(f)\leq\delta$. The function $\psi_f\in L^2(\Gamma_f, d\sigma_{\Gamma_f})$ is defined by $\psi_f(\underline{x})=\psi(f^{-1}(\underline{x}))$, $\underline{x}\in\Gamma_f$.

In this section we consider two cases (I) and (II). Let S_t be the solution of the SPDE (1.1) with initial distribution $\mu\in\mathcal{P}(\boldsymbol{L}^2_r)$ (or $\in\mathcal{P}(\hat{\boldsymbol{C}}_{\bar{r}})$) satisfying $E^\mu[|S|_r^2]<\infty$ (or $E^\mu[|||S|||_r^2]<\infty$) in the case (I) (or the case (II), resp.), where $\mathcal{P}(\boldsymbol{E})$ stands for the family of all Borel probability measures on $\boldsymbol{E}$. We shall prove the following two theorems.

Theorem 3.1. *(i) For every $t>0$ and $G\in\mathcal{V}$, S_t is weakly C^p, $p=(m-l-1)\wedge(2m-n-1)$, at $\Gamma=\partial G$ and especially $Y_{i,t}(\psi)\equiv Y_{i,t}(\psi;\Gamma)=$*

$\frac{d^i}{dh^i} F_{S_t}(h, \psi; \Gamma)|_{h=0}$, $0 \leq i \leq p$, *exists* a.s.

(ii) $Y_t = \{Y_{i,t}(\psi)\}_{i=0}^p$ *satisfies* $(RC)_\Gamma^s$ *for every* s: $0 < s < (m-l) \wedge (2m - n + \frac{\kappa}{2})$, *where* $\kappa = 0$ *in the case (I) and* $\kappa = 1$ *in the case (II).*

(iii) $Y_{i,t}(\psi; \Gamma)$ *is mean-square continuous in* Γ *for every* $t > 0$ *and* $0 \leq i \leq p$.

Theorem 3.2. *S_t, $t>0$, satisfies $(RC)_G^s$ for every s: $0<s<(m-l)\wedge(2m-n)$.*

As the first step to the proof of Theorem 3.1 we briefly mention how the SPDE (1.1) changes its form under the coordinate transform in the variable x. Let $x' = g(x)$ be an orientation-preserving C^∞-diffeomorphism of $\boldsymbol{R}^d \to \boldsymbol{R}^d$ satisfying $g(x) = x$ for $|x| \geq M$ with some $M > 0$; the class of such g's will be denoted by $\mathbf{Diff}_0$. We set $J(x)$ and $J'(x')$ the Jacobians of the map's g and g^{-1}, respectively, so that $dx' = J(x)dx$, $dx = J'(x')dx'$ and $J(x) \cdot J'(x') = 1$ if $x' = g(x)$. We say $\varphi \sim \varphi'$ for $\varphi = \varphi(x)$ and $\varphi' = \varphi'(x') \in C_0^\infty(\boldsymbol{R}^d)$ if $\varphi'(x') = J'(x')\varphi(g^{-1}(x'))$, $x' \in \boldsymbol{R}^d$, or equivalently if $\varphi(x) = J(x)\varphi'(g(x))$, $x \in \boldsymbol{R}^d$. Suppose the operators $\mathcal{A}$, $\{\mathcal{B}_j\}_{j=1}^J$, $\{C_j\}_{j=1}^J$ and the functions $B = \{B_j(x, S)\}_{j=1}^J$ satisfying the assumptions in Sect. 1 are given. We define new operators $\mathcal{A}'$, $\{\mathcal{B}_j'\}_{j=1}^J$, $\{C_j'\}_{j=1}^J$ and functions $B' = \{B_j'(x', S')\}_{j=1}^J$ by $\mathcal{A}'u(x') = \mathcal{A}\{u(g(x))\}|_{x=g^{-1}(x')}$, $\mathcal{B}_j'u(x') = \mathcal{B}_j\{u(g(x))\}|_{x=g^{-1}(x')}$, $C_j'u(x') = C_j\{\sqrt{J(x)}u(g(x))\}|_{x=g^{-1}(x')}$, for $u = u(x') \in C_0^\infty(\boldsymbol{R}^d)$ and $B_j'(x', S') = B_j(g^{-1}(x'), S'(g(\cdot)))$, respectively. Let $\{w_t'^j\}_{j=1}^J$ be a family of processes taking values in the space of generalized functions on $\boldsymbol{R}^d$ defined from the system of independent $\{\mathcal{F}_t\}$-c.B.m.'s $\{w_t^j\}_{j=1}^J$ on $L^2(\boldsymbol{R}^d)$ by the relation $\langle w_t'^j, \varphi' \rangle = \langle w_t^j, \varphi/\sqrt{J} \rangle$, $\varphi \sim \varphi'$. Then it is not difficult to prove the following assertions (1)–(5):

(1) Similarly to $\mathcal{A}$, $\{\mathcal{B}_j\}_{j=1}^J$ and $\{C_j\}_{j=1}^J$, the operators $\mathcal{A}'$, $\{\mathcal{B}_j'\}_{j=1}^J$ and $\{C_j'\}_{j=1}^J$ are differential operators of order $2m$, n and l, respectively, with coefficients belonging to the class $C_b^\infty(\boldsymbol{R}^d)$.

(2) The operator $\mathcal{A}'$ satisfies the assumption (A.1).

(3) If the functions $B = \{B_j(x, S)\}_{j=1}^J$ satisfy (A.3.1) (or (A.3.2)), then $B' = \{B_j'(x', S')\}_{j=1}^J$ also satisfy the same assumption (A.3.1) (or (A.3.2), resp.).

