Kasue, A. and Washio, T.
Osaka J. Math.
27 (1990), 899–928

GROWTH OF EQUIVARIANT HARMONIC MAPS AND HARMONIC MORPHISMS

Dedicated to Prof. Tadashi Nagano on his 60th birthday

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(Received December 4, 1989)

0. Introduction

The purpose of the present paper is to study the growth of certain harmonic maps in relation with the geometry of the domains and ranges.

Let $\phi: M \to N$ be a harmonic map between complete noncompact Riemannian manifolds M and N. We fix a point o of M (resp. a point o' of N) and denote by r_M (resp. r_N) the distance to o in N (resp. o' in N). Set $\mu(\phi; t) := \max\{r_N(\phi(x)) : x \in M, r_M(x) = t\}$. We want to know the growth of ϕ, or the asymptotic behavior of $\mu(\phi; t)$ as t goes to infinity. We first recall the following result by Cheng [8] (cf. also [3] [31: Chap. 6]): Suppose that M has nonnegative Ricci curvature and N is a Hadamard manifold, namely, N is a simply connected and nonpositively curved manofod manifold. Then the energy density $e(\phi)$ of the map ϕ satisfies: $e(\phi)(o) \leq c_m \mu(\phi: t)^2 t^2$, where c_m is a constant depending only on the dimension m of M. It follows that ϕ is a constant map if ϕ has sublinear growth, that is, $\liminf_{t\to\infty} \mu(\phi; t)/t = 0$. We are interested in a (nonconstant) harmonic map $\phi: M \to N$ which has linear growth, namely, which has the property that $\limsup_{t\to\infty} \mu(\phi; t)/t < +\infty$. For instance, it turns out that a harmonic map $\phi: M \to N$ of linear growth must be totally geodesic if M has volume growth of at most quadratic order (cf. [9]). It has been also proved in [24] that a d-closed harmonic 1-form of bounded length on M must be parallel if the sectional curvature of M is nonnegative and decays quadratically. Moreover Li and Tam [26] have shown that the dimension on the space of linear growth harmonic functions on M is less than or equal to $k+1$ if the volume of the metric ball of radius t around o is bounded by ct^k for some constant c.

On the other hand, we can construct a noncompact complete manifold M of positive Ricci curvature and a harmonic map $\phi_F: M \to F$ of bounded energy density from M onto a complete manifold F of nonnegative Ricci curvature (cf. Example in Section 2). It turns out from the construction that ϕ is a *harmonic marphism* from M onto F with *totally geodesic fibers*, namely, it is a map carrying the germs of harmonic functions to the germs of harmonic functions such that the inverse image of ϕ for each p of F is a totally geodesic submanifold. For example, in case $F=\boldsymbol{R}$, $\phi_{\boldsymbol{R}}: M\to\boldsymbol{R}$ is a harmonic function of linear growth which is not totally geodesic, and in case $F=S^1$, $\phi_{S^1}: M\to S^1$ defines a harmonic 1-form of bounded length which is not parallel. Moreover we observe that if F admits a harmonic function h of linear growth, then the composition $h\circ\phi_F: M\to\boldsymbol{R}$ is also a harmonic function of linear growth on M. It is not clear whether there exist (nonconstant) harmonic maps with linear growth between manifolds of nonnegative Ricci curvature and Hadamard manifods of negative curvature.

We shall explain briefly the contents of this paper. In Section 1, we construct equivariant harmonic maps by solving certain ordinary differential equations and discuss their growth in some cases (cf. Theorem 1.1). Section 2 is devoted to the study of harmonic morphisms and their growth. For example, we shall give a lower bound for the growth of harmonic morphisms under certain conditions (cf. Theorem 2.10). In Section 3, we consider harmonic maps with linear growth between manifolds of nonnegative Ricci curvature and Hadamard manifolds and get sufficient conditions for such maps to be totally geodesic (cf. Theorem 3.2).

1. Examples of equivariant harmonic maps and their growth

In this section, we shall first show some examples of equivariant harmonic maps and then discuss the asymptotic behavior of them at infinity. See e.g., [4: Chap. 6] for a general theory on equivariant harmonic maps.

1.1. Let us first consider simple equivariant harmonic maps between rotationally symmetric spaces. To begin with, take a smooth function η on $[0, \infty)$ such that

$$\eta(0)=0\,,\quad \eta'(0)=1 \quad\text{and}\quad \eta>0 \qquad \text{on}\quad (0,\infty)\,, \tag{1.1}$$

and also a smooth function ξ on $[0, \infty)$ with the same property (1.1) as η. We denote by g_η a Riemannian metric on $\boldsymbol{R}^m$ which can be expressed as $g_\eta=dt^2+\eta(t)^2d\theta^2$ in the polar coordinates (t, θ). Let us denote by $\boldsymbol{R}^m{}_\eta$ (resp., $\boldsymbol{R}^m_\eta(T)$) the Riemannan manifold $(\boldsymbol{R}^m, g_\eta)$ (resp., the metric ball of $\boldsymbol{R}^m{}_\eta$ with radius T around the origin) for simplicity. Let $\phi: S^{m-1}\to S^{n-1}$ be a harmonic map from the unit sphere S^{m-1} of $\boldsymbol{R}^m$ to the unit sphere S^{n-1} of $\boldsymbol{R}^n$ with constant energy $\boldsymbol{e}$. For a positive smooth function $\alpha_1(t)$ on $(0, T)$ $(0<T\leq+\infty)$, define a map $F_1: \boldsymbol{R}^m{}_\eta(T)\setminus\{o\}\to\boldsymbol{R}^n{}_\xi$ by

$$F_1(t, \theta)=(\alpha_1(t), \phi(\theta))\,.$$

Then it turns out from direct computations that F_1 is harmonic if and only if α_1 satisfies an ordinary differential equation:

$$(E_1) \qquad \alpha_1''(t)+\frac{1}{t}P_1(t)\alpha_1'(t)-\frac{1}{t^2}Q_1(t, \alpha_1(t)) = 0$$

on $(0, T)$, where $P_1(t)=(m-1)t\eta'(t)\eta(t)^{-1}$ and $Q_1(t, s)=et^2\eta(t)^{-2}\xi(s)\xi'(s)$.

Let us next consider equivariant harmonic maps from the Riemannian 4-manifold $\boldsymbol{R}^4{}_{\eta_1\eta_2\eta_3}$ described below to $\boldsymbol{R}^3{}_\xi$ or $\boldsymbol{R}^4{}_{\xi_1\xi_2\xi_3}$. Let Z_1, Z_2, Z_3 be a left invariant orthonormal frame field on the unit 3-sphere S^3 such that $[Z_1, Z_2]=2Z_3$, $[Z_2, Z_3]=2Z_1$, $[Z_3, Z_1]=2Z_2$. We denote by Ω_i $(i=1, 2, 3)$ the dual forms of Z_i and consider a Riemannian metric $g_{\eta_1\eta_2\eta_3}$ on $\boldsymbol{R}^4$ of the form:

$$g_{\eta_1\eta_2\eta_3} = dt^2+\eta_1(t)^2\Omega_1{}^2+\eta_2(t)^2\Omega_2{}^2+\eta_3(t)^2\Omega_3{}^2 ,$$

where η_i $(i=1, 2, 3)$ satisfy (1.1). As before, $\boldsymbol{R}^4{}_{\eta_1\eta_2\eta_3}$ (resp., $\boldsymbol{R}^4{}_{\eta_1\eta_2\eta_3}(T)$) stands for the Riemannian manifold $(\boldsymbol{R}^4, g_{\eta_1\eta_2\eta_3})$ (resp., the metric ball around the origin with radius T).

Let ψ: $S^3\to S^2$ be the Hopf fibering and assume that $\operatorname{Ker}\psi_*=Z_3$. We consider the case $\eta_1=\eta_2$, and set $\eta=\eta_1$ and $\lambda=\eta_3$. Given a smooth function α_2: $(0, T)\to(0, \infty)$ $(0<T\leq+\infty)$, define a map F_2: $\boldsymbol{R}^4{}_{\eta\eta\lambda}(T)/\{o\}\to\boldsymbol{R}^3{}_\xi$ by

$$F_2(t, \theta) = (\alpha_2(t), \psi(\theta)) .$$

Then direct computations show that F_2 is harmonic if and only if α_2 satisfis the following ordinary differential equation:

$$(E_2) \qquad \alpha_2''(t)+\frac{1}{t}P_2(t)\alpha_2'(t)-\frac{1}{t^2}Q_2(t, \alpha_2(t)) = 0$$

on $(0, T)$, where $P_2(t)=t\{2\eta'(t)\eta(t)^{-1}+\lambda'(t)\lambda(t)^{-1}\}$ and $Q_2(t, s)=8t^2\eta(t)^{-2}\xi(s)\xi'(s)$.

Let F_3: $\boldsymbol{R}^4{}_{\eta_1\eta_2\eta_3}(T)\setminus\{o\}\to\boldsymbol{R}^4{}_{\xi_1\xi_2\xi_3}$ be a map difined by

$$F_3(t, \theta) = (\alpha_3(t), \theta) ,$$

where $\alpha_3(t)$ is a positive smooth function on $(0, T)$. Then the equation for the harmonicity of the map F_3 is given by

$$(E_3) \qquad \alpha_3''(t)+\frac{1}{t}P_3(t)\alpha_3'(t)-\frac{1}{t^2}Q(t, \alpha_3(t)) = 0$$

on $(0, T)$, where $P_3(t)=t\sum_{i=1}^3 \eta_j'(t)\eta_i(t)^{-1}$ and $Q_3(t, s)=t^2\sum_{i=1}^3 \eta_i(t)^{-2}\xi_i'(s)\xi_i(s)$.

Finally let us consider equivariant harmonic maps between $\boldsymbol{R}^2$-bundles over 2-sphere S^2 with certain metrics. Take first a smooth function g on $[0, \infty)$ such that

$$(1.2) \qquad g(0) = 1 , \quad g'(0) = 0 , \quad \text{and} \quad g>0 \quad \text{on} \quad [0, \infty)$$

and also a smooth function h on $[0, \infty)$ with the same property (1.2) as g. Let

η be a function satisfying (1.1). Then define a Reimannian metric $\hat{G}_{g,h,\eta}$ on $S^2\times \boldsymbol{R}^2$ by

$$\hat{G}_{g,h,\eta}=g(t)^2(\Omega_1{}^2+\Omega_2{}^2)+h(t)^2\Omega_3{}^2+dt^2+\eta(t)^2d\theta^2\,,$$

where $\{\Omega_1, \Omega_2, \Omega_3\}$ are, as before, left invariant 1-forms over the unit 3-sphere S^3 in $\boldsymbol{R}^4=\boldsymbol{C}^2$. Given an integer k, we consider the quotient space L^k, or the $\boldsymbol{R}^2$-bundle over S^2, by the action of S^1 on $S^3\times\boldsymbol{R}^2$: $(p, t, \theta)e^{i\omega}=(pe^{i\omega}, t, \theta-k\omega)$. Then the metric $\hat{G}_{g,h,\eta}$ descends to L^k and define a Riemannin metric $G_{g,h,\eta}$ on L^k, since the action preserves the metric. We denote by $L^k{}_{g,h,\eta}$ and π_k: $(S^3\times\boldsymbol{R}^2, \hat{G}_{g,h,\eta})\to L^k{}_{g,h,\eta}$, respectively, the resulting Riemannian manifold $(L^k, G_{g,h,k})$ and the Riemannian submersion. Moreover the radial function t on $S^3\times\boldsymbol{R}^2$, which is the distance function to $S^3\times\{0\}$ with respect to $\hat{G}_{g,h,\eta}$, descends also to the quotient manifold $L^k{}_{g,h,\eta}$ and defines the distance function to the zero section $S^2{}_0$ of L^k with respect to $G_{g,h,\eta}$. We put $L^k{}_{g,h,\eta}(T)=\{x\in L^k{}_{g,h,\eta}: t(x)<T\}$. For a pair (k, l) of integers such that k divides l, i.e., $l=nk$ for some integer n, we take two Riemannian manifolds $L^k{}_{g,h,\eta}$ and $L^l{}_{\tilde{g},\tilde{h},\tilde{\eta}}$ described just above, and consider a map F_4: $L_{g,h,\eta}(T)\backslash S^2{}_0\to L^l{}_{\tilde{g},\tilde{h},\tilde{\eta}}$ defined by

