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A NOTE ON Γ_G-SPACES

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Introduction. In [3], I introduced the notion of a special Γ_G-space and showed that every special Γ_G-space A functorially determines a G-spectrum S_GA such that the associated infinite loop G-space $\Omega^\infty S_GA$ is an equivariant group completion of the G-space $A(\mathbf{1})$. On the other hand, Hauschild, May and Waner [1] established an equivariant infinite loop space machine based on the notion of a (special) Γ-G-space. The purpose of this note is to show that these two notions are canonically equivalent, although their definitions appear to be rather different.

1. For a finite group G, let Γ_G denote the category of based finite G-sets and based maps. We endow Γ_G with the standard G-action

$$(g,f)\mapsto {}^gf=gfg^{-1} \qquad \text{for} \quad g\in G, f\in \operatorname{mor}\Gamma_G .$$

For the trivial group $G=e$, Γ_e is equivalent to the opposite of Segal's Γ [2], and so we denote $\Gamma=\Gamma_e$. Then Γ can be regarded as the full subcategory of Γ_G consisting of trivial G-sets. In fact, we have $\Gamma\subset(\Gamma_G)^G$ because every based map between trivial G-sets is automatically G-equivariant. As usual we denote by $\mathbf{n}$ the trivial G-set $\{0, 1, \cdots, n\}\in\Gamma$.

DEFINITION. A Γ-G-space is a functor from Γ to the category of based G-spaces and based G-maps. A Γ_G-space is a G-equivariant functor from Γ_G to the category of based G-sets and based maps equipped with the standard G-action.

Let us denote by

$$\Gamma_G\textbf{-Space} \quad (\text{resp. } \Gamma\text{-}G\textbf{-Space})$$

the category of Γ_G-spaces (resp. Γ-G-spaces) with G-equivariant natural transformations (resp. natural ransformations) as morphisms. We will show that there is an adjoint equivalence between Γ_G-**Space** and Γ-G-**Space**.

If A is a Γ_G-space then its restriction to Γ becomes a Γ-space because every morphism $f\colon \mathbf{m}\to\mathbf{n}$ in Γ induces a G-map $A(f)\colon A(\mathbf{m})\to A(\mathbf{n})$. Thus we have a functor

$$R\colon \Gamma_G\textbf{-Space} \to \Gamma\text{-}G\textbf{-Space}$$

induced by the inclusion $\Gamma \subset \Gamma_G$.

On the other hand, there is a functor

$$E\colon \Gamma\text{-}G\textbf{-Space} \to \Gamma_G\textbf{-Space}$$

which takes each Γ-G-space X to the functor

$$EX\colon\ S \mapsto S \otimes_\Gamma X = \coprod_n \mathrm{Map}_0(\boldsymbol{n}, S) \times X(\mathbf{n})/\sim .$$

Here we identify $(sf, x) \in \mathrm{Map}_0(\mathbf{m}, S) \times X(\mathbf{m})$ with $(s, X(f)(x)) \in \mathrm{Map}_0(\mathbf{n}, S) \times X(\mathbf{n})$ for every $f\colon \mathbf{m} \to \mathbf{n}$ in Γ, and define a G-action on $S \otimes_\Gamma X$ by

$$g[s, x] = [{}^g s, gx] = [gs, gx] \qquad \text{for} \quad g \in G,\ [s, x] \in S \otimes_\Gamma X .$$

(The second equality follows from the fact that the objects of Γ are trivial G-sets.)

Then for any $g \in G$ and $f\colon S \to T$ in Γ_G we have $EX({}^g f) = {}^g EX(f)$ because

$$\begin{aligned} EX({}^g f)[s, x] &= [{}^g f s, x] \\ &= [gfg^{-1}s, x] \\ &= g[fg^{-1}s, g^{-1}x] \\ &= gEX(f)[g^{-1}s, g^{-1}x] \\ &= gEX(f)(g^{-1}[s, x]) \\ &= {}^g EX(f)[s, x] . \end{aligned}$$

Thus EX becomes a Γ_G-space.

It is evident that the G-homeomorphisms

$$X(\mathbf{n}) \to \mathbf{n} \otimes_\Gamma X = REX(\mathbf{n})\,, \quad x \mapsto [\mathrm{id}_{\mathbf{n}}, x]$$

define a natural isomorphism $\mathrm{Id} \to RE$. On the other hand, there is a natural isomorphism $ER \to \mathrm{Id}$ given by the G-homeomorphisms

$$ERA(S) = S \otimes_\Gamma RA \to A(S)$$

which takes the class of $(s, a) \in \mathrm{Map}_0(\mathbf{n}, S) \times A(\mathbf{n})$ to $A(s)(a) \in A(S)$. This shows that E is a left adjoint of R. Hence

Theorem 1. *The restriction* $R\colon \Gamma_G$**-Space**$\to \Gamma$-G**-Space** *is an equivalence of categories.*