(4) $\{w_t'^j\}_{j=1}^J$ is a system of independent $\{\mathcal{F}_t\}$-c.B.m.'s on $L^2(\boldsymbol{R}^d)$.

(5) Assume $S_t(x)$ is a solution of the SPDE (1.1) and define S_t' by $S_t'(x') = S_t(g^{-1}(x'))$, $x' \in \boldsymbol{R}^d$. Then S_t' is a solution of the SPDE (1.1) with $\mathcal{A}$, $\{\mathcal{B}_j\}_{j=1}^J$, $\{C_j\}_{j=1}^J$, $B = \{B_j(x, S)\}_{j=1}^J$ and $\{w_t^j\}_{j=1}^J$ replaced by $\mathcal{A}'$, $\{\mathcal{B}_j'\}_{j=1}^J$, $\{C_j'\}_{j=1}^J$, $B' = \{B_j'(x', S')\}_{j=1}^J$ and $\{w_t'^j\}_{j=1}^J$, respectively.

Now we turn to the proof of Theorem 3.1. For given $G \in \mathcal{V}$, we can find a covering $\{O_i \in \mathcal{V}\}_{i=1}^N$ of $\Gamma = \partial G$ and $\{g_i \in \mathbf{Diff}_0\}_{i=1}^N$ in such a way that

$$g_i(O_i \cap \Gamma) = \{x' = (\underline{x}', x_d') \in \boldsymbol{R}^d;\ |\underline{x}'| < 1,\ x_d' = 0\},$$
$$g_i(x) = (g_i(\underline{x}), h) \quad \text{if } x = \underline{x} + h \cdot n(\underline{x}),\ \underline{x} \in O_i \cap \Gamma,\ |h| < h_0,$$

for each i with some $h_0>0$. Furthermore there exists a partition of unity $\{\alpha_i\}_{i=1}^N$ on Γ such that $\alpha_i\in C^\infty(\Gamma)$, supp $\alpha_i\subset O_i\cap\Gamma$ and $\sum_{i=1}^N \alpha_i=1$ on Γ. Therefore it may be sufficient to verify the conclusion by assuming $\psi=0$ a.e. on $\Gamma\backslash O_i$ for some i. For such ψ, however, by changing the variable in the integral according as the map g_i, we obtain

$$F_{S_t}(h, \psi; \Gamma) = \int_{\partial \boldsymbol{R}_+^d} \psi(g_i^{-1}(\underline{x}'))\, S_t'((\underline{x}', h))\, j'(\underline{x}')\, d\underline{x}' , \quad |h|<h_0$$

with $j'(\underline{x}')\equiv j_\Gamma'(\underline{x}')=(d\sigma)\circ g_i^{-1}/d\underline{x}'\in C^\infty(\partial\boldsymbol{R}_+^d)$; i.e., $j'(\underline{x}')\,d\underline{x}'$ is the image measure of $d\sigma(\underline{x})$ by the map g_i. Here S_t' is the process defined by $S_t'(x')=S_t(g_i^{-1}(x'))$, $x'\in\boldsymbol{R}^d$, and satisfies an SPDE of the form (1.1) again as we have already noticed. This means that for completing the proof of Theorem 3.1 we can assume from the beginning

$$G = \boldsymbol{R}_+^d = \{x = (\underline{x}, x_d)\in\boldsymbol{R}^d,\ x_d>0\}$$
$$\Gamma = \partial\boldsymbol{R}_+^d = \{x\in\boldsymbol{R}^d,\ x_d = 0\}\cong\boldsymbol{R}^{d-1}$$

and

$$(3.1)\qquad \psi\in L^2(\partial\boldsymbol{R}_+^d, d\underline{x}) \quad\text{and}\quad \psi(\underline{x}) = 0 \quad\text{a.e. on}\quad \{\underline{x}\in\partial\boldsymbol{R}_+^d;\ |\underline{x}|>1\}\ .$$

We prepare a fundamental lemma which gives bounds on certain integral operators. Let $H_r^s(\Gamma)$, $s,r\in\boldsymbol{R}$, be the Hilbert space defined similarly to $H_r^s(\boldsymbol{R}^d)$; see Remark 2.1-(i), in which we replace χ with its restriction on Γ. We denote $\boldsymbol{L}_r^p=L^p(\boldsymbol{R}^d, e^{-rp\chi(x)}\,dx)$, $r\in\boldsymbol{R}$, $p\geq 1$, the Banach space having the norm $||S||_{\boldsymbol{L}_r^p}=\{\int_{\boldsymbol{R}^d}|S(x)|^p e^{-rp\chi(x)}\,dx\}^{1/p}$. Let $\mathcal{E}=\sum_{|\alpha|\leq k} e_\alpha D^\alpha$ be a differential operator of order k with coefficients $e_\alpha\in C_b^\infty(\boldsymbol{R}^d)$. We associate with $\mathcal{E}$ a linear operator $T=T_{i,t,h}$, $i\in\boldsymbol{Z}_+$, $t\in(0, T]$, $|h|<h_0$, acting on the class of functions ψ satisfying (3.1):

$$(3.2)\qquad T_{i,t,h}\,\psi(y) = \int_\Gamma \psi(\underline{x})\, D_h^i\, \mathcal{E}_y\, q(t, \underline{x}^h, y)\, d\underline{x}\,, \quad y\in\boldsymbol{R}^d\,,$$

where $D_h^i=\dfrac{d^i}{dh^i}$ and $\underline{x}^h=(\underline{x}, h)\in\Gamma\times\boldsymbol{R}$.