$$F_4(\pi_k(p, t, \theta))=\pi_l(f(p), \alpha_4(t), \phi_n(\theta))\,,$$

where f: $S^3\to S^3$ is a symmetry of the Hopf fibration (i.e., a unitary transformation or the composition of a unitary transformation and the conjugation), α_4 is a positive smooth function on $(0, T)$, and ϕ_n: $S^1\to S^1$ is the rotation given by $\phi_n(e^{i\omega})=e^{in\omega}$. Then direct computations show that F_4: $L^k{}_{g,h,\eta}(T)\backslash S^2{}_0\to L^l{}_{\tilde{h},\tilde{g},\tilde{\eta}}$ is harmonic if and only if α_4 satisfies

$$(E_4)\qquad \alpha_4''(t)+\frac{1}{t}P_4(t)\alpha_4'(t)-\frac{1}{t^2}Q_4(t, \alpha_4(t))=0$$

on $(0, T)$, where

$$P_4(t)=t\Big(2\frac{g'(t)}{g(t)}+\frac{k^2\eta(t)^3h'(t)+h(t)^3\eta'(t)}{h(t)\eta(t)(k^2\eta(t)^2+h(t)^2)}\Big)$$

$$Q_4(t, s)=t^2\Big(2\frac{\tilde{g}(s)\tilde{g}'(s)}{g(t)^2}+\frac{n^2(k^2\eta(t)^2+h(t)^2)(l^2\tilde{\eta}(s)^4\tilde{h}(s)\tilde{h}'(s)+\tilde{h}(s)^4\tilde{\eta}(s)\tilde{\eta}'(s))}{h(t)^2\eta(t)^2(l^2\tilde{\eta}(s)^2+\tilde{h}(s)^2)^2}\Big)\,.$$

We note here that $P_i(t)$ and $Q_i(t, s)$ $(i=1, 2, 3, 4)$ have the following properties:

$$(1)\qquad \lim_{t\to 0}P_i(t)=p_i+1\,,$$

where $p_1=m-2$, $p_2=p_3=2$ and $p_4=0$;

$$(2)\qquad \lim_{t,s\to 0}\frac{1}{s}Q_i(t, s)=q_i\,,$$

where $q_1=e$, $q_2=8$, $q_3=3$ and $q_4=n^2$;

(3) there exist positive numbers t_0 and s_0 such that

$$
\begin{aligned}
(1.3) \qquad & Q_i(t, 0) = 0\,, \quad Q_i(0, s) > 0 \qquad (0 < s \leq s_0)\,, \\
& 0 < Q_i(t, s_1) \leq Q_i(t, s_2) \qquad (0 < t \leq t_0,\ 0 < s_1 < s_2 < s_0)\,.
\end{aligned}
$$

In the last two inequalities for the case: $i=4$, it is assumed that

$$
(1.4) \qquad \tilde{g}(s) \geq 0 \quad \text{and} \quad \tilde{h}'(s) \geq 0 \qquad \text{on} \quad [0, s_0]\,.
$$

Let us now state some results on the existence of certain solutions $\alpha_i(t)$ of equation (E_i) $(i=1, 2, 3, 4)$ and their asymptotic behavior as t goes to zero and also tends to infinity, under some conditions. The proofs will be found in the next subsection.

We assume first that $n \neq 0$ for the case of $i=4$. Fix an $i \in \{1, 2, 3, 4\}$. Then given two positive numbers t_0 and s_0 for which (1.3) and (1.4) hold, there exists a monotonically increasing, positive solution α_i: $(0, t_0] \to (0, \infty)$ with $\alpha_i(t_0)=s_0$ such that

$$
(1.5) \qquad
\begin{aligned}
& \limsup_{t\to 0} \frac{\log \alpha_i(t)}{\log t} \leq \mu(p_i, q_i) \\
& \limsup_{t\to 0} \frac{\log \alpha_i'(t)}{\log t} \leq \mu(p_i, q_i) - 1\,,
\end{aligned}
$$

where $\mu(p, q) = \frac{1}{2}\{-p+\sqrt{p^2+4q}\}$ (cf. Lemmas 1.3 in 1.2). Hence the harmonic map F_i defined by α_i as before turns out to be a continuous weakly harmonic maps defined around the origin o for the cases: $i=1, 2, 3$ and the zero-section $s^2{}_0$ for the case: $i=4$. Thus the fundamental regularity theory (cf. e.g., [11], [17]) shows that F_i is actually smooth over the origin for the cases: $i=1, 2, 3$ and the zero-section $S^2{}_0$ for the case: $i=4$.

In what follows, we assume that the solution α_i is defined maximally on $(0, T_i)$ for some $T_i \in (t_0, \infty]$. Then we have the following

Proposition 1.1. *Let P_i, Q_i and α_i be as above. Suppose that $Q_i(t, s)$ is non-negative on $(0, \infty)\times(0, \infty)$. Then α_i' is positive on $(0, T_i)$. In particular, $\alpha_i(t)$ tends to infinity as t goes to T_i if T_i is finite. Moreover suppose that for some constants $A_i \in [-\infty, +\infty)$ and $B_i \in [0, \infty]$, $P_i(t)$ converges to A_i as t goes to infinity and $Q_i(t, s)/s$ tends to B_i as t and s go to infinity. Then the following assertions hold:*

(1) *If $-\infty < A_i < +\infty$ and $0 < B_i < +\infty$, or if $0 \leq A_i < 1$ and $0 \leq B_i < +\infty$, then $T_i = +\infty$ and*

$$
\lim_{t\to\infty} \frac{\log \alpha_i(t)}{\log t} = \mu(A_i - 1, B_i) \qquad (\in (0, \infty))\,.
$$

(2) *If* $B_i=+\infty$, *then*

$$\lim_{t\to T_i}\frac{\log\alpha_i(t)}{\log t}=+\infty .$$

Moreover $T_i=+\infty$ *if* $Q_i(t,s)\le B_i(t)s$ *for some continuous function* $B_i(t)$ *on* $(0,\infty)$.

(3) *If* $1\le A_i<+\infty$ *and* $B_i=0$, *then* $T_i=+\infty$ *and*

$$\lim_{t\to\infty}\frac{\log\alpha_i(t)}{\log t}=0 .$$

(cf. Lemmas 1.4 and 1.7 in 1.2).

Before concluding this subsection, let us consider the case: $i=4$ and determine the order $\mathrm{ord}(\alpha_4)=\mu(A_4-1,B_4)$ of α_4 in some cases. For simplicity, we assume that

$$g(t)=at+a',\quad h(t)=bt+b',\quad \eta(t)=ct+c',$$

for large t and

$$\tilde g(s)=\tilde as+\tilde a',\quad \tilde h(s)=\tilde bs+\tilde b',\quad \tilde\eta(s)=\tilde cs+\tilde c'$$

for large s, where a, $\tilde a$, b, $\tilde b$, c, and $\tilde c$ are nonnegative constants and the others are arbitrary constants.

Case 1: $a>0$, $\tilde a\ge 0$, $b>0$, $\tilde b>0$, $c<0$, $\tilde c>0$.

$$\mathrm{ord}(\alpha_4)=-1+\left[1+2\left(\frac{\tilde a}{a}\right)^2+n^2\frac{\tilde b^2\tilde c^2(k^2c^2+b^2)}{b^2c^2(l^2\tilde c^2+\tilde b^2)}\right]^{1/2}$$

Case 2: $a=0$, $\tilde a>0$, $b>0$, $\tilde b>0$, $c>0$, $\tilde c>0$.

$$\mathrm{ord}(\alpha_4)=+\infty .$$

Case 3: $a=\tilde a=0$, $b>0$, $\tilde b>0$, $c>0$, $\tilde c>0$.

$$\mathrm{ord}(\alpha_4)=n\,\frac{\tilde b\tilde c}{bc}\left(\frac{k^2c^2+b^2}{l^2\tilde c^2+\tilde b^2}\right)^{1/2}.$$

Case 4: $a>0$, $\tilde a\ge 0$, $b=\tilde b=c=\tilde c=0$.

$$\mathrm{ord}(\alpha_4)=\frac{1}{2}\left[-1+\left(1+8\left(\frac{\tilde a}{a}\right)^2\right)^{1/2}\right].$$

Case 5: $\tilde a>0$, $a=b=\tilde b=c=\tilde c=0$.

$$\mathrm{ord}(\alpha_4)=+\infty .$$

Case 6: $b=c=0$, $\tilde b>0$, $\tilde c>0$.

$$\operatorname{ord}(\alpha_4) = +\infty .$$

Case 7: $b>0$, $c>0$, $\tilde{b}=\tilde{c}=0$.

$$\operatorname{ord}(\alpha_4) = \begin{cases} 0 & \text{if} \quad a = \tilde{a} = 0 , \\ -1+\left[1+2\left(\dfrac{\tilde{a}}{a}\right)^2\right]^{1/2}, & \text{otherwise} . \end{cases}$$

1.2. In this subsection, we shall consider a nonlinear ordinary differential equation which contains equations (E_i) $(i=1, 2, 3, 4)$ as special cases.

Given a smooth function $P(t)$ on $[0, \infty)$ and a smooth function $Q(t, s)$ on $[0, \infty)\times[0, \infty)$, we consider an equation as follows:

$$(E_0) \qquad \alpha''(t)+\frac{1}{t}P(t)\alpha'(t)-\frac{1}{t^2}Q(t, s) = 0 .$$

In what follows, we assume that for some $p_*\geq 0$, $q_*>0$, $t_0>0$, and $s_0>0$,

$$(1.3) \qquad \begin{aligned} &\lim_{t\to 0} P(t) = p_*+1 \qquad \text{if} \quad p_*>0 , \\ &\frac{1}{t}\,|P(t)-1| \ \text{ is bounded on } (0, t_0) \qquad \text{if} \quad p_* = 0 , \end{aligned}$$

$$(1.4) \qquad \begin{aligned} &Q(t, 0) = 0 \ (0<t<t_0), \quad Q(0, s)>0 \ (0<s<s_0) , \\ &\lim_{t\to 0,\, s\to 0} \frac{1}{s}Q(t, s)=q_*, \ \text{and} \\ &0<Q(t, s_1)\leq Q(t, s_2) \qquad \text{for} \quad t\in(0, t_0),\ s_1 \text{ and } s_2\in(0, s_0) \text{ with } s_1\leq s_2 . \end{aligned}$$

We want a positive solution $\alpha(t)$ of equation (E_0) which converges monotonically to zero as t goes to zero. To begin with, let us reparametrize equation (E_0) with parameter $u=\log t\in(-\infty, +\infty)$ as follows:

$$(E_0') \qquad \beta''(u)+(P(e^u)-1)\beta'(u)-Q(e^u, \beta(u)) = 0 .$$

Let us take two positive numbers t_0 and s_0 for which (1.3) and (1.4) hold, and set $u_0=\log t_0$. For any $v\in\boldsymbol{R}$, we denote by β_v a unique solution of (E_0') subject to the conditions: $\beta_v(u_0)=s_0$ and $\beta_v'(u_0)=v$. Define a set $\mathcal{A}$ of $\boldsymbol{R}$ by $\mathcal{A}=\{v$: $\beta_v(u)$ decreases monotonically to zero in finite time as u decreases from $u_0\}$, and put $v_0=\inf\mathcal{A}$. Then $v_0>0$ and $\mathcal{A}$ is open (cf. [4: Chap.6]). Moreover we have

Lemma 1.1. *For any u_0 and $s_0>0$ as above, the solution $\beta=\beta_{v_0}$ of equation (E_0') has the following properties:*

(1) $\beta>0$, $\beta'>0$ *on* $(-\infty, u_0]$,

(2) $\lim_{u\to-\infty}\beta(u) = \lim_{u\to-\infty}\beta'(u) = \lim_{u\to-\infty}\beta''(u)=0.$

Proof. The first assertion (1) follows from the same arguments as in [4:

Chap. 6, Lemma 6.1.4]. As for the second one, we first prove that $\lim_{u\to-\infty}\beta'(u)=0$. Fix a positive constant r_* and choose $v_0\in(-\infty, u_0]$ so that $p_*+r_*>0$ and $P(e^u)<p_*+1+r_*$ on $(-\infty, v_0]$. For any $v_1\in(-\infty, v_0-1]$, let $\gamma(u)$ be a unique solution of equation:

$$\gamma''(u)+(p_*+r_*)\gamma'(u) = 0$$

subject to the conditions: $\gamma(v_1)=\beta(v_1)$ and $\gamma'(v_1)=\beta'(v_1)$. Then $\gamma(u)=\beta(v_1)+(p_*+r_*)^{-1}\beta'(v_1)[1-\exp\{(p_*+r_*)(v_1-u)\}]$. Define a function R by

$$R(u) = \beta'(u)\gamma(u)-\beta(u)\gamma'(u)\,.$$

Then we have

$$\text{(1.5)}\quad R'(u) = -(p_*+r_*)R(u)+(p_*+1+r_*-P(e^u))\beta'(u)\gamma(u)+Q(e^u, \beta(u))\gamma(u)\,.$$

Hence $R(v_1)=0$ and $R'(v_1)>0$. Moreover $R\geq 0$ on $[v_1, v_0]$. In fact, if $R>0$ on (v_1, v_2) and $R(v_2)=0$ for some $v_2\in(v_1, v_0)$, then $R'(v_2)\leq 0$, which contradicts (1.5). Since $(\beta/\gamma)'=R/\gamma^2>0$ on (v_1, v_0), we have $\beta\geq\gamma$ on $[v_1, v_0]$. Thus we get

$$\text{(1.6)}\quad \begin{aligned}\beta(v_1+1)-\beta(v_1)&\geq\gamma(v_1+1)-\gamma(v_1)\\ &=\frac{\beta'(v_1)}{p_*+r_*}\{1-e^{-p_*-s_*}\}\end{aligned}$$

for any $v_1\in(-\infty, v_0-1]$. Suppose that $\limsup_{u\to-\infty}\beta'(u)>0$. Then there exists a positive constant δ and a sequence $\{v_j\}_{j=1,2,\dots}$ such that $\beta'(v_j)>\delta$ and $v_{j+1}\leq v_j-1$. It follows from (1.6) that for each j,

$$\beta(v_j+1)-\beta(v_j)\geq\frac{\delta}{p_*+r_*}\{1-e^{-q_*-r_*}\}\,.$$

This is absurd, because β is positive. Thus we have shown that $\lim_{u\to-\infty}\beta'(u)=0$. Set $\beta_*=\lim_{u\to-\infty}\beta(u)$. Then we have

$$\begin{aligned}0 &= \lim_{u\to-\infty}\beta''(u) = \lim_{u\to-\infty}\{-(P(e^u)-1)\beta'(u)+Q(e^u, \beta(u))\}\\ &= Q(0, \beta_*)\,,\end{aligned}$$

and hence $\beta_*=0$ by (1.4). This proves the second assertion (2) of the lemma. //

Lemma 1.2. *Let u_0, s_0 and β be as in Lemma 1.1. Given $\varepsilon>0$, suppose that $|s^{-1}Q(e^u, s)-q_*|<\varepsilon$, and $|P(e^u)-1-p_*|<\varepsilon$ on $(-\infty, u_0]\times(-\infty, s_0]$. Then $\beta(u)\leq c\exp\delta_\varepsilon u$ and $\beta'(u)\leq c\exp\delta_\varepsilon u$ on $(-\infty, u_0]$, for some positive constant c, where $\delta_\varepsilon=\frac{1}{2}\{-(p_*+\varepsilon)+\sqrt{(p_*+\varepsilon)^2+4(q_*-\varepsilon)}\}$.*

Proof. For any $v\in(-\infty, u_0]$, let $\gamma_{v,\varepsilon}(u)$ be a unique solution of equation: $\gamma_{v,\varepsilon}'(u)=\delta_\varepsilon\gamma_{v,\varepsilon}(u)$, subject to the condition: $\gamma_{v,\varepsilon}(v)=\beta(v)$. That is, $\gamma_{v,\varepsilon}(u)=\beta(v)\exp\delta_\varepsilon(u-v)$. Then we claim that $\beta'(v)\geq\gamma_{v,\varepsilon}'(v)=\delta_\varepsilon\beta(v)$. In fact, we have

$$
\begin{aligned}
\gamma_{v,\varepsilon}''(u) &= \delta_\varepsilon^2\gamma_{v,\varepsilon}(u) \\
&= -(p_*+\varepsilon)\delta_\varepsilon\gamma_{v,\varepsilon}(u)+(q_*-\varepsilon)\gamma_{v,\varepsilon}(u) \\
&= -(p_*+\varepsilon)\gamma_{v,\varepsilon}'(u)+(q_*-\varepsilon)\gamma_{v,\varepsilon}(u) \\
&< -(p_*+\varepsilon)\gamma_{v,\varepsilon}'(u)+Q(e^u, \gamma_{v,\varepsilon}(u)).
\end{aligned} \tag{1.7}
$$

Suppose that $\beta'(v)<\gamma_{v,\varepsilon}'(v)$. Then by (1.7) we get

$$
\begin{aligned}
\gamma_{v,\varepsilon}''(v)&<-(p_*+\varepsilon)\beta'(v)+Q(e^v, \beta(v)) \\
&<-(P(e^v)-1)\beta'(v)+Q(e^v, \beta(v)) = \beta''(v)\,.
\end{aligned}
$$

Hence we have

$$\gamma_{v,\varepsilon}<\beta\,, \quad \gamma_{v,\varepsilon}'>\beta'\,, \quad \gamma_{v,\varepsilon}''<\beta'' \tag{1.8}$$

on $[v', v)$ for some $v'\in(-\infty, v]$. If (1.8) holds on $(-\infty, v)$, then $(\gamma_{v,\varepsilon}-\beta)''<0$ on $(-\infty, v)$, so that $(\gamma_{v,\varepsilon}-\beta)'(u)>(\gamma_{v,\varepsilon}-\beta)'(v)>0$ for any $u\in(-\infty,v)$. This yields a contradiction, because $\lim_{u\to-\infty}\beta'(u)=\lim_{u\to-\infty}\gamma_{v,\varepsilon}'(u)=0$. Therefore there exists $v_1\in(-\infty, v)$ such that (1.8) ceases to hold at v_1. Then it turns out that $\gamma_{v,\varepsilon}(v_1)\leq\beta(v_1)$, $\gamma_{v,\varepsilon}'(v_1)\geq\beta'(v_1)$ and $\gamma_{v,\varepsilon}''(v)=\beta''(v_1)$. By (1.7) and (1.4), we obtain

$$
\begin{aligned}
\gamma_{v,\varepsilon}''(v_1)&<-(p_*+\varepsilon)\gamma_{v,\varepsilon}'(v_1)+Q(e^{v_1}, \gamma_{v,\varepsilon}(v_1)) \\
&<-(p_*+\varepsilon)\beta'(v_1)+Q(e^{v_1}, \beta(v_1)) = \beta''(v_1)\,.
\end{aligned}
$$

This yields a contradiction. Thus we have shown that $\beta'(v)\geq\gamma_{v,\varepsilon}'(v)=\delta_\varepsilon\beta(v)$ on $(-\infty, u_0]$, which implies that $\beta(v)\leq c_1\exp\delta_\varepsilon u$ on $(-\infty, u_0]$ for some constant $c_1>0$. As for the estimate on β', consider first the case $p_*>0$ and then assume that $p_*>\varepsilon$. Then, we have

$$
\begin{aligned}
\beta''(u)&\leq-(p_*-\varepsilon)\beta'(u)+(q_*+\varepsilon)\beta(u) \\
&\leq-(p_*-\varepsilon)\delta_\varepsilon\beta(u)+(q_*+\varepsilon)\beta(u) \\
&\leq c_1(-(p_*-\varepsilon)\delta_\varepsilon+q_*+\varepsilon)e^{\delta_\varepsilon u}
\end{aligned}
$$

on $(-\infty, u_0]$. Integrating the both sides, we obtain

$$\beta'(u)\leq c_1(-(p_*-\varepsilon)\delta_\varepsilon+q_*+\varepsilon)\delta_\varepsilon^{-1}e^{\delta_\varepsilon u}$$

on $(-\infty, u_0]$. As for the case $p_*=0$, we have by (1.3)

$$\beta''(u)\leq c_2(e^u\beta'+\beta) \tag{1.9}$$

for some constant $c_2>0$. Integrating the both sides, we get

$$\beta'(u)\leq c_4 e^{\delta_\varepsilon(1)u} \tag{1.10}$$

on $(-\infty, u_0]$, for some constant $c_3>0$, where $\delta_\varepsilon(1)=\min\{1, \delta_\varepsilon\}$. If $\delta_\varepsilon>\delta_\varepsilon(1)$, then inserting (1.10) into (1.9) and integrating the resulting inequality, we obtain

$$\beta'(u)\leq c_4 e^{\delta_\varepsilon(2)u}$$

for some constant $c_4>0$, where $\delta_\varepsilon(2)=\min\{2, \delta_\varepsilon\}$. Thus repeating the same argument, we have

$$\beta'(u)\leq c_5 e^{\gamma_\varepsilon u}$$

on $(-\infty, u_0]$ for some constant $c_5>0$. This completes the proof of Lemma 1.2. //

Let us now return to equation (E_0). Let β and u_0 be as in Lmema 1.2. Define a solution $\alpha(t)$ of equation (E_0) by $\alpha(t)=\beta(\log t)$. Then by Lemma 1.2, we have

Lemma 1.3. *Let α be as above. Then α satisfies*
(1) $\alpha(t_0)=s_0$ $(t_0=e^{u_0})$, $\alpha>0$, $\alpha'>0$ *on* $(0, t_0]$;
(2) $\displaystyle\limsup_{t\to 0}\frac{\log\alpha(t)}{\log t}\leq\mu(p_*, q_*)$;
(3) $\displaystyle\limsup_{t\to 0}\frac{\log\alpha'(t)}{\log t}\leq\mu(p_*, q_*)-1$,
where $\mu(p_*, q_*)=\dfrac{1}{2}\{-p_*+\sqrt{p_*^2+4q_*}\}$ (≥ 0).

In what follows, we shall study the asymptotic behavior at infinity of solutions of equation (E_0) under certain conditions. Let t_0 and s_0 be positive numbers and $\alpha(t)$ a solution of equation (E_0) with $\alpha(t_0)>s_0$ and $\alpha'(t_0)>0$. Here $\alpha(t)$ is assumed to be defined at least on $[t_0, T_\alpha)$, where $T_\alpha=\sup\{T: \alpha$ is positive and bounded on $[t_0, T]\}$ $(\leq+\infty)$. In order to study the asymptotic behavior at infinity of α, we employ elementary comparison arguments just used in the proof for the preceding lemmas. Let A and $B\geq 0$ be two numbers chosen appropiately later and γ a unique solution of equation:

$$\gamma''(t)+\frac{A}{t}\gamma'(t)-\frac{B}{t^2}\gamma(t)=0\,, \tag{1.11}$$

subject to the condition: $\gamma(t_0)=\alpha(t_0)$ and $\gamma'(t_0)=\alpha'(t_0)$. Define a function R_{AB}: $[t_0, T_\alpha)\to\boldsymbol{R}$ by

$$R_{AB}(t)=\alpha(t)\gamma'(t)-\alpha'(t)\gamma(t)\,.$$

Then obviously $R_{AB}(t_0)=0$ and

$$R_{AB}'(t) = -\frac{1}{t}P(t)R_{AB}(t)+\frac{1}{t}(P(t)-A)\alpha(t)\gamma'(t)+\frac{1}{t^2}(B\alpha(t)-Q(t, \alpha(t))\gamma(t)\,.$$

Lemma 1.4. *Let α be as above. Suppose that $Q(t, s)\geq 0$ on $[t_0, T_\alpha)\times[s_0, \infty)$. Then $\alpha'>0$ on $[t_0, T_\alpha)$. In particular, $\lim_{t\to T_\alpha}\alpha(t)=+\infty$ if $T_\alpha<+\infty$.*