Theorem 1 implies that for a Γ_G-space A, the G-space $A(S)$ $(S \in \Gamma_G)$ can be reconstructed from those $A(\mathbf{n})$ $(\mathbf{n} \in \Gamma)$. To see this, let us choose a bijection $f\colon S \to \mathbf{n}$ with $S \in \Gamma_G$ and $\mathbf{n} \in \Gamma$. Then f determines a homomorphism $\rho\colon G \to \mathrm{Map}_0(\mathbf{n}, \mathbf{n}) = \Sigma_n$ such that

$$\begin{array}{ccc} S & \xrightarrow{f} & \mathbf{n} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \rho(g)} \\ S & \xrightarrow[f]{} & \mathbf{n} \end{array}$$

commutes for every $g \in G$. Let $A(\mathbf{n})_\rho$ denote the based G-space having the underlying space $A(\boldsymbol{n})$ and equipped with the G-action

$$(g, a) \mapsto A(\rho(g))(ga) \qquad \text{for} \quad g \in G,\ a \in A(\mathbf{n}).$$

(This formula in fact gives a G-action because $A(\rho(g))\colon A(\mathbf{n}) \to A(\mathbf{n})$ are G-maps.)
Then we have

Proposition 2. *$A(f)\colon A(S) \to A(\mathbf{n})_\rho$ is a G-homeomorphism.*

Proof. First note that ${}^g f = fg^{-1}$ holds for any $g \in G$ because $\mathbf{n}$ has the trivial G-action. Now, for every $g \in G$ and $a \in A(\mathbf{n})$ we have

$$\begin{aligned} A(f)(ga) &= A(\rho(g) f g^{-1})(ga) \\ &= A(\rho(g)) A({}^g f)(ga) \\ &= A(\rho(g))\, {}^g A(f)(ga) \\ &= A(\rho(g))(gA(f)(g^{-1}ga)) \\ &= A(\rho(g))(gA(f)(a)). \end{aligned}$$

This shows that $A(f)\colon A(S) \to A(\mathbf{n})_\rho$ is a G-map. Since $A(f)$ has the inverse $A(f^{-1})$, we conclude that $A(f)$ is a G-homeomorphism.

2. Proposition 2 enables us to restate the definition of a special Γ_G-space in terms of the associated Γ-G-space, and so, to compare with the definition of a special Γ-G-space given by Hauschild, May and Waner [1].

First recall the definition of a special Γ_G-space. Let A be a Γ_G-space such that for all $S \in \Gamma_G$, $A(S)$ has the G-homotopy type of a based G-CW complex. For each based G-set $S \in \Gamma_G$ let us consider the based map

$$P_S\colon A(S) \to \mathrm{Map}_0(S, A(\mathbf{1})) = A(\mathbf{1})^{S_-}, \quad a \mapsto \{A(p_s)(a)\}$$

where $S_- = S - \{\text{point}\}$ and for every $s \in S_-$, p_s denotes the based map $S \to \mathbf{1} = \{0, 1\}$ such that $p_s^{-1}(1) = s$. Then it is easily observed that P_S becomes a G-map, although each p_s is not necessarily G-equivariant.

DEFINITION. A is called a special Γ_G-space if
(1) for every $S \in \Gamma_G$ the based G-map $P_S\colon A(S) \to A(\mathbf{1})^{S_-}$ is a G-homotopy equivalence.

Notice that if we take S=point then the condition (1) says that A(point) is

G-contractible. (Thus the condition (a) of [3, Definition 1.3] can be regarded as a special case of the condition (b).)

For every homomorphism $\rho: G \to \Sigma_n$ let us denote $A(\mathbf{1})^n_\rho = \mathrm{Map}_0(\mathbf{n}_\rho, A(\mathbf{1}))$; that is, the n-fold product $A(\mathbf{1})^n$ equipped with G-action

$$(g, \{a_j\}) \mapsto \{ga_{\rho(g^{-1})(j)}\} \qquad \text{for} \quad g \in G, \ \{a_j\} \in A(\mathbf{1})^n .$$

Then, by Proposition 2, we have

Proposition 3. *Let A be a Γ_G-space such that $A(S)$ has the G-homotopy type of a based G-CW complex for every $S \in \Gamma_G$. Then A is special if and only if*
(2) *for every $n \geq 0$ and every homomorphism $\rho: G \to \Sigma_n$ the based G-map $P_{\mathbf{n}}: A(\mathbf{n})_\rho \to A(\mathbf{1})^n_\rho$ is a G-homotopy equivalence.*

We now turn to the definition of a specia Γ-G-space [1]. Let X be a Γ-G-space. For each n, we endow $X(\mathbf{n})$ with the $G \times \Sigma_n$-action

$$((g, \sigma), x) \mapsto X(\sigma)(gx) \quad \text{for} \quad (g, \sigma) \in G \times \Sigma_n, \ x \in X(\mathbf{n}) .$$

Then the canonical map $P_{\mathbf{n}}: X(\mathbf{n}) \to X(\mathbf{1})^n$ can be regarded as a $G \times \Sigma_n$-map.