Lemma 3.1. *For $i\in\boldsymbol{Z}_+$, $t\in(0, T]$, $|h|<h_0$, $s\geq 0$ and $r, r'\in\boldsymbol{R}$ such that $r>r'$, we have*

$$(3.3)\qquad ||T_{i,t,h}||_{H_r^{-s}(\Gamma)\to\boldsymbol{L}_r^2}\leq \text{const}\ t^{-(2s+2k+2i+1)/4m}\ ,$$

$$(3.4)\qquad ||T_{i,t,h}||_{H_{r'}^{-s}(\Gamma)\to\boldsymbol{L}_r^1}\leq \text{const}\ t^{-(s+k+i)/2m}\ ,$$

where const *can be taken independently of t and $s\in[0, s_0]$ for every $s_0>0$.*

Proof. First we note (see [1, p50]) that, if $s\in\boldsymbol{Z}_+$,

$$||\tilde{\psi}||_{H^{-s}(\Gamma)}^2 = \inf\,\{\sum_{|\alpha|\leq s} ||\tilde{\psi}_\alpha||_{L^2(\Gamma)}^2\}\ ,$$

for $\tilde{\psi}\in H^{-s}(\Gamma)$, where the infimum is taken over all representations of $\tilde{\psi}$ such as

$$\tilde{\psi}=\sum_{|\alpha|\leq s}(-1)^{|\alpha|}D_{\underline{x}}^{\alpha}\tilde{\psi}_{\alpha},\ \tilde{\psi}_{\alpha}\in L^2(\Gamma)\,.$$

For $\psi=\tilde{\psi}e^{r\chi}\in H_r^{-s}(\Gamma)$ with $\tilde{\psi}\in H^{-s}(\Gamma)$ of this form, we have

$$\begin{aligned}\|T_{i,t,h}\psi\|_{L_r^2}^2&=\int_{R^d}e^{-2r\chi(y)}dy\Big[\sum_{|\alpha|\leq s}\int_\Gamma\tilde{\psi}_\alpha(\underline{x})D_{\underline{x}}^{\alpha}\{e^{r\chi(\underline{x})}D_h^i\mathcal{E}_y q(t,\underline{x}^h,y)\}d\underline{x}\Big]^2\\&\leq\text{const }t^{-(s+i+k)/m}\sum_{|\alpha|\leq s}\int_{R^d}e^{-2r\chi(y)}dy\Big\{\int_\Gamma|\tilde{\psi}_\alpha(\underline{x})|\bar{q}(t,\underline{x}^h,y)e^{r\chi(\underline{x})}d\underline{x}\Big\}^2\\&\leq\text{const }t^{-(s+i+k)/m}\sum_{|\alpha|\leq s}\int_{R^d}e^{-2r\chi(y)}dy\Big\{\int_\Gamma\tilde{\psi}_\alpha(\underline{x})^2\bar{q}(t,\underline{x}^h,y)e^{2r\chi(\underline{x})}d\underline{x}\Big\}\\&\qquad\times\Big\{\int_\Gamma\bar{q}(t,\underline{x}^h,y)d\underline{x}\Big\}\\&\leq\text{const }t^{-(2s+2i+2k+1)/2m}\sum_{|\alpha|\leq s}\|\tilde{\psi}_\alpha\|_{L^2(\Gamma)}^2\,,\end{aligned}$$

and this proves that the domain of $T_{i,t,h}$ can be extended to the space $H_r^{-s}(\Gamma)$ and (3.3) holds when $s\in\boldsymbol{Z}_+$. In this calculation the first inequality is derived from the estimate (2.1) and the third by using (2.3) and

$$\int_\Gamma\bar{q}(t,\underline{x}^h,y)d\underline{x}\leq\text{const }t^{-1/2m},\ y\in\boldsymbol{R}^d,\ 0<t\leq T\,.$$

The estimate (3.3) for general $s\geq0$ can be derived by using the interpolation technique. Indeed, apply the result of Calderòn [3, Paragraph 4] to the operator $T_{i,t,h}$ by noting that $H^{-s}(\Gamma)$ is a space of linear interpolation [13, p36 (or p32) and p92] between $H^{-n}(\Gamma)$ and $H^{-n-1}(\Gamma)$ when $n<s<n+1$, $n\in\boldsymbol{Z}_+$. The estimate (3.4) is shown similarly. □

We set $F_t^{(k)}(h,\psi)\equiv F_t^{(k)}(h,\psi;\Gamma)=F_{S_{t,k}}(h,\psi;\Gamma)$, $k=1,2,3$, for fixed $t>0$. The advantage to introduce the notion of weak differentiability consists in treating the stochastic term $S_{t,2}$ so that we expose calculations mainly on this term in the following.

Lemma 3.2. *(i)* $F_t^{(2)}(\cdot,\psi)\in\bigcap_{\delta>0}C^{m-l-1/2-\delta}((-h_0,h_0))$ (a.s.). *Therefore* $S_{t,2}$ *is weakly* C^{m-l-1} *at* $\Gamma=\partial\boldsymbol{R}_+^d$ *and* $Y_{i,t}^{(2)}(\psi)\equiv Y_{i,t}^{(2)}(\psi;\Gamma)=\dfrac{d^i}{dh^i}F_t^{(2)}(h,\psi)|_{h=0}$, $0\leq i\leq m-l-1$, *exists* a.s.
(ii) $Y_t^{(2)}=\{Y_{i,t}^{(2)}(\psi)\}_{i=0}^{m-l-1}$ *satisfies* $(RC)_\Gamma^s$ *for every* s: $0<s<m-l$.