Proof. Suppose that $\alpha'(t)>0$ on $[t_0, t_1)$ and $\alpha'(t_1)=0$, for some $t_1\in(t_0, T_\alpha)$. Take two numbers A and B as above so that $P(t)<A$ on $[t_0, t_1]$ and $B=0$. Then $R_{A0}(t_0)=0$ and $R_{A0}'(t_0)<0$. We claim that $R_{A0}(t)<0$ on $(t_0, t_1]$. In fact if we have $t_2\in(t_0, t_1]$ such that $R_{A0}(t)<0$ on (t_0, t_2) and $R_{A0}(t_2)=0$, then

$$R_{AO}'(t_2)= \frac{1}{t_2}(P(t_2)-A)\alpha(t_2)\gamma'(t_2)-\frac{1}{t_2}Q(t_2, \alpha(t_2))\gamma(t_2)<0\,.$$

This yields a contradiction. Hence we have $R_{A0}(t)<0$ on $(t_0, t_1]$, that is,

$$\alpha(t)'\gamma(t)<\alpha'(t)\gamma(t)$$

on $(t_0, t_1]$. Putting $t=t_1$, we get

$$0<\alpha(t_1)\gamma'(t_1)<\alpha'(t_1)\gamma(t_1) = 0\,.$$

This is absurd. Thus we have shown that $\alpha'>0$ on $[t_0, T_\alpha)$. //

Lemma 1.5. *Let α be as above. Suppose that for some A and $B\geq 0$, $P(t)<A$ on $[t_0, \infty)$ and $Q(t, s)\geq Bs$ on $[t_0, \infty)\times[s_0, \infty)$, or for some A and $B\geq 0$, $P(t)\leq A$ on $[t_0, \infty)$ and $Q(t, s)>Bs$ on $[t_0, \infty)\times[s_0, \infty)$. Then:*

$$\liminf_{t\to T_\alpha}\frac{\log\alpha(t)}{\log t}\geq\mu(A-1, B)\,,$$

where $\mu(A-1, B)=\frac{1}{2}\{-(A-1)+\sqrt{(A-1)^2+4B}\}$.

Proof. Obviously $R_{AB}(t_0)=0$ and $R_{AB}'(t_0)<0$. The same arguments as in the proof of Lemma 1.4 show that $R_{AB}<0$ on (t_0, T_α), namely, $\alpha(t)\gamma'(t)<\alpha'(t)\gamma(t)$ on (t_0, T_α). Then it turns out that $\gamma(t)<\alpha(t)$ on (t_0, T_α), which implies that

$$\liminf_{t\to T_\alpha}\frac{\log\alpha(t)}{\log t}\geq\liminf_{t\to\infty}\frac{\log\gamma(t)}{\log t} = \mu(A-1, B)\,.$$

This completes the poof of Lemma 1.5. //

Lemma 1.6. *Let α be as above. Suppose that for some A and $B\geq 0$, $P(t)>A$ on $[t_0, \infty)$ and $Bs\geq Q(t, s)\geq 0$ on $[t_0, \infty)\times[s_0, \infty)$, or for some A and*

$B>0$, $P(t)\geq A$ on $[t_0, \infty)$ and $Bs>Q(t, s)\geq 0$ on $[t_0, \infty)\times[s_0, \infty)$. Then, $T_\alpha=+\infty$ and

$$\limsup_{t\to\infty} \frac{\log \alpha(t)}{\log t} \leq \mu(A-1, B).$$

Proof. Obviously $R_{AB}(t_0)=0$ and $R_{AB}'(t_0)>0$. The same arguments as in the proof of Lemma 1.4 again show that $R_{AB}>0$ on (t_0, T_α), namely, $\alpha(t)\gamma'(t)>\alpha'(t)\gamma(t)$ on (t_0, T_α). Hence $\gamma(t)>\alpha(t)>\alpha(t_0)$ on (t_0, T_α). This shows that $T_\alpha=+\infty$ and

$$\limsup_{t\to\infty} \frac{\log \alpha(t)}{\log t} \leq \lim_{t\to\infty} \frac{\log \gamma(t)}{\log t} = \mu(A-1, B).$$

This completes the proof of Lemma 6. //

By Lemmas 1.5 and 1.6, we have

Lemma 1.7. *Let α and T_α be as before. Suppose that $Q(t, s)\geq 0$ for any $t>0$ and $s>0$, and suppose that for some $A\in[-\infty, +\infty)$ and $B\in[0, +\infty]$, $P(t)$ converges to A as t goes to infinity and $Q(t, s)/s$ tends to B as t and s go to infinity.*

(1) *If $-\infty<A<+\infty$ and $0<B\leq+\infty$, or if $0\leq A<1$ and $0\leq B<+\infty$, then $T_\alpha=+\infty$ and*

$$\lim_{t\to\infty} \frac{\log \alpha(t)}{\log t} = \mu(A-1, B) \qquad (\in(0, \infty)).$$

(2) *If $B=+\infty$, then*

$$\lim_{t\to\infty} \frac{\log \alpha(t)}{\log t} = +\infty.$$

Moreover $T_\alpha=+\infty$ if $Q(t, s)\leq B(t)s$ for some continuous function $B(t)$.

(3) *If $1\leq A<+\infty$ and $B=0$, then $T_\alpha=+\infty$ and*

$$\lim_{t\to\infty} \frac{\log \alpha(t)}{\log t} = 0.$$

2. Growth of harmonic morphisms

In this section, we shall first show a generalization of the O'Neill's curvature formula for Riemannian submersions to horizontally conformal maps, and then discuss the growth of harmonic morphisms (cf. e.g., [4] [11] for the general theory on such maps).

2.1. Let $\phi\colon M\to N$ be a smooth map between Riemannian manifolds (M, g) and (N, h). For a point x of M, we set $\mathcal{V}_x=\ker(d\phi_x)$. The space $\mathcal{V}_x$ is called the vertical space at x. Let $\mathcal{H}_x$ denote the orthogonal complement of $\mathcal{V}_x$ in the tangent space T_xM. For a tangent vector $E \in T_xM$, we denote by $\mathcal{V}E$ and $\mathcal{H}E$, respectively, the vertical component and the horizontal component of E. Let $\mathcal{V}$ and $\mathcal{H}$, respectively, denote the corresponding vertical and horizontal distributions in the tangent bundle TM. We say that ϕ is *horizontally conformal* if, for each point $x \in M$ at which $d\phi_x \neq 0$, the restriction $d\phi_{x|\mathcal{H}_x}: \mathcal{H}_x \to T_{\phi(x)}N$ is conformal and surjective. Thus for some nonnegative function λ_ϕ on M, $\phi^*h_{|\mathcal{H}\times\mathcal{H}} = \lambda_\phi^2 g_{|\mathcal{H}\times\mathcal{H}}$. The function λ_ϕ is called the dilation of ϕ. Then λ_ϕ^2 is smooth on M and actually equal to $e(\phi)/n$, where $n = \dim N$. Fuglede [13] and Ishihara [20] showed that a smooth map $\phi: M \to N$ is a harmonic morphism if and only if ϕ is both harmonic and horizontally conformal. Here ϕ is called a *harmonic morphism* if for every function f harmonic on an open subset U of N, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(U)$.

A horizontal conformal map is obviously a Riemannian submersion if (and only if) the dilation is constantly equal to 1. For Riemannian submersions, we have the O'Neill's formula on curvature [28]. We will first state the corresponding formula for horizontally conformal maps, after some definitions and auxial results which follow from the same direct computations as in [28].

Let $\phi: M \to N$ be a horizontally conformal map betwen Riemannian manifolds (M, g) and (N, h). We call the points $x \in M$ where $d\phi_x = 0$ the critical points of ϕ and denote by C_ϕ (resp., M_0) the critical points (resp., the complement of C_ϕ, namely $M_0 = \{x \in M: \lambda_\phi^2(x) > 0\}$, where λ_ϕ is the dilation of ϕ). After [28], we define two tensors T and A of type (1,2) over M_0 by

$$
\begin{aligned}
T_E F &= \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \\
A_E F &= \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F\,,
\end{aligned}
$$

for vector fields E and F on M_0, where ∇ denotes the Levi-Civita connection of M. Then we have

Lemma 2.1 (cf. [28]). (1) *Both T and A are skew-symmetric operators on the tangent space of M reversing horizontal and vertical subspaces.*

(2) $T_E = T_{\mathcal{V}E}$, *and* $A_E = A_{\mathcal{H}E}$.

(3) *For vertical vector fields V and W, T is symmetric, i.e., $T_V W = T_W V$. For horizontal vector fields X and Y,*

$$A_X Y = \frac{1}{2}\mathcal{V}[X, Y] + \frac{1}{2}g(X, Y)\mathcal{V}\Lambda_\phi\,,$$

where Λ_ϕ denotes a vector field on M_0 defined by

$$\Lambda_\phi = \operatorname{grad}\log\lambda_\phi^2\,.$$

A *basic vector field* is by definition a horizontal vector field X on M_0 which is ϕ-related to a vector field X_* on N, namely $d\phi_x(X) = X_{*\phi(x)}$ for all $x \in M_0$.

Lemma 2.2 (cf. [28]). *Suppose X and Y are, respectively, basic vector fields on M_0 which are ϕ-related to X_* and Y_*. Then:*

(1) $\lambda_\phi^2 g(X, Y)=h(X_*, Y_*)\circ\phi$.

(2) $\mathcal{H}[X, Y]$ *is basic and ϕ-related to* $[X_*, Y_*]$.

(3) *The basic vector field which is ϕ-related to $\nabla^*_{X_*}Y_*$ is given by*

$$\mathcal{H}\nabla_X Y+\frac{1}{2}\{g(\Lambda_\phi, X)Y+g(\Lambda_\phi, Y)X-g(X, Y)\mathcal{H}\Lambda_\phi\},$$

where ∇^* denotes the Levi-Civita connection of N.

For linearly independent vectors E and F, we denote by $P_{E,F}$ the tangent plane spanned by these two vectors. Moreover K_M, K_N, and $\hat{K}$, respectively, stand for the sectional curvature of M, N, and the fibres (in M_0). Then making use of Lemmas 2.1 and 2.2, we can derive the curvature formula stated below for horizontally conformal maps as in [28].

Theorem 2.3. *Let $\phi: M\to N$ be a horizontally conformal map between Riemannian manifolds M and N. Then for orthonormal horizontal vectors X, Y, and orthonormal vertical vectors V, W on M_0, one has the following relations:*

(1) $K_M(P_{V,W})=\hat{K}(P_{V,W})-g(T_V V, T_W W)+g(T_V W, T_V W)$.

(2) $K_M(P_{X,V})=g(A_X V, A_X V)-g(T_V X, T_V X)+g((\nabla_X T)_V V, X)$
$$-\frac{1}{2}g(\Lambda_\phi, V)^2+\frac{1}{2}g(\nabla_V \mathcal{V}\Lambda_\phi, V).$$

(3) $K_M(P_{X,Y})=\lambda_\phi^2 K_N(P_{X_*,Y_*})-3g(A_X Y, A_X Y)$
$$-\frac{1}{4}\{g(\mathcal{V}\Lambda_\phi, \mathcal{V}\Lambda_\phi)-g(\mathcal{H}\Lambda_\phi, \mathcal{H}\Lambda_\phi)+g(\Lambda_\phi, X)^2+g(\Lambda_\phi, Y)^2\}$$
$$+\frac{1}{2}\{g(\nabla_X\mathcal{H}\Lambda_\phi, X)+g(\nabla_Y\mathcal{H}\Lambda_\phi, Y)\},$$

where $X_*=d\phi(X)$ and $Y_*=d\phi(Y)$.

2.2. Before showing a few applications of Theorem 2.3, let us first make some observations. Let $\phi: M\to N$ be a horizontally conformal map and $\nabla d\phi$ the second fundamental form of ϕ. Then the tension field $\tau(\phi)$ of ϕ, i.e., the trace of $\nabla d\phi$, is given by

$$\tau(\phi) = d\phi\left(\left(1-\frac{n}{2}\right)\Lambda_\phi-(m-n)\eta\right)$$

on M_0, where $n=\dim N$, $m=\dim M$, and η is the mean curvature normal of the fibres, namely, $\eta=(m-n)^{-1}\,\mathrm{trace}\, T_{|\mathcal{V}}$. Hence if $n\geq 3$ then it follows that any two of the conditions below imply the other one:

(i) ϕ is harmonic on M.