Definition. X is called a *special Γ-G-space* if
(3) for each n, $P_{\mathbf{n}}$ induces an ordinary weak homotopy equivalence on passage to K-fixed points for those subgroups K of $G \times \Sigma_n$ whose intersection with Σ_n is the trivial group; that is, $K = \{(h, \rho(h)) \mid h \in H\}$ for some subgroup H of G and homomorphism $\rho: H \to \Sigma_n$.

In other words, X is a special Γ-G-space if and only if $P_{\mathbf{n}}: X(\mathbf{n})_\rho \to X(\mathbf{1})^n_\rho$ is a weak H-equivalence for every subgroup H and every homomorphism $\rho: H \to \Sigma_n$. Thus (3) implies, in particular,
(4) for every n and every homomorphism $\rho: G \to \Sigma_n$, $P_n: X(\mathbf{n})_\rho \to X(\mathbf{1})^n_\rho$ is a weak G-equivalence;
or equivalently,
(5) for every $S \in \Gamma_G$, $P_S: EX(S) \to EX(\mathbf{1})^{S^-}$ is a weak G-equivalence.

Conversely we can prove that (3) follows from the weaker condition (4) in the following way. By Proposition 2 again, it suffices to show that if X satisfies (5) then for every based finite H-set U, $P_U: EX(U) = U \otimes_\Gamma X \to EX(\mathbf{1})^{U^-}$ is a weak H-equivalence. Let $S \in \Gamma_G$ be a based G-set which contains U as an H-invariant subset (e.g., $S = G_+ \wedge_H U$). Then S can be written as the union $S = U \vee V$ of based H-sets U and $V = S - U_-$, and we have a commutative diagram of based H-spaces

$$\begin{tikzcd}
EX(S) \arrow[r, "{(EX(p),\, EX(q))}"] \arrow[d, "P_S"'] & EX(U) \times EX(V) \arrow[d, "P_U \times P_V"] \\
EX(\mathbf{1})^{S^-} \arrow[r, equal] & EX(\mathbf{1})^{U^-} \times EX(\mathbf{1})^{V}
\end{tikzcd}$$

where p and q denote the projections $U\vee V\to U$ and $U\vee V\to V$ respectively.

We will show that $(EX(p), EX(q))$ is a weak H-equivalence; that is

$$(EX(p), EX(q))_*: \pi_*^K EX(S) \to \pi_*^K EX(U)\oplus\pi_*^K EX(V)$$

is an isomorphism for every subgroup K of H. Since P_S is a weak G-equivalence, this implies that $P_U\times P_V$ is a weak H-equivalence, and hence P_U becomes a weak H-equivalence for any U.

Let i and j be the inclusions $U\to U\vee V$ and $V\to U\vee V$ respectively, and let us consider the commutative diagram

$$\begin{tikzcd}
EX(U\vee V) \arrow[r, "{(EX(p),\, EX(q))}"] \arrow[d, "EX(i\vee j)"'] & EX(U)\times EX(V) \arrow[d, "EX(i)\times EX(j)"] \\
EX(S\vee S) \arrow[r, "{(EX(\mathrm{pr}_1),\, EX(\mathrm{pr}_2))}"] \arrow[d, "EX(p\vee q)"'] & EX(S)\times EX(S) \arrow[d, "EX(p)\times EX(q)"] \\
EX(U\vee V) \arrow[r, "{(EX(p),\, EX(q))}"'] & EX(U)\times EX(V)
\end{tikzcd}$$

Then $(EX(\mathrm{pr}_1), EX(\mathrm{pr}_2))$ is a weak G-equivalence by the assumption, and $EX(i\vee j)$ (resp. $EX(i)\times EX(j)$) is a section of $EX(p\vee q)$ (resp. $EX(p)\times EX(q)$). It is now easy to see that the composite

$$EX(p\vee q)_*(EX(\mathrm{pr}_1), EX(\mathrm{pr}_2))_*^{-1}(EX(i)_*\oplus EX(j)_*)$$

gives the inverse of $(EX(p), EX(q))_*$. This proves that (4) implies (3).

Especially, we have

Corollary. *Let X be a Γ-G-space such that $X(\mathbf{n})_\rho$ has the G-homotopy type of a based G-CW complex for every $\mathbf{n}\in\Gamma$ and $\rho: G\to\Sigma_n$. Then X is a special Γ-G-space in the sense of Hauschild, May and Waner* [1] *if and only if X is the restriction of some special Γ_G-space.*

In view of the equivalence Γ_G-**Space**$\approx\Gamma$-G-**Space**, this corollary says that the notion of special Γ_G-space is essentially the same with the notion of special Γ-G-Space. (The only difference lies in the fact that we impose the restriction that special Γ_G-spaces have values in the G-spaces having the G-homotopy types of based G-CW complexes.)

References

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