Proof. Denoting the operator $T_{i,t,h}$ with $\mathcal{E}=C_j^*$ by $T_{i,j,t,h}^{(2)}$, $1\leq j\leq J$, we set

$$Y_{i,t}^{(2)}(h,\psi)=\sum_{j=1}^J\int_0^t\int_{R^d}T_{i,j,t-u,h}^{(2)}\psi(y)\,dw_u^j(y)\,dy\,,\quad|h|<h_0,\ 0\leq i\leq m-l-1\,.$$

Note that the RHS is well-defined since (3.3) with $s=r=0$ implies

$E[\{Y_{i,t}^{(2)}(h, \psi)\}^2] < \infty$ if $0 \leq i \leq m-l-1$. Moreover, for $-h_0 < h < h' < h_0$, we have

$$(3.5)\quad E[\{Y_{i,t}^{(2)}(h', \psi) - Y_{i,t}^{(2)}(h, \psi)\}^2] = \sum_{j=1}^{J} \int_0^t \|T_{i,j,u,h'}^{(2)} \psi - T_{i,j,u,h}^{(2)} \psi\|_{L^2(\mathbf{R}^d)}^2 \, du\,.$$

However, the estimate (3.3) with $s=r=0$ implies the following two bounds:

$$(3.6)\quad \|T_{i,j,u,h'}^{(2)} \psi - T_{i,j,u,h}^{(2)} \psi\|_{L^2(\mathbf{R}^d)}^2 \leq \text{const}\ (h'-h)^{2a}\, u^{-(2l+2i+2a+1)/2m}\, \|\psi\|_{L^2(\Gamma)}^2$$

for $a=0$ and 1. We have used that the LHS of (3.6) $= \|\int_h^{h'} T_{i+1,j,u,\hbar}^{(2)} \psi \, d\hbar\|_{L^2}^2$ for deriving (3.6) when $a=1$. It is then easy to see that (3.6) holds for every $a \in [0, 1]$. Especially when $i=m-l-1$, by choosing $0<a<1/2$, (3.5) and (3.6) prove $Y_{m-l-1,t}^{(2)}(\cdot, \psi) \in \bigcap_{\delta>0} C^{1/2-\delta}((-h_0, h_0))$ a.s.; use Kolmogorov's regularization theorem by noting that $\{Y_{m-l-1,t}^{(2)}(\cdot, \psi)\}$ forms a Gaussian system. The assertion (i) is therefore verified, since $\dfrac{d^i}{dh^i} F_t^{(2)}(h, \psi) = Y_{i,t}^{(2)}(h, \psi)$. On the other hand, the assertion (ii) is an immediate consequence of (3.3) with $r=0$. In fact, we have

$$\begin{aligned} E[\{Y_{i,t}^{(2)}(\psi)\}^2] &= \sum_{j=1}^{J} \int_0^t \|T_{i,j,u,0}^{(2)} \psi\|_{L^2(\mathbf{R}^d)}^2 \, du \\ &\leq \text{const}\ \Big\{\int_0^t u^{-(s+l)/m}\, du\Big\}\ \|\psi\|_{H^{-s+i+1/2}(\Gamma)}^2 \end{aligned}$$

and the integral in the RHS converges for s: $0<s<m-l$. □

REMARK 3.2. We explain an intuitive meaning of the regularity condition: If $Y_{i,t}^{(2)}$ is non-random, then $(RC)_\Gamma^s$, $s<m-l$, implies that $Y_{i,t}^{(2)}(\cdot) \in \bigcap_{s<m-l} H^{s-i-1/2}(\Gamma) \subset \bigcap_{\delta>0} H^{1/2-\delta}(\Gamma) \subset L^2(\Gamma)$ for $0 \leq i \leq m-l-1$. This means that $\{Y_{i,t}^{(2)}\}_{i=0}^{m-l-1}$ can be treated as if L^2-functions on Γ, at least in the stochastic sense. Compare this with the result of Corollary 2.1-(ii).

Lemma 3.3. *For every $t>0$ and $0 \leq i \leq m-l-1$, $\{Y_{i,t}^{(2)}(\psi; \Gamma); \psi, \Gamma\}$ is mean-square continuous in Γ.*

Proof. Let $\{O_i\}$ and $\{g_i\}$ be the same as before. We assume $\operatorname{supp} \psi \subset O_i$ and denote simply $g=g_i$. Then, for a C^∞-diffeomorphism f: $\mathbf{R}^d \to \mathbf{R}^d$, we have

$$(3.7)\qquad F_t^{(2)}(h, \psi_f; \Gamma_f) = \int_{\partial \mathbf{R}_+^d} \psi(g^{-1}(\underline{x}'))\, S_{t,2}'(z_f(\underline{x}', h))\, j_f'(\underline{x}')\, d\underline{x}'$$

where $S_{t,2}'(x') = S_{t,2}(g^{-1}(x'))$; $z_f(\underline{x}', h) = g(\underline{x} + h \cdot n_{\Gamma_f}(\underline{x})) \in \mathbf{R}^d$, $\underline{x} = f(g^{-1}(\underline{x}')) \in \Gamma_f$, for $\underline{x}' \in \partial \mathbf{R}_+^d$, $h \in \mathbf{R}$ and $j_f'(\underline{x}') = (d\sigma_{\Gamma_f}) \circ (g \circ f^{-1})^{-1} / d\underline{x}'$. As we have seen already, $S_{t,2}'$ has a similar form to $S_{t,2}$ so that we write simply $S_{t,2}$ instead of $S_{t,2}'$ in the following. From (3.7) we have

$$\begin{aligned} Y_{i,t}^{(2)}(\psi_f; \Gamma_f) &= D_h^i F_t^{(2)}(0, \psi_f; \Gamma_f) \\ &= \sum_{\alpha \in \mathbf{Z}_+^d;\ |\alpha|=i} \frac{i!}{\alpha_1! \cdots \alpha_d!} \int_{\partial \mathbf{R}_+^d} (D^\alpha S_{t,2})\, (z_f(\underline{x}', 0))\, \Psi_f(\underline{x}')\, d\underline{x}' \end{aligned}$$