(ii) ϕ has minimal fibres on M_0.

(iii) grad λ_ϕ^2 is vertical

(cf. [4: Chap. 7]). We mention here a theorem by Fuglede [14], which states that a nonconstant harmonic morphism $\phi: M \to N$ is an open map and furthermore if grad λ_ϕ^2 is vertical, then the set of singular points of ϕ is empty, that is, $\phi: M \to N$ is a submersion.

Let $\phi: M \to N$ be a nonconstant harmonic morphism with grad λ_ϕ^2 vertical. Let $\eta: [a, b] \to N$ be a regular curve lying in the image of ϕ. For any point $x \in \phi^{-1}(\eta(a))$, take the horizontal lift $\bar{\eta}_x: [a, b] \to M$ of η with $\bar{\eta}_x(a) = x$, and then define a map $\mathcal{P}_{ab}: \phi^{-1}(\eta(a)) \to \phi^{-1}(\eta(b))$ by $\mathcal{P}_{ab}(x) = \bar{\eta}_x(b)$. Since grad λ_ϕ^2 is vertical, $d\lambda_\phi^2(\bar{\eta}_x(t))/dt = 0$, and hence $\lambda_\phi^2 \circ \mathcal{P}_{ab} = \lambda_\phi^2$. Moreover it is easy to see that if M is complete, then so is N and ϕ is surjective. Now suppose that the fibres are totally geodesic. Then $\mathcal{P}_{ab}$ induces an isometry between two fibers $\phi^{-1}(\eta(a))$ and $\phi^{-1}(\eta(b))$, and hence $|\mathrm{grad}\,\lambda_\phi^2| \circ \mathcal{P}_{ab} = |\mathrm{grad}\,\lambda_\phi^2|$. In fact, let $\sigma: (c, d) \to \phi^{-1}(\eta(a))$ be a regular curve in $\phi^{-1}(\eta(a))$, and define a map $F: [a, b] \times (c, d) \to M$ by $F(t, s) = \bar{\eta}_{\sigma(s)}(t)$. Set $V = \partial F/\partial s$ and $X = \partial F/\partial t$. Then obviously V is vertical and X is horizontal. Since ϕ has totally geodesic fibres, we have

$$\begin{aligned}\frac{d}{dt} g(V, V) &= 2g(\nabla_X V, V)\\ &= 2g(\nabla_V X, V)\\ &= 2g(T_V X, V) = 0\,.\end{aligned}$$

This proves that $\mathcal{P}_{ab}$ is an isometry from $\phi^{-1}(\eta(a))$ onto $\phi^{-1}(\eta(b))$. Thus we have shown

Lemma 2.4. *Let $\phi: M \to N$ be a nonconstant harmonic morphism with* grad λ_ϕ^2 *vertical.*

(1) *If M is complete, then so is N and ϕ is surjective.*

(2) *If the fibres of ϕ are totally geodesic, then the map $\mathcal{P}_{ab}: \phi^{-1}(\eta(a)) \to \phi^{-1}(\eta(b))$ defined as above induces an isometry and furthermore one has*

$$\lambda_\phi^2 \circ \mathcal{P}_{ab} = \lambda_\phi^2\,, \quad |\mathrm{grad}\,\lambda_\phi^2| \circ \mathcal{P}_{ab} = |\mathrm{grad}\,\lambda_\phi^2|\,.$$

2.3. We are now in a position to show an example of a harmonic morphism of a complete noncompact manifold of positive Ricci curvature, and then give a few applications of Theorem 2.3 (3), in connection with the example.

EXAMPLE. Let us denote by $\boldsymbol{R}^k_\eta$ the Riemannian manifold $\boldsymbol{R}^k$ equipped with a rotationally symmetric metric $g_\eta = dr^2 + \eta(r)^2 d\theta^2$. Assume that k is greater than or equal to 3 and take a smooth function $\xi_1(t)$ on $[0, \infty)$ such that $\xi_1(t) = t$ on $[0, 1]$, $\xi_1(t) > 0$ on $[1, 2]$ and $\xi_1(t) = t^{-1-\alpha}$ on $[2, \infty)$, where $\alpha \in (0, 1)$ is a constant. Then define a smooth function $\xi_2(t)$ on $[0, \infty)$ by $\xi_2(t) = \int_t^\infty \xi_1(s) ds$, and set

$$\eta(r) = \frac{1}{2}r + \frac{1}{2a}\int_0^r \xi_2(t)dt\,,$$

where $a = \xi_2(0) = \int_0^\infty \xi_1(t)dt$. We choose two constants $\alpha, \beta \in (0, 1)$ so that $k - \beta > 2 + \alpha$, and define a smooth function $f(r)$ on $\boldsymbol{R}^k$ by

$$f(r) = (b + r^2)^{(\beta+2-k)/2} + c\,,$$

where b and c are positive constants sufficiently large. Given a complete manifold F of nonnegative Ricci curvature, we consider the warped product $M = \boldsymbol{R}^k{}_\eta \times_f F$ of $\boldsymbol{R}^k{}_\eta$ and F with a warping function $f(r)$. We denote by ϕ_F the projection of M onto F. Then M has positive Ricci curvature and ϕ defines a harmonic morphism from M onto F with bounded energy density and totally geodesic fibers. We note that if F is flat and compact, then the sectional curvature of M decays quadratically in the absolute values.

The first application of Theorem 2.3 is an immediate consequence from the assertion (3):

Proposition 2.5. *Let ϕ: $M \to N$ be a (nonconstant) harmonic morphism with* grad $\lambda_\phi{}^2$ *vertical, where λ_ϕ denotes the dilation of ϕ. Suppose that the sectional curvature K_M is nonnegative. Then the sectional curvature K_N is also nonnegative on the image $\phi(M)$. Moreover if $K_N(\pi) = 0$ for a plane π tangent to N at a point of the image $\phi(M)$, then λ_ϕ is constant, i.e., ϕ is a Riemannian submersion up to homothety.*

We shall state further results.

Proposition 2.6. *Let ϕ: $M \to N$ be as in Proposition* 2.5. *Suppose that the following conditions hold:*

(1) *M is complete and N is noncompact.*
(2) *The Ricci curvature* Ricci_M *of M is nonnegative.*
(3) *The scalar curvature* Scal_N *of N is nonpositive.*
(4) *The sectional curvature K_M of M satisfies*

$$K_M \geq -\frac{c}{r_M{}^{2+\varepsilon}}\,,$$

where c and ε are positive constants, and r_M stands for the distance on M to a fixed point o of M. Then ϕ is totally geodesic and N is Ricci-flat.

See [30] for totally geodesic maps.

Proposition 2.7. *Let ϕ: $M \to N$ be as in Proposition* 2.5. *Suppose that M is complete and N is noncompact, and suppose that M has nonnegative sectional* *curvature and the fibres of ϕ are totally geodesic. Then the dilation λ_ϕ of ϕ is constant, i.e., ϕ is a Reimannian submersion up to homothety.*

Proposition 2.8. *Let ϕ: $\boldsymbol{R}^m \to N$ be a harmonic morphism of Euclidean space $\boldsymbol{R}^m$ onto a Riemannian manifold N of dimension $n \geq 3$. Suppose the fibers are totally geodesic, i.e., affine subspaces of codimension n. Then N is an affine subspace and ϕ is the orthogonal projection.*

Proof of Proposition 2.6. To bigin with, we state the Weitzenböck formula for harmonic morphisms, which reads

$$\frac{1}{2n}\Delta_M \lambda_\phi^2 = |\nabla d\phi|^2 + \lambda_\phi^2 \operatorname{trace} \mathrm{Ricci}_{M|\mathcal{H}} - \lambda_\phi^4 \mathrm{Scal}_N \circ \phi\,, \tag{2.1}$$

where trace $\mathrm{Ricci}_{M|\mathcal{H}}$ denotes the trace of the Ricci tensor of M on the horizontal distribution $\mathcal{H}$ and Scal_N stands for the scalar curvature of N. On the other hand, in the case of grad λ_ϕ^2 vertical, as we noted in 2.2, $d\phi$ has maximal rank $n = \dim N$ everywhere. Moreover by Theorem 2.3 (3), we have

$$\begin{aligned}\sum_{1\leq i<j\leq n} K_M(P_{X_i, X_j}) = \frac{\lambda_\phi^2}{2}\mathrm{Scal}_N \circ \phi - 3\sum_{2\leq i<j\leq n} g(A_{X_i}X_j, A_{X_i}X_j) \\ -\frac{n(n-1)}{8} g(\Lambda_\phi, \Lambda_\phi)\,,\end{aligned} \tag{2.2}$$

where $\{X_1, \cdots, X_n\}$ is an orthonormal basis of the horizontal subspace.

Now it turns out from the completeness of M and the assumption (4) on the sectional curvature of M that for large R, the outside of the metric ball $B_M(R)$ around o with radius R is homeomorphic to $[R, \infty) \times \partial B_M(R)$ and furthermore any pair of points belonging to the same connected component of $M \backslash B_M(R)$ can be joined by a smooth curve which lies in $M \backslash B_M(R)$ and the length of which is bounded by $c_M R$, where c_M is a positive constant depending only on M (cf. [23]). Based on this observation, we shall show that λ_ϕ must be constant. In fact, take two points x, y in a connected component, say $\mathcal{E}$, of $M \backslash B_M(R)$ and assume that $r_M(x) \geq r_M(y)$ $(\geq R)$. Let γ: $[0, a] \to M$ be a distance minimizing geodesic joining o to x $(a = r_M(x))$. Since we have by the assumptions (3), (4) and (2.2)

$$|\Lambda_\phi| \leq \frac{c_1}{(1+r_M)^{1+\delta}} \tag{2.3}$$

on M_0 for some positive constants c_1 and δ, we get

$$\frac{d}{dt}\log \lambda_\phi^2(\gamma(t)) \leq \frac{c_1}{(1+t)^{1+\delta}}$$

and hence integrating the both sides, we obtain

$$\frac{\lambda_\phi^2(\gamma(t))}{\lambda_\phi^2(\gamma(s))} \leq \exp \frac{c_1}{\delta}\left(\frac{1}{(1+s)^\delta}-\frac{1}{(1+t)^\delta}\right) \tag{2.4}$$

$(0 \leq s \leq t \leq a)$. This shows that

$$\lambda_\phi^2(x) \leq c_2 \lambda_\phi^2(o)$$

for some positive constant c_2. Let us take a smooth arclength-parametrized curve σ: $[0, b] \to M$ which lies in $M/B_M(d)$ $(d=r_M(y))$ and joins y to $\gamma(d)$. Then it follows from (2.3) again that

$$\frac{\lambda_\phi^2(\gamma(d))}{\lambda_\phi^2(y)} \leq \exp \frac{c_1 b}{(1+d)^{1+\delta}} \leq \exp \frac{c_1 c_M d}{(1+d)^{1+\delta}}\,. \tag{2.5}$$

Hence by (2.4) and (2.5) ,we have

$$\frac{\lambda_\phi^2(x)}{\lambda_\phi^2(y)} \leq \exp\left(\frac{c_1}{\delta}\left(\frac{1}{(1+d)^\delta}-\frac{1}{(1+a)^\delta}\right)+\frac{c_1 c_M d}{(1+d)^{1+\delta}}\right).$$

This implies that $\lambda_\phi^2(x)$ goes to a constant $e_{\mathcal{E}}$ as $x \in \mathcal{E}$ tends to infinity. Moreover for any point $x \in M$, there is a smooth curve $\bar{\gamma}_x$: $[0, \infty) \to M$ which is the horizontal lift of a ray γ in N starting at $\phi(x)$, since N is complete and noncompact. Then

$$\frac{d}{dt}\lambda_\phi^2(\bar{\gamma}_x(t)) = g(\nabla\lambda_\phi^2, \dot{\bar{\gamma}}_x(t)) = 0\,,$$

so that we have

$$\lambda_\phi^2(x) = \lim_{t\to\infty} \lambda_\phi^2(\bar{\gamma}_x(t)) = e_{\mathcal{E}}\,,$$

if $\bar{\gamma}_x(t)$ goes to infinity through $\mathcal{E}$. This proves that λ_ϕ is constant. Now it turns out from the assumption (2) and the Weitzenböck formula (2.1) that ϕ is totally geodesic. //

