Fujita, Y.
Osaka J. Math.
27 (1990), 797-804

INTEGRODIFFERENTIAL EQUATION WHICH INTERPOLATES THE HEAT EQUATION AND THE WAVE EQUATION (II)

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(Received November 27, 1989)

1. Introduction

In the present paper we are concerned with the integrodifferential equation

$$(\mathrm{IDE})_\alpha \qquad u(t,x)=\phi(x)+\frac{t^{\alpha/2}}{\Gamma\left(1+\frac{\alpha}{2}\right)}\psi(x)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\Delta u(s,x)\,ds \qquad t>0,\ x\in\boldsymbol{R}$$

for $1\leq\alpha\leq 2$. Here $\Gamma(x)$ is the gamma function and $\Delta=(\partial/\partial x)^2$. When $\psi\equiv 0$, $(\mathrm{IDE})_1$ is reduced to the heat equation. For $\alpha=2$, $(\mathrm{IDE})_2$ is just the wave equation and its solution $u_2(t,x)$ has the expression called the d'Alembert's formula:

$$u_2(t,x)=\frac{1}{2}[\phi(x+t)+\phi(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}\psi(y)\,dy\,.$$

The present paper is the continuation of [6]; the aim of the present paper, which is different from that of [6], is to investigate the structure of the solution of $(\mathrm{IDE})_\alpha$ by its decomposition for every α, $1\leq\alpha\leq 2$.

In Theorem B below, we shall show that $(\mathrm{IDE})_\alpha$ has the unique solution $u_\alpha(t,x)$ $(1\leq\alpha\leq 2)$ expressed as

$$(1)\qquad u_\alpha(t,x)=\frac{1}{2}\boldsymbol{E}[\phi(x+Y_\alpha(t))+\phi(x-Y_\alpha(t))]+\frac{1}{2}\boldsymbol{E}\int_{x-Y_\alpha(t)}^{x+Y_\alpha(t)}\psi(y)\,dy$$

where $Y_\alpha(t)$ is continuous, nondecreasing and nonnegative stochastic process with Mittag-Leffler distributions of order $\alpha/2$, and $\boldsymbol{E}$ stands for the expectation. We remark that the expression (1) has the same form as that of the d'Alembert's formula.

In Theorem A below, we shall consider the decomposition of $u_\alpha(t,x)$ $(1\leq\alpha\leq 2)$. We decompose u_α into two functions u_α^+ and u_α^- defined by

$$(2)\qquad u_\alpha^+(t,x)=\frac{1}{2}\boldsymbol{E}[\phi(x-Y_\alpha(t))-\Psi(x-Y_\alpha(t))]$$

and

(3) $$u_\alpha^-(t,x)=\frac{1}{2}\boldsymbol{E}[\phi(x+Y_\alpha(t))+\Psi(x+Y_\alpha(t))]$$

where $\Psi(x)=\int_0^x \psi(y)\,dy$. It is easy to see that $u_\alpha=u_\alpha^+ + u_\alpha^-$. The function u_α^+, which consists of the element $x-Y_\alpha(t)$, represents the disturbance moving into the positive direction of the x—axis. Similarly, the function u_α^-, which consists of the element $x+Y_\alpha(t)$, represents the disturbance moving into the negative direction of the x—axis. Furthermore the functions u_α^+ and u_α^- are characterized as the unique solutions of the integrodifferential equations

$$(\mathrm{IDE})^+_{\alpha/2}\quad u(t,x)+\frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{(\alpha/2)-1}\nabla u(s,x)\,ds=\frac{1}{2}[\phi(x)-\Psi(x)]$$

and

$$(\mathrm{IDE})^-_{\alpha/2}\quad u(t,x)-\frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{(\alpha/2)-1}\nabla u(s,x)\,ds=\frac{1}{2}[\phi(x)+\Psi(x)]$$

respectively, where $\nabla=(\partial/\partial x