where $\Psi_f(\underline{x}')=\psi(g^{-1}(\underline{x}'))\,\{D_h z_f(\underline{x}',0)\}^{\alpha} j'_f(\underline{x}')$; recall the definition of σ^{α} for $\sigma\in\boldsymbol{R}^d$ and $\alpha\in\boldsymbol{Z}^d_+$ given in (A.1). It is easily seen that $D_h z_f(\underline{x}',0)=\nabla g(\underline{x})\cdot n_{\Gamma_f}(\underline{x})\in\boldsymbol{R}^d$, $\underline{x}=f(g^{-1}(\underline{x}'))$. Therefore we obtain

$$|Y^{(2)}_{i,t}(\psi_f;\Gamma_f)-Y^{(2)}_{i,t}(\psi;\Gamma)|\le\text{const}\sum_{|\alpha|=i}\{I_\alpha+II_\alpha\}$$

with

$$I_\alpha=\left|\int_{\partial\boldsymbol{R}^d_+}(D^\alpha S_{t,2})(\underline{x}')\,\{\Psi_f(\underline{x}')-\Psi(\underline{x}')\}\,d\underline{x}'\right|$$

$$II_\alpha=\left|\int_{\partial\boldsymbol{R}^d_+}\{(D^\alpha S_{t,2})(z_f(\underline{x}',0))-(D^\alpha S_{t,2})(\underline{x}')\}\,\Psi_f(\underline{x}')\,d\underline{x}'\right|$$

where $\Psi(\underline{x}')=\psi(g^{-1}(\underline{x}'))\,\{n(\underline{x}')\}^{\alpha} j'(\underline{x}')$, $n(\underline{x}')=(0,\cdots,0,1)\in\boldsymbol{R}^d$ is the inner normal unit vector at $\underline{x}'\in\partial\boldsymbol{R}^d_+$ and $j'(\underline{x}')=(d\sigma_\Gamma)\circ g^{-1}/d\underline{x}'$. However,

$$E[I^2_\alpha]=\sum_{j=1}^J\int_0^t du\int_{\boldsymbol{R}^d}dy\left[\int_{\partial\boldsymbol{R}^d_+}D^\alpha_x C^*_{j,y}\,q(t-u,\underline{x}',y)\,\{\Psi_f(\underline{x}')-\Psi(\underline{x}')\}\,d\underline{x}'\right]^2$$
$$\le\text{const}\,\|\Psi_f-\Psi\|^2_{L^2(\partial\boldsymbol{R}^d_+)}\int_0^t u^{-(2i+2l+1)/2m}\,du\,.$$

In fact, the inequality in this formula can be shown similarly to (3.3) (replace D^i_h by D^α and take $s=r=0$). Now the integral in the RHS converges if $i\le m-l-1$ and $\|\Psi_f-\Psi\|_{L^2}\to0$ as $\delta_\Gamma(f)\to0$; note that $n(\underline{x}')=\nabla g(\underline{x})\cdot n_\Gamma(\underline{x})$, $\underline{x}=g^{-1}(\underline{x}')\in\Gamma$. On the other hand,

$$E[II^2_\alpha]=\sum_{j=1}^J\int_0^t du\int_{\boldsymbol{R}^d}dy\left[\int_{\partial\boldsymbol{R}^d_+}\{D^\alpha_x C^*_{j,y}\,q(t-u,z_f(\underline{x}',0),y)\right.$$
$$\left.-D^\alpha_x C^*_{j,y}\,q(t-u,\underline{x}',y)\}\,\Psi_f(\underline{x}')\,d\underline{x}'\right]^2$$
$$\le\text{const}\sup_{\underline{x}'\in\partial\boldsymbol{R}^d_+\cap\{\text{supp}\,\psi\circ g^{-1}\}}|z_f(\underline{x}',0)-\underline{x}'|^{2a}\|\Psi_f\|^2_{L^2(\partial\boldsymbol{R}^d_+)}\int_0^t u^{-(2i+2l+2a+1)/2m}\,du$$

for every $a\in[0,1]$. This estimate is shown similarly to (3.6). Notice that $\delta_\Gamma(f)\to0$ implies $|z_f(\underline{x}',0)-\underline{x}'|\to0$ (uniformly on bounded sets) and also the integral in the RHS converges if $i\le m-l-1$ and $0<a<1/2$. The proof is completed. □

Now we conclude the proof of Theorem 3.1: It is only remained to investigate the terms $S_{t,1}$ and $S_{t,3}$. However, since the calculations are similar to those for the term $S_{t,2}$, we mention briefly here. Assuming again that $\Gamma=\partial\boldsymbol{R}^d_+$ and $\psi\in L^2(\Gamma)$ satisfies (3.1), we have

$$D^i_h F^{(3)}_t(h,\psi)=\sum_{j=1}^J\int_0^t du\int_{\boldsymbol{R}^d}B_j(y,S_u)\,T^{(3)}_{i,j,t-u,h}\,\psi(y)\,dy$$

where the operator $T^{(3)}_{i,j,t,h}$ is defined by the formula (3.2) with $\mathcal{E}$ replaced by $\mathcal{B}^*_j$. In the case (I), using the estimate (3.3) with $r=-\bar{r}$, it is proved that