Proof of Proposition 2.7. By Theorem 2.3 (3) and Lemma 2.4 (1), N is a complete manifold of nonnegative curvature K_N. Since N is assumed to be noncompact, there exists a sequence of points $\{p_j\}$ of N and tangent planes P_i at p_i such that $K_N(P_i)$ goes to zero as i tends to infinity. Let p be any fixed point of N and take smooth paths η_i: $[0, b_i] \to N$ joining p with p_i. Then it follows from Theorem 2.3 (3) and Lemma 2.4 (2) that for any $x \in \phi^{-1}(p)$,

$$\begin{aligned}|\operatorname{grad}\lambda_\phi^2|^2(x) &= |\operatorname{grad}\lambda^2|^2 \circ \mathcal{P}_{0,b_i}(x) \\ &\leq 4\lambda_\phi^8(x) K_N(P_i)\,,\end{aligned}$$

where $\mathcal{P}_{0,b_i}$: $\phi^{-1}(p) \to \phi^{-1}(p_i)$ is the isometry as in Lemma 2.4. Thus $\operatorname{grad}\lambda_\phi^2$

vanishes everywhere on M. This completes the proof of Proposition 2.7. //

Proof of Proposition 2.8. Since the fibres are affine subspaces, it is easy to see that N is diffeomorphic to $\boldsymbol{R}^n$. Thus it turns out from Proposition 2.7 that the fibres are pallarel, and hence ϕ is totally goedesic. This completes the proof of Proposition 2.8. //

REMARK. In Propositions 2.7 and 2.8, if we assume, instead of the fibres being totally geodesic, that the second fundamental form α of the fibres satisfies

$$\sup_{\phi^{-1}(p)} |\alpha| \leq \frac{c}{r_N(p)^{1+\varepsilon}} \qquad (p \in N)$$

for some constants $c>0$ and $\varepsilon>0$, then the same assertions hold. See [5] for related results to Proposition 2.8.

2.4. As an application of the Weitzenböck formula (2.1) and a generalized maximum principle, we can derive Schwarz lemma for harmonic morphisms. To be precise, let $\phi: M \to N$ be a harmonic morphism. Suppose M is complete, the Ricci curvature of M is bounded from below by a nonpositive constant $-k_1$ and the scalar curvature of N is bounded from above by a *negative constant* $-k_2$. Then the dilation λ_ϕ of ϕ satisfies:

$$\sup \lambda_\phi{}^2 \leq n \frac{k_1}{k_2}$$

where $n = \dim N$ (cf. [32]).

We are also able to give an upper bound of the growth of a harmonic morphism, comparing the decay order of the Green functions of the domain and the target.

Proposition 2.9. *Let $\phi: M \to N$ be a (nonconstant) harmonic morphism between complete noncompact Riemannian manofolds M and N. Suppose that the conditions below hold:*

(1) *The Ricci curvature* Ricci_M *of M satisfies:*

$$\mathrm{Riccin}_M \geq -\frac{c}{r_M{}^{2+\varepsilon}}, \tag{2.6}$$

where c and ε are positive constants.

(2) *The dimension n of N is greater than or equal to 3, N is simply connected, and the sectional curvature of N is nonpositive.*

Then one has

$$\limsup_{t\to\infty} \frac{\log \mu(\phi, t)}{\log t} \leq \frac{m-2}{n-2},$$

where $m=\dim M$, $\mu(\phi, t)=\max\{r_N(\phi(x)) : x\in M, r_M(x)=t\}$, *and* r_N *stands for the distance in* N *to a fixed point* o' *of* N.

Proof. Let us consider the Dirichlet problem outside the metric ball $B_N(a)$ of N around o' with radius a: $\Delta_N u=0$ on $N\backslash B_N(a)$, $u=1$ on $\partial B_N(a)$. Then there exists a unique solution u of the problem which satisfies:

$$u(y)\leq\frac{c_1}{r_N{}^{n-2}(y)} \tag{2.7}$$

for some positive constant c_1. In fact, the Rauch comparison theorem says that the Laplaaian $\Delta_N r_N$ of the distance function r_N is bounded from below by $(n-1)r_N{}^{-1}$, so that $r_N{}^{2-n}$ is superharmonic (i.e., $\Delta_N r_N{}^{2-n}\leq 0$) (cf. e.g. [16]). Hence by the maximum principle, the solution u_s of the Dirichlet problem: $\Delta_N u_s=0$ on $B_N(s)\backslash B_N(a)$, $u_s=1$ on $\partial B_N(a)$ and $u_s=0$ on $\partial B_N(s)$, is bounded from above by $a^{n-2}r_N{}^{2-n}$. Thus $u=\lim_{s\to\infty} u_s$ satisfies inequality (2.7). By setting $u(y)=1$ on $B_N(a)$, we assume that u is defined on N. Then the composition $u\circ\phi$ is a positive superharmonic function on M. Let us now consider the same Dirichlet problem outside the metric ball $B_M(b)$ around $o\in M$ with radius b: $\Delta_M v=0$ on $M\backslash B_M(b)$, $v=1$ on $\partial B_M(b)$. Taking the radius a sufficiently large, we may assume that $B_M(b)\subset\phi^{-1}(B_N(a))$. Then there exists a unique solution v of the problem such that

$$v\leq u\circ\phi \qquad \text{on} \quad M\backslash B_M(b)\,. \tag{2.8}$$

On the other hand, by the assumption (2.6), we have

$$\frac{c_2}{r_M{}^{m-2}}\leq v \qquad \text{on} \quad M\backslash B_M(b) \tag{2.9}$$

for some positive constant c_2 (cf. [22, 24]). Thus we have by (2.7) and (2.8)

$$\log\mu(\phi, t)\leq\frac{m-2}{n-2}\log t+c_3$$

for some positive constant c_3 and large t. This proves Proposition 2.9. //

REMARKS. (1) In case $\mathrm{Ricci}_M\geq 0$, estimate (2.9) is due to Calabi [7]. Our condition (2.6) is rather technical, but from the view point of the problem discussed here, it seems to be natural (cf. [16, 22, 23, 24]). (2) Proposition 2.9 is a generalization of Proposition 8.1.1 in [4] where harmonic morphisms of homogenuous polynomials between Euclidean spaces are discussed. In particular, the upper bound in Proposition 2.9 is sharp as noted in [4].

2.5. Finally we shall show a lower bound for the growth of a harmonic morphism under certain conditions.

Let M be a complete noncompact Riemannian manifold. We assume that M is connected at infinity, namely, for any compact set K, there is a compact set $\tilde{K}$ such that $K\subset\tilde{K}$ and $M\backslash\tilde{K}$ is connected. Since M is complete, M is connected at infinity if and only if for any metric ball $B_M(t)$ centered at a point $o\in M$ of radius t, $M\backslash B_M(t)$ has only one noncompact component, say $\sum(t)$. We fix a point o of M as a base point and denote by $\operatorname{diam}(\partial\sum(t), \sum(t))$ the diameter of $\partial\sum(t)$ measured with respect to the *intrinsic distance* of the open manifold $\sum(t)$. Set

$$\delta_\infty=\limsup_{t\to\infty}\frac{1}{t}\operatorname{diam}(\partial\textstyle\sum(t),\ \sum(t))\in[0,\ \infty]\,.$$

We shall now prove the following

Theorem 2.10 *Let ϕ: $M\to N$ be a (nonconstant) harmonic morphism between complete noncompact Riemannian manifolds M and N both of which are connected at infinity. Suppose that the conditions below hold:* (1) *The Ricci curvature $Ricci_M$ of M satisfies:*

$$\operatorname{Ricci}_M\geq-\frac{c}{{r_M}^{2+\varepsilon}}$$

for some positive constants c and ε,

(2) *The dimension of N is equal to* 2 *and N has finite total curvature $\int K_N dA<+\infty$.*

Then if $\delta_\infty>0$ or $2\pi\chi(N)-\int K_N dA>0$, one has

$$\liminf_{t\to\infty}\frac{\log\mu(\phi,\ t)}{\log t}\geq\Big(\chi(N)-\frac{1}{2\pi}\int K_N\,dA\Big)\log\left[\frac{\exp(c_M\delta_\infty)+1}{\exp(c_M\delta_\infty)-1}\right],$$

where $\mu(\phi,\ t)$ is as in Proposition 2.9, *$\chi(N)$ denotes the Euler characteristic of N and δ_∞ is defined as above.*

REMARKS. (1) The classical Cohn-Vossen inequality says that $2\pi\chi(N)\geq\int K_N dA$. Finn [12] and Huber [18, 19] studied the difference: $\chi(N)-\int K_N dA/2\pi$ from the view point of conformal geometry. On the other hand, Shiohama [29] discussed the same problem in a different way and showed that

$$\begin{aligned}\chi(N)-\frac{1}{2\pi}\int K_N\,dA&=\lim_{t\to\infty}\frac{\operatorname{Length}(\partial B_N(t))^2}{4\pi\operatorname{Area}(B_N(t))}\\&=\lim_{t\to\infty}\frac{\operatorname{Area}(B_N(t))}{\pi t^2}\\&=\lim_{t\to\infty}\frac{\operatorname{Length}(\partial B_N(t))}{2\pi t}\,.\end{aligned}$$

(2) If we replace the Ricci curvature of M with the sectional curvature in the condition (1) of Theorem 2.10, then the scaled metric spheres $\{\partial B_M(t), t^{-1}d_t\}$ (where d_t is the intrinsic distance of $\partial B_M(t)$ induced from that of M) converge to a compact metric space $M(\infty)$ with respect to the Gromov-Hausdorff distance, and δ_∞ is equal to the diameter of $M(\infty)$ (cf. [23]). (3) When M has nonnegative Ricci curvature and $M \backslash B_M(t)$ is homeomorphic to $(0, \infty)\times\partial B_M(t)$ for large t, δ_∞ is finite (cf. [1]).

The proof of Theorem 2.10 is carried out by the same idea as in [24]. To begin with, we shall show

Lemma 2.11. *Let M be as in Theorem 2.10 and let h be a harmonic function defined outside a compact set K of M such that h is not bounded from below nor above. For large t, set $\overline{m}(h, t)=\max\{h(x): x\in\partial B_M(t)\}$, $\underline{m}(h, t)=\min\{h(x): x\in\partial B_M(t)\}$, and $\mu(h, t)=\overline{m}(h, t)-\underline{m}(h, t)$. Then,*

$$\limsup_{t\to\infty}\frac{\log\mu(h, t)}{\log t}\geq\log\left[\frac{\exp(c_m\delta_\infty)+1}{\exp(c_m\delta_\infty)-1}\right],$$

where c_m is a positive constant depending only on the dimension m of M.