$F_t^{(3)}(\cdot,\psi)\in\bigcap_{\delta>0}C^{2m-n-1/2-\delta}((-h_0,h_0))$ and $Y_t^{(3)}=\{Y_{i,t}^{(3)}(\psi)\equiv\dfrac{d^i}{dh^i}F_t^{(3)}(0,\psi)\}_{i=0}^{2m-n-1}$ satisfies $(RC)_\Gamma^s$ for $0<s<2m-n$. In the case (II), we see from (3.4) with $r=-\bar{r}$ that $F_t^{(3)}(\cdot,\psi)\in\bigcap_{\delta>0}C^{2m-n-\delta}((-h_0,h_0))$ and $Y_t^{(3)}$ satisfies $(RC)_\Gamma^s$ for $0<s<2m-n+\dfrac{1}{2}$ (although Lemma 2.5 has already proved this Hölder property for $F_t^{(3)}(\cdot,\psi)$). On the other hand, it is easy to show that $S_{t,1}$, $t>0$, is weakly C^p for arbitrary $p\in\boldsymbol{Z}_+$ (actually $S_{t,1}(\cdot)\in C^\infty(\boldsymbol{R}^d)$, see Lemma 2.1) and $Y_t^{(1)}=\{Y_{i,t}^{(1)}(\psi)\equiv\dfrac{d^i}{dh^i}F_t^{(1)}(h,\psi)\,|_{h=0}\}_{i=0}^{p}$ satisfies $(RC)_\Gamma^s$ for arbitrary $s>0$ if the initial distribution μ satisfies $E^\mu[|S_0|_r^2]<\infty$ with some $r>0$; we use (3.3) by taking $\mathcal{E}=$identity. The mean-square continuity of $Y_{i,t}^{(k)}(\psi;\Gamma)$, $k=1, 3$, in Γ is shown similarly to $Y_{i,t}^{(2)}(\psi;\Gamma)$.

The proof of Theorem 3.2 is similar to that of Theorem 3.1-(ii) and rather simpler. Actually it is a consequence of the following lemma.

Lemma 3.4. *For each $t>0$, $S_{t,k}$, $k=1, 2, 3$, satisfy the regularity condition $(RC)_G^s$ for s such that $s>0$ if $k=1$, $0<s<m-l$ if $k=2$ and $0<s<2m-n$ if $k=3$.*

Proof. For $t\in(0, T]$ and $\psi\in L^2(G)$, we set

$$T_t\psi(y)=\int_G \mathcal{E}_y\, q(t,x,y)\,\psi(x)\,dx,\ y\in\boldsymbol{R}^d\,, \tag{3.8}$$

where $\mathcal{E}$ is a differential operator of order k as before. Then the following estimate can be shown:

$$\|T_t\|_{H^{-s}(G)\to L_r^2}\leq\text{const}\ t^{-(s+k)/2m},\ s\geq0,\ s\neq\text{integer}+\frac{1}{2},\ r\in\boldsymbol{R}\,. \tag{3.9}$$

Indeed this is derived in a similar manner to the verification of (3.3) by using the estimate (2.1) first for $s\in\boldsymbol{Z}_+$ and then by the interpolation technique for general $s\geq0$ (see [13, p71] especially for the condition $s\neq\text{integer}+\frac{1}{2}$). Now the conclusion for $S_{t,2}$ follows by noting an equality

$$E[\langle S_{t,2},\psi\rangle_G^2]=\sum_{j=1}^J\int_0^t\|T_{j,u}^{(2)}\psi\|_{L^2(\boldsymbol{R}^d)}^2\,du\,,$$

where $T_{j,t}^{(2)}$ is the operator T_t defined by (3.8) with $\mathcal{E}=\mathcal{C}_j^*$. We use (3.9) by taking $r=0$. The case of $k=3$ can be discussed similarly, since we have

$$\langle S_{t,3},\psi\rangle_G=\sum_{j=1}^J\int_0^t du\int_{\boldsymbol{R}^d}B_j(y,S_u)\,T_{j,t-u}^{(3)}\psi(y)\,dy\,,$$

where $T_{j,t}^{(3)}$ is an operator defined by (3.8) with $\mathcal{E}=\mathcal{B}_j^*$. The case of $k=1$ is also similar and easy. □

4. Formulation as the martingale problem

Here we consider only the case (I). The case (II) can be treated similarly. Let $\mathcal{D}$ be the class of all tame functions on $\boldsymbol{L}_r^2$, namely $\Psi\in\mathcal{D}$ iff it has the form:

$$\Psi(S)=\psi(\langle S,\varphi_1\rangle,\cdots,\langle S,\varphi_k\rangle)\,,\quad S\in\boldsymbol{L}_r^2\,, \tag{4.1}$$

with $k=1,2,\cdots$, $\psi=\psi(\alpha_1,\cdots,\alpha_k)\in C_b^2(\boldsymbol{R}^k)$ and $\varphi_1,\cdots,\varphi_k\in C_0^\infty(\boldsymbol{R}^d)$, where $\langle S,\varphi\rangle=\int_{\boldsymbol{R}^d}S(x)\,\varphi(x)\,dx$. With the operators $\mathcal{A}$, $\{\mathcal{B}_j\}_{j=1}^J$, $\{C_j\}_{j=1}^J$ and the functions $B=\{B_j(x,S)\}_{j=1}^J$ we associate an operator $\mathcal{L}$ defined on $\mathcal{D}$: For $\Psi\in\mathcal{D}$ having the form (4.1),

$$\begin{aligned}\mathcal{L}\Psi(S)=&\sum_{i=1}^k\frac{\partial\psi}{\partial\alpha_i}(\langle S,\varphi_1\rangle,\cdots,\langle S,\varphi_k\rangle)\\&\times\{-\langle S,\mathcal{A}^*\varphi_i\rangle+\sum_{j=1}^J\langle B_j(\cdot,S),\mathcal{B}_j^*\varphi_i\rangle\}\\&+\frac{1}{2}\sum_{i,i'=1}^k\frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_{i'}}(\langle S,\varphi_1\rangle,\cdots,\langle S,\varphi_k\rangle)\langle C\varphi_i,\varphi_{i'}\rangle,\quad S\in\boldsymbol{L}_r^2\,,\end{aligned} \tag{4.2}$$