Proof. It suffices to show the lemma in case $\delta_\infty<+\infty$. For large t, we take two points p_t, q_t of $\partial B_M(t)$ such that $h(p_t)=\overline{m}(h, t)$ and $h(q_t)=\underline{m}(h, t)$. By the maximum principle, both p_t and q_t belong to $\partial\Sigma(t)$. Join q_t to p_t by an arclength-parametrized smooth curve τ_t: $[0, a_t]\to\Sigma(t)$ whose length a_t satisfies: $a_t\leq\operatorname{diam}(\partial\Sigma(t), \Sigma(t))+\varepsilon_1(t)$, where $\varepsilon_1(t)$ goes to zero as t tends to infinity. Let us fix here a posotive integer k which is greater than δ_∞ and set $p_{t,i}=\tau_t(ia_t/3k)$ $(i=0, 1, \cdots, 3k)$. Observe that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{a_t}{t}\leq\delta_\infty\\ (2.10)\qquad &\limsup_{t\to\infty}\frac{1}{t}\operatorname{dis}_M(p_{t,i}, p_{t,i+1})\leq\frac{\delta_\infty}{3k}<\frac{1}{3}\,.\end{aligned}$$

By the assumption on h, $\overline{m}(h, t)$ is monotone increasing and hence $\overline{m}(h, (\delta_\infty+3/2)t)-h$ is a positive harmonic function on the metric ball $B_M(p_{t,i}, t/2)$ around $p_{t,i}$ with radius $t/2$. By applying the Harnack inequality due to Cheng and Yau [9] to $\overline{m}(h, (\delta_\infty+3)t)-h$, and by the assumption on the Ricci curvature, we have

$$\overline{m}(h, (\delta_\infty+3)t)-h(p_{t,i+1})\leq\exp\left(c_m(1+\varepsilon_2(t))\frac{a_t}{3kt}\right)\{\overline{m}(h, (\delta_\infty+3)t)-h(p_{t,i})\}$$

where $\varepsilon_2(t)$ goes to zero as t tends to infinity. This implies that

$$(2.11)\quad \overline{m}(h, (\delta_\infty+3)t)-\underline{m}(h, t)\leq \exp\left(c_m(1+\varepsilon_2(t)\frac{a_t}{t}\right)\{\overline{m}(h, (\delta_\infty+3)t)-\overline{m}(h, t)\}.$$

Moreover by the assumption on h, $\underline{m}(h, t)$ is monotone decreasing and $h-\underline{m}(h, (\delta_\infty+3)t)$ is a positive harmonic function on $B_M(p_{t,i}, t/2)$. Hence the same reasoning as above shows that

$$(2.12)\quad \overline{m}(h, t)-\underline{m}(h, (\delta_\infty+3)t)\leq \exp\left(c_m(1+\varepsilon_2(t))\frac{a_t}{t}\right)\{m(h, t)-m(h, (\delta_\infty+3)t)\}.$$

Now it follows from (2.11) and (2.12) that

$$\mu(h, (\delta_\infty+3)t)+\mu(t)\leq \exp\left(c_m(1+\varepsilon_2(t))\frac{a_t}{t}\right)\{\mu(h, (\delta_\infty+3)t)-\mu(t)\}\ .$$

This implies that

$$\mu(h, t)\leq \frac{\exp(c_m(1+\varepsilon_2(t))a_t/t)-1}{\exp(c_m(1+\varepsilon_2)t))a_t/t)+1}\mu(h, (\delta_\infty+3)t)\,.$$

Thus it turns out from the above inequality, (2.10) and the standard iteration argument that

$$\limsup_{t\to\infty}\frac{\log\mu(h, t)}{\log t}\geq \log\left[\frac{\exp(c_m\delta_\infty)+1}{\exp(c_m\delta_\infty)-1}\right].$$

This completes the proof of Lemma 2.11. ///

Proof. of Theorem 2.10. Since N has finite total curvature $\int K_N dA$, it follows from [18] that the end of N is conformally equivalent to that of $\boldsymbol{C}$, to be precise, there is a conformal diffeomrophism Ψ: $N\backslash K\to \boldsymbol{C}\backslash D_R$ from the complement of a compact set K in N onto that of a disk $D_R=\{z\in\boldsymbol{C}: |z|\leq R\}$. Then applying the argument in Theorems 11 and 13 of [12] and Théorèm 1 of [19] to $N\backslash K$, we have

$$(2.13)\qquad \lim_{x\in N\to\infty}\frac{\log r_N(x)}{\log|\Psi(x)|}=\chi(N)-\frac{1}{2\pi}\int K_N dA\,,$$

where r_N denotes the distance to a fixed point in N. Moreover there exist harmonic functions f and g on N such that both $|f-Re(\Psi)|$ and $|g-Im(\Psi)|$ are bounded on $N\backslash K$ (cf. [3: Chap.III)]). Define a harmonic map $\hat{\Psi}$: $N\to\boldsymbol{C}$ by $\hat{\Psi}=f+\sqrt{-1}g$. Then by Lemma 2.11, we get

$$(2.14)\qquad \liminf_{t\to\infty}\frac{\log\mu(\hat{\Psi}\circ\phi, t)}{\log t}\geq \log\left[\frac{\exp(c_m\delta_\infty)+1}{\exp(c_m\delta_\infty)-1}\right].$$

Thus Theorem 2.10 follows from (2.13) and (2.14). This completes the proof of the theorem. //

The same argument as in the proof of Theorem 2.10 will yield the following

Proposition 2.12. *Let $\phi: M \to N$ be a (nonconstant) harmonic morphism between complete noncompact Riemannian manifolds M and N both of which are connected at infinity. Suppose the following conditions hold:*

(1) *ϕ has at most linear growth.*

(2) *M has nonnegative Ricci curvature.*

(3) *The dimension of N is equal to 2, the Gaussian curvature K_N of N is nonpositive and the total curvature $\int K_N dA$ is finite.*

Then N must be flat.

Proof. We observe first that N is conformally equivalent to $\boldsymbol{C}$, because M admits no nonconstant harmonic functions. Hence there exists a conformal diffeomrophism $\Psi: N \to \boldsymbol{C}$ such that

$$\lim_{x\in N\to\infty} \frac{\log |\Psi(x)|}{\log r_N(x)} = \left(1-\frac{1}{2\pi}\int K_N dA\right)^{-1}.$$

Therefore by the condition (1), we have

$$\begin{aligned}\liminf_{t\to\infty} \frac{\log \mu(\Psi\circ\phi, t)}{\log t} &= \liminf_{t\to\infty} \frac{\log \mu(\Psi\circ\phi, t)}{\log \mu(\phi, t)} \frac{\log \mu(\phi, t)}{\log t} \\ &\leq \left(1-\frac{1}{2\pi}\int K_N dA\right)^{-1}.\end{aligned}$$

The right side of the above inequality is greater than or equal to 1, because of the Cheng's theorem [8] mentioned in the introduction. Thus we see that the total curvature of N vanishes, which implies that N must be flat. This completes the proof of Proposition 2.12.. //

Corollary. *Let $\phi: M \to N$ be as in Proposition* 2.12. *Suppose the conditions* (1) *and* (3) *of the proposition hold, and suppose that*

(2)′ *the sectional curvature K_M of M is nonnegative and decays quadratically (i.e., $0 \leq K_M \leq c/r_M^2$).*

Then ϕ is totally geodesic.

Proof. This follows from Proposition 2.12 and Theorem B in [24] quoted in the introduction. //

3. Harmonic maps of linear growth

Let M be a complete noncompact Riemannian manifold of nonnegative Ricci curvature, and let N be a Hadamard manifold. In Theorem 3.2 below, we shall give sufficient conditions for a nonconstant harmonic map $\phi: M \to N$ to be totally geodesic. The main ingrudients in the proof of Theorem 3.2 are a scaling argument and harmonic coordinates with specific properties (cf. e.g., [2,6,21,24,25]). Especially we make use of the following recent result due to Anderson [2]:

Fact 3.1 ([2: Main Lemma 2.1]). *Let $X=(X, G)$ be a Riemannian manifold (not necessarily complete) of dimension d such that*

$$|\mathrm{Ricci}_X| \leq \lambda\,, \quad \mathrm{inj}_X \geq i_0 > 0\,.$$

Then given any $C>1$ and $\sigma \in (0, 1)$, there is an $\varepsilon_0 = \varepsilon_0(\lambda, i_0, d, \sigma)$ with the following property: given any $x \in X$, there is a harmonic coordinate system $U=(u_1, \cdots, u_d)$ defined on the metric ball $B_X(\varepsilon(x))$ of X around x with radius $\varepsilon(x)$ such that $U(x)=0$ and if $G_{ij}=G(\nabla u_i, \nabla u_j)$, then $G_{ij}(x)=\delta_{ij}$ and

$$C^{-1} I \leq G(y) \leq CI \qquad (\textit{as bilinear forms})$$
$$\varepsilon(x)^{1+\sigma} \| G_{ij}(y) \|_{C^{1,\sigma}} \leq C$$

for all $y \in B_X(\varepsilon(x))$, where

$$\frac{\varepsilon(x)}{\mathrm{dis}\,(x, \partial X)} \geq \varepsilon_0 > 0\,.$$

We shall now prove the following

Theorem 3.2. *Let M be a complete noncompact manifold of nonnegative Ricci curvature and N a Hadamard manifold. Let $\phi: M \to N$ be a nonconstant harmonic map between M and N. Suppose the conditions below hold:*

(1) *The Ricci curvature Ricci_M of M satisfies*

$$(0 \leq)\, \mathrm{Ricci}_M \leq \frac{c_1}{r_M{}^2} \tag{3.1}$$

for some constant $c_1 > 0$, and moreover M has maximal volume growth, i.e.,

$$\mathrm{Vol}\,(B_M(t)) \geq c_2 t^m \tag{3.2}$$

for some constant $c_2 > 0$, where $m = \dim M$.

(2) *The sectional curvature K_N of N satisfies*

$$-\frac{c_3}{r_N{}^2} \leq K_N \qquad (\leq 0) \tag{3.3}$$

for some constant $c_3 > 0$.

(3) *The map ϕ satisfies*

$$c_4(r_M - 1) \leq r_M \circ \phi \leq c_5(r_M + 1) \tag{3.4}$$

for some constants $c_4>0$ *and* $c_5>0$*, and moreover* ϕ *has maximal rank* n $(=\dim N)$ *outside a compact set* K*, i.e.,*

$$\text{(3.5)} \qquad \textit{the rank of } d\phi_x = n \qquad (x\in M\backslash K)\,.$$

Then both M *and* N *are isometric to Euclidean m-space* $\boldsymbol{R}^m$ *and* ϕ *is an affine map.*

We make here some remarks before the proof of Theorem 3.2.

REMARKS. (1) As mentioned in the introduction, the second inequality in (3.4) is equivalent to the condition that the energy density $e(\phi)$ of ϕ is bounded on M. Moreover $e(\phi)$ is subharmonic on M, because of the Weitzenböck formura which reads:

$$\text{(3.6)} \qquad \frac{1}{2}\Delta_M e(\phi) = |\nabla d\phi|^2 + \sum_{i=1}^m \langle \phi_* \mathrm{Ricci}_M\, e_i,\ \phi_* e_i\rangle - \sum_{i,j=1}^m \langle R^N(\phi_* e_i,\ \phi_* e_j)\phi_* e_j,\ \phi_* e_i\rangle$$

where $\{e_1, \cdots, e_m\}$ is an orthonormal basis at the point under consideration on M.

(2) By a theorem due to Croke [10] and by the nonnegativity of the Ricci curvature of M and the condition (3.2) on the volume growth of M, the injectivity radious $\mathrm{inj}_M(x)$ of M at a point $x\in M$ satisfies

$$\text{(3.7)} \qquad \mathrm{inj}_M(x) \geq c_6 r_M(x)$$

for some constant $c_6>0$.

Proof of Theorem 3.2. We shall divide the proof into three steps.

Step 1. Given a positive number t, consider the scaled metric $g_t=t^{-2}g_M$ of the metric g_M of M. Denote by $B_t^M(x, a)$ (resp., $A_t^M(b, c)$) the metric ball around a point x of radius a with respect to g_t (resp., the annular domain $B_t^M(o, b)\backslash B_t^M(o, c)$ $(b<c)$). Given a number $k>1$, by (3.7), we can apply Fact 3.1 to $A_t^M(k, k^{-1})$ and find constants a and b *independent of* t such that for any $x\in A_t^M(k, k^{-1})$, there is a harmonic coordinate system $U=(u_1, \cdots, u_m)$ on $B_t^M(x, a)$ which has the property described in Fact 3.1 and the image of which contains the Euclidean m-ball $\boldsymbol{B}^m(b)$ of radius b, *i.e.* $U(B_t^M(x, a))\supset \boldsymbol{B}^m(b)=\{w\in \boldsymbol{R}^m: |w|<b\}$. Moreover since M has nonnegative Ricci curvature and maximal volume growth (3.2), we can employ the simple covering argument based on the Bishop comparison theorem (cf. [15]) and obtain a finite collection of balls $\{B_t^M(x_i, a)\}_{1\leq i\leq I}$ the union of which covers $A_t^M(k, k^{-1})$ and the number I of which is bounded by a constant *independent of* t. Therefore it turns out from a version of the Gromov's Lipschitz convergence theorem and the standard diagonal argument that arbitrary divergent sequences $\{t_j\}$ and $\{k_j\}$ respectively contain some subsequences, denoted by the same letters, for which $A_{t_j}^M(k_j, k_j^{-1})$ converges to an m-manifold $C^*(M(\infty))$ of a $C^{1,\sigma}$-metric g_∞ in the

$C^{1,\tau}$-topology $(0<\sigma<\tau<1)$ as j goes to infinity (cf. e.g., [6, 25]). As for N, the same observations are valid. To be precise, if we take two (large) numers t and $\hat{k}$, and consider the scaled metric $h_t=t^{-2}h_N$ of the metric h_N of N, then we can find positive constants $\hat{a}$ and $\hat{b}$ *independent of* t such that for any $y\in A_t^N(\hat{k}, \hat{k}^{-1})$, there is a harmonic coordinate system $V=(v_1, \cdots, v_n)$ on $B_t^N(y, \hat{a})$ which has the property described in Fact 3.1 and the image of which contains the Euclidean n-ball $\boldsymbol{B}^n(\hat{b})$ of radius $\hat{b}$. Moreover by (3.4) and (3.5), we will assume that

$$A_t^N(ck, (ck)^{-1})\subset\phi(A_t^M(k, k^{-1})\subset A_t^N(\hat{c}k, (\hat{c}k)^{-1})\subset A_t^N(\hat{k}, \hat{k}^{-1}) \tag{3.8}$$

for some constants c and $\hat{c}$ *independent of* t. Therefore it turns out again that arbitrary divergent sequences $\{t_i\}$ and $\{\hat{k}_j\}$ respectively contain some subsequences, denoted by the same letters, for which $A_{t_j}^N(\hat{k}_j, \hat{k}_j^{-1})$ converges to an n-manifold $C^*(N(\infty))$ of a $C^{1,\sigma}$-metric h_∞ in the $C^{1,\tau}$-topology as j goes to infinity. (In this case, it is easy to see that the limit Riemannian manifold $C(N(\infty))$ is a unique tangent cone at infinity of N (cf. [23]).)

Step 2. Set $e_\infty=\sup e(\phi)$ and take a family of points $\{x_t\}$ in M such that $r_M(x_t)=t$ and $e(\phi)(x_t)$ goes to e_∞ as t tends to infinity. In what follows, we will assume that $e_\infty a<\hat{a}$ and $B_t^N(\phi(x), \hat{a})\subset A_t^N(\hat{k}, \hat{k}^{-1})$ for any $x\in A_t^M(k, k^{-1})$, where a, $\hat{a}$, k, and $\hat{k}$ are as in Step 1. Let $U_t=(u_1, \cdots, u_m)$ be a harmonic coordinate system on $B_t^M(x_t, a)$ with the property described in Fact 3.1 and let $V_t=(v_1, \cdots, v_n)$ be such a system on $B_t^N(\phi(x_t), \hat{a})$. Set $\phi_t=V_t\circ\phi\circ U_t^{-1}$ and assume that $\phi_t(\boldsymbol{B}^m(b))\subset\boldsymbol{B}^n(\hat{b})$, where b and $\hat{b}$ are as in Step 1. Then the components $\{\phi_t^\alpha\}_{\alpha=1,\cdots,n}$ of ϕ_t satisfy

$$\textstyle\sum_{i,k=1}^m g_t^{ij}\left(\frac{\partial^2\phi_t^\alpha}{\partial u_i\partial u_j}-\sum_{l=1}^m {}^M\Gamma_{t,ij}^{\,l}\frac{\partial\phi_t^\alpha}{\partial u_l}+\sum_{\beta,\gamma=1}^n {}^N\Gamma_{t,\beta\gamma}^{\,\alpha}\frac{\partial\phi_t^\beta}{\partial u_i}\frac{\partial\phi_t^\gamma}{\partial u_j}\right)=0\,,$$

where $\{{}^M\Gamma_{t,ij}^{\,k}\}$ (resp., $\{{}^N\Gamma_{t,\beta\gamma}^{\,\alpha}\}$) are the Cristoffel symbols of g_t (resp. h_t) with respect to the harmonic coordinates U_t (resp., V_t). It follows from the standard elliptic regularity theory that the $C^{2,\alpha}$-norms of ϕ_t^α are bounded uniformly in t. Thus for any divergent sequence $\{t_j\}$, there exist a subsequence, denoted by the same letters $\{t_j\}$, a $C^{1,\sigma}$ metric g_∞ on $\boldsymbol{B}^m(b)$, a $C^{1,\sigma}$ metric h_∞ on $\boldsymbol{B}_u(\hat{b})$ and a $C^{2,\sigma}$ map ϕ_∞: $\boldsymbol{B}^m(b)\to\boldsymbol{B}^n(\hat{b})$ such that $U_{t_j*}g_{t_j}$ (resp., $V_{t_j*}h_{t_j}$) converges to g_∞ (resp., h_∞) in the $C^{1,\tau}$ topology and so does $\{\phi_{t_j}\}$ to ϕ_∞ in the $C^{2,\tau}$ topology. Moreover if we use the apriori estimates in the Sobolev space, then we may assume that the limit metrics g_∞ and h_∞ have, respectively, the curvature tensors R_∞^M and R_∞^N in L^2-sense, to which the curvature tensors of g_{t_j} and h_{t_j}, respectively, converge *weakly* in L^2-sense. Thus it follows that the limit map ϕ_∞ defines a harmonic map with respect to the limit metrics and the energy density $e(\phi_\infty)$ of ϕ_∞ satisfies weakly the Weitzenböck formula (3.6) in L^2-sense. Applying the maximum principle to $e(\phi_\infty)$, we see that $e(\phi_\infty)$ must be constant, because $e(\phi_\infty)(o)=\max e(\phi_\infty)=e_\infty$. This implies that ϕ_∞ is totally geodesic, i.e., the second fundamental form of ϕ_∞ with respect to the limit metrics vanishes identically. Thus in particular, it turns out that $e(\phi)(x)$ converges to the constant e_∞ as $x \in M$ tends to infinity and the second fundamental form $\nabla d\phi$ of satisfies: $\lim_{x\to\infty} r_M(x)|\nabla d\phi|(x)=0$.

Step 3. Let $\{t_j\}$, $\{k_j\}$ and $\{\hat{k}_j\}$ be divergent sequences as in Step 1 such that (3.8) is kept for any j, and $A^M_{t_j}(k_j, k_j^{-1})$ and $A^N_{t_j}(\hat{k}_j, \hat{k}_j^{-1})$ converge respectively to the Riemannian manifolds $C^*(M(\infty))$ and $C^*(N(\infty))$ as j goes to infinity. As observed in Step 2, we may assume that as j goes to infinity, the restriction ϕ_{t_j} of ϕ to $A^M_{t_j}(k_j, k_j^{-1})$ converges to a totally geodesic map, say again ϕ_∞, from $C^*(M(\infty))$ to $C^*(N(\infty))$. Moreover it follows from (3.8) that ϕ_∞ is a diffeomrophism. Hence by the Weitzenböck formula (3.6), we see that both $C^*(M(\infty))$ and $C^*(N(\infty))$ are isometric to $\boldsymbol{R}^m\setminus\{o\}$. This shows in particular that $m=n$ and

$$\lim_{t\to\infty}\frac{\mathrm{Vol}(B_M(t))}{\omega_m t^m}=\lim_{t\to\infty}\frac{\mathrm{Vol}(B_N(t))}{\omega_m t^m}=1$$

where ω_m denotes the volume of the unit sphere in $\boldsymbol{R}^m$ (cf. [24,25]). Thus it turns out from the equality discussion of Bishop and Rauch comparisom theorems that both M and N are isometric to $\boldsymbol{R}^m$. Hence it is obvious that ϕ is an affine map. This completes the proof of Theorem 3.2. //

References

[1] U. Abresch and D. Gromoll: *On complete manifolds with non-negative Ricci curvature,* J. Amer. Math. Soc., **3** (1990), 355–374.

[2] M.T. Anderson: *Convergence and rigidity of manifolds under Ricci curvature bounds,* to appear in Invent. math.

[3] L.H. Ahlfors and L. Sario: Riemann Surfaces, Princeton University Press, Princeton N.J., 1960.

[4] P. Baird: Harmonic Maps with Symmetry, Harmonic Morphisms and Deformations of Metrics, Research Notes in Math. 87, Pitman, 1983.

[5] P. Baird and J.C. Wood: *Bernstein theorems for harmonic morphisms from $\boldsymbol{R}^3$ and S^3*, Math. Ann. **280** (1988), 579–603.

[6] S. Bando, A. Kasue and H. Nakajima: *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth,* Invent. math. **97** (1989), 313–349.

[7] E. Calabi: *An extension of E. Hopf's maximum principle with an application to Riemannian geometry,* Duke Math. J. **25** (1958), 45–56.

[8] S.Y. Cheng: *Liouville theorems for harmonic maps,* Proc. Symp. Pure Math. 36 (1980), 119–128.

[9] S.Y. Cheng and S.T. Yau: *Differential equations on Riemannian manifolds and their geometric applications,* Comm. Pure Appl. Math. **28** (1975), 333–354.

[10] C. Croke: *Some isoperimetric inequalities and eigenvalue estimates,* Ann. Sci. Ecole Norm. Sup. **14** (1980), 249–260.

[11] J. Eells and L. Lemaire: *Another report on harmonic maps,* Bull. London Math. Soc. **20** (1988), 385–524.

[12] R. Finn: *On a class of conformal metrics, with application to differential geometry in the large,* Comment. Math. Helv. **40** (1965), 1–30.

[13] B. Fuglede: *Harmonic morphisms between Riemannian manifolds,* Ann. Inst. Fourier, Grenoble **28** (1978), 107–144.

[14] ————: *A criterion of non-vanishing differential of a smooth map,* Bull. London Math. Soc. **14** (1982), 98–102.

[15] M. Gromov: Structure métrique pour les variétés riemanniennes, redige par J. Lafontaine et P. Pansu, Textes Math. No. 1, Edic-Fernand and Nathan, Paris, 1981.

[16] R.E. Greene and H. Wu: Function Theory on Manifolds Which Possess a Pole, Lecture Notes in Math. 699, Springer-Verlag, 1979.

[17] S. Hildebrandt: *Harmonic mappings of Riemannian manifolds,* Harmonic maps and Minimal Immersions (Montecatini 1984), Lecture Notes in Math. 1161, 1–117, Springer, Berlin, 1985.

[18] A. Huber: *On subharmonic functions and differential geometry in the large,* Comment. Math. Helv. **32** (1957), 13–72.

[19] ————: *Métrique conformes complètes et singularités isoleés de fonctions sous-harmoniques,* C.R. Acad. des Sci. Paris, **260** (1965), 6267–6268.

[20] T. Ishihara: *A mapping of Riemannian manifolds which preserves harmonic functions,* J. Math. Kyoto Univ. **19** (1979), 215–229.

[21] J. Jost and H. Karcher: *Geometrische Methoden zur Gewinnung von a priori-Schranken für harmonische Abbildungen,* Manuscripta Math **40** (1982), 27–77.

[22] A. Kasue: *A Laplacian comparison theorem and function theoretic properties of a complete Riemannian manifold,* Japan. J. Math. **8** (1982), 309–341.

[23] ————: *A compactification of a manifold with asymptotically nonnegative curvature,* Ann. Ecole Norm. Sup. Paris, **21** (1988), 593–622.

[24] ————: *Harmonic functions with growth conditions on a manifold of asymptotically nonnegative curvature* II, Recent Topics in Differential and Analytic Geometry, ed. by T. Ochiai, Advanced Studies in Pure Math. 18, 1989.

[25] ————: *A convergence theorem for Riemannian manifolds and some applications,* Nagoya Math. J. **114** (1989), 21–51.

[26] P. Li and L.F. Tam: *Linear growth harmonic functions on a complete manifold,* J. Diff. Geometry **29** (1989), 421–425.

[27] P. Nabonnard: *Sur les variétés riemanniannes complétes a courbure de Ricci positive,* C.R. Acad. des Sci. Paris, **291** (1981), 591–593.

[28] B. O'Neill: *The fundamental equations of a submersion,* Mich. Math. J. **13** (1966), 459–469.

[29] K. Shiohama: *Total curvature and minimal areas of complete open surfaces,* Proc. Amer. Math. Soc. **94** (1985), 310–316.

[30] J. Vilms: *Totally geodesic maps,* J. Diff. Geometry **4** (1970), 73–79.

[31] H. Wu: The Bochner Technique in Differential Geometry, E. Horwood, London, 1987.

[32] S.T. Yau: *A general Schwarz lemma for Kähler manifolds,* Amer. J. Math. **100** (1978), 197–203.

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