where $C=\sum_{j=1}^J C_j C_j^*$. This operator may be written as

$$\begin{aligned}\mathcal{L}\Psi(S)=&\langle-\mathcal{A}S+\sum_{j=1}^J\mathcal{B}_j\{B_j(\cdot,S)\},D\Psi(\cdot,S)\rangle\\&+\frac{1}{2}\sum_{j=1}^J(\mathrm{Tr}\,C_j^*\otimes C_j^*\,D^2\Psi)(S)\,,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}D\Psi(x,S)&=\sum_{i=1}^k\frac{\partial\psi}{\partial\alpha_i}(\langle S,\varphi_1\rangle,\cdots,\langle S,\varphi_k\rangle)\,\varphi_i(x)\\D^2\Psi(x,y,S)&=\sum_{i,i'=1}^k\frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_{i'}}(\langle S,\varphi_1\rangle,\cdots,\langle S,\varphi_k\rangle)\,\varphi_i(x)\,\varphi_{i'}(y)\\(\mathrm{Tr}\,C_j^*\otimes C_j^*\,D^2\Psi)(S)&=\int_{\boldsymbol{R}^d}C_{j,x}^*\,C_{j,y}^*\,D^2\Psi(x,y,S)\,|_{x=y}\,dx\,.\end{aligned}$$

In the first term of the RHS of (4.3), $\langle\cdot,\cdot\rangle$ should be understood in the sense of generalized functions. We sometimes denote the operator $\mathcal{L}$ by $\mathcal{L}_B$ in order to indicate its dependence on the function B.

In this section we take $\Omega=C([0,\infty),\boldsymbol{L}_r^2)$ equipped with the uniform topology and denote the coordinate process by $S_t(\omega)=\omega_t$ for $\omega\in\Omega$. Set $\mathcal{F}_t=\sigma\{S_u;0\le u\le t\}$ and $\mathcal{F}=\vee_{t\ge0}\mathcal{F}_t$ as usual. We call a probability measure P on $(\Omega,\mathcal{F})$ a solution of $\mathcal{L}$-m.p. (martingale problem for $\mathcal{L}$) starting from $S\in\boldsymbol{L}_r^2$ if $P(S_0=S)=1$ and $\Psi(S_t)-\int_0^t\mathcal{L}\Psi(S_u)\,du$, $t\ge0$, is $(P,\{\mathcal{F}_t\})$-martingale for every $\Psi\in\mathcal{D}$.

Proposition 4.1. *The distribution of the solution* $S_t\in\Omega$ (a.s.) *of the SPDE* (1.1) *is a solution of the* $\mathcal{L}$*-m.p.*

This proposition is a consequence of the following lemma. Indeed we may only calculate the stochastic differential $d\Psi(S_t)$ by using the (finite-dimensional) Itô's formula.

Lemma 4.1. *The solution $S_t(x)$ of the SPDE* (1.1) *satisfies the same equation in the sense of generalized functions, i.e., for every $\varphi\in C_0^\infty(\boldsymbol{R}^d)$, we have*

$$\langle S_t, \varphi\rangle = \langle S_0, \varphi\rangle + \int_0^t \{\langle S_u, -\mathcal{A}^*\varphi\rangle + \sum_{j=1}^J \langle B_j(\cdot, S_u), \mathcal{B}_j^*\varphi\rangle\}\, du + \sum_{j=1}^J \langle w_t^j, C_j^*\varphi\rangle$$

Proof. First we note that similarly to the operator $-\mathcal{A}$ its adjoint $-\mathcal{A}^*$ also generates a strongly continuous semigroup $\{T_t^*\}_{t\geq 0}$ on the space $\boldsymbol{L}_r^2$ for every $r\in\boldsymbol{R}$ and it is actually given by $T_t^*\varphi(y)=\int_{\boldsymbol{R}^d} q(t, x, y)\,\varphi(x)\,dx$, $\varphi\in\boldsymbol{L}_r^2$. This is because the operator $\dfrac{\partial}{\partial t}+\mathcal{A}^*$ is uniformly parabolic in the sense of Petrovskiǐ as well and its fundamental solution q^* is given by the formula $q^*(t, x, y)=q(t, y, x)$ (see [5]). Therefore we have from (2.2)

$$\langle S_t, \varphi\rangle = \langle S_0, T_t^*\varphi\rangle + \sum_{j=1}^J \int_0^t \langle C_j^*\, T_{t-u}^*\,\varphi, dw_u^j\rangle + \sum_{j=1}^J \int_0^t \langle B_j(\cdot, S_u), \mathcal{B}_j^*\, T_{t-u}^*\,\varphi\rangle\, du\,.$$

The conclusion now follows easily by taking the stochastic differential of the both sides of this equality. □

The assertion of the converse direction of Proposition 4.1 is partially given by the next lemma.

Lemma 4.2. *Let P be the solution of $\mathcal{L}$-m.p. Define a process m_t taking values in the space of generalized functions on $\boldsymbol{R}^d$ by*

$$m_t = S_t - S_0 + \int_0^t \mathcal{A}S_u\, du - \sum_{j=1}^J \int_0^t \mathcal{B}_j\{B_j(\cdot, S_u)\}\, du\,. \tag{4.4}$$

Then, for every $\varphi\in C_0^\infty(\boldsymbol{R}^d)$, $m_t(\varphi)\equiv\langle m_t, \varphi\rangle$ and $m_t(\varphi)^2-t\langle \mathcal{C}\varphi, \varphi\rangle$ are $\{\mathcal{F}_t\}$-martingales on $(\Omega, \mathcal{F}, P)$. Especially the quadratic variation of $m_t(\varphi_1)$ and $m_t(\varphi_2)$, $\varphi_1, \varphi_2\in C_0^\infty(\boldsymbol{R}^d)$, is given by $\langle m_t(\varphi_1), m_t(\varphi_2)\rangle = t\langle \mathcal{C}\varphi_1, \varphi_2\rangle$.

Proof. By introducing a sequence of functions $\Psi_M(S)=\psi_M(\langle S, \varphi\rangle)\in\mathcal{D}$, $M\nearrow\infty$, with $\psi_M\in C_b^2(\boldsymbol{R})$ such that $\psi_M(\alpha)=\alpha$ if $|\alpha|\leq M$, we can prove $m_t(\varphi)$ and $m_t(\varphi)^2-t\langle \mathcal{C}\varphi, \varphi\rangle$ are $\{\mathcal{F}_t\}$-local martingales. However, this implies the conclusion (use the martingale characterization of Brownian motions). □

The next subject is to establish the well-posedness of the $\mathcal{L}$-m.p., i.e., the existence and uniqueness of solutions of $\mathcal{L}$-m.p. starting from every $S\in\boldsymbol{L}_r^2$.

If the converse assertion of Lemma 4.1 (i.e., the solution of (1.1) in the sense of generalized functions is also the solution of the stochastic integral equation (2.2), cf. [11]) is established, then the argument of the type of Yamada and Watanabe might work since the pathwise uniqueness of solutions to the SPDE (1.1) has been proved (Theorem 2.1). However, in our situation, a quite simple proof of the uniqueness for $\mathcal{L}$-m.p. is possible based on the fact that the diffusion coefficients $\{C_j\}$ are non-random. Let us begin with the case of $B=0$ for which the well-posedness was shown essentially by Holley and Stroock [10]:

Lemma 4.3. *Let P be a solution of $\mathcal{L}_0$-m.p. (i.e., $B_j=0$ for $1\leq j\leq J$). Then we have*

$$(4.5)\quad E^P[e^{\sqrt{-1}\langle S_T,\varphi\rangle}|\mathcal{F}_t]=\exp\{\sqrt{-1}\langle S_t,\varphi_{T-t}\rangle-\frac{1}{2}\int_0^{T-t}\langle C\varphi_u,\varphi_u\rangle du\}\ ,$$

for $0\leq t\leq T$, $\varphi\in C_0^\infty(\boldsymbol{R}^d)$, where $\varphi_t=T_t^*\varphi$. Especially, $\mathcal{L}_0$-m.p. is well-posed.

Proof. The existence of solutions for $\mathcal{L}_0$-m.p. is already verified (Proposition 4.1). The uniqueness follows from (4.5) and this equality is shown by observing that

$$Y_t=\exp\{\sqrt{-1}\langle S_t,\varphi_{T-t}\rangle+\frac{1}{2}\int_0^t\langle C\varphi_{T-u},\varphi_{T-u}\rangle du\},\ 0\leq t\leq T\ ,$$

is a martingale with respect to P; use Lemma 4.2 and Itô's formula. □

For treating general B, we introduce a map Θ on the space Ω defined by $(\Theta S.)_t=S_t-S_{t,3}(\cdot\,;S.)$, $t\geq0$, $S.\in\Omega$ and denote by Ω_S, $S\in\boldsymbol{L}_r^2$, the family of all $S.\in\Omega$ satisfying $S_0=S$. The space Ω_S is equipped with the natural topology induced from Ω.

Lemma 4.4. *The map $\Theta\colon\Omega_S\to\Omega_S$ is bijective and continuous. Moreover its inverse is measurable.*

Proof. The continuity and one-to-one property of Θ follow immediately from Lemma 2.6. For verifying the onto property of Θ, we have only to solve the equation $S_t=\bar{S}_t+S_{t,3}(\cdot\,;S.)$ for given $\bar{S}.\in\Omega_S$. To this end, we can use the usual method of successive approximation. The map which gives the each step of this approximation is clearly continuous, so that the limit giving the map Θ^{-1} is measurable. □

REMARK 4.1. Denote by $\boldsymbol{L}_{r,w}^2$ the space $\boldsymbol{L}_r^2$ equipped with the weak topology. Let Ω_w and $\Omega_{S,w}$ be the spaces defined similarly to Ω and Ω_S, respectively, but with $\boldsymbol{L}_r^2$ replaced by $\boldsymbol{L}_{r,w}^2$. Then we can also prove that $\Theta\colon\Omega_{S,w}\to\Omega_{S,w}$ is bijective and bi-measurable for each $S\in\boldsymbol{L}_r^2$.

Theorem 4.1. *The $\mathcal{L}_B$-m.p. is well-posed.*

Proof. The image measure $P\circ\Theta^{-1}$ of an arbitrary solution P of $\mathcal{L}_B$-m.p. solves the $\mathcal{L}_0$-m.p. In fact, noting Lemma 4.4, we have only to check that $\Psi((\Theta S.)_t)-\int_0^t \mathcal{L}_0 \Psi((\Theta S.)_u)\,du$ is a martingale with respect to $(P, \{\mathcal{F}_t\})$ for every $\Psi\in\mathcal{D}$, and this is shown by using Lemma 4.2 and Itô's formula. Therefore Lemma 4.3 proves the uniqueness of solutions of $\mathcal{L}_B$-m.p. starting from each $S\in \boldsymbol{L}_r^2$, while Proposition 4.1 shows the existence of solutions. □

REMARK 4.2. The $\mathcal{L}_B$-m.p. considered on the space Ω_w is also well-posed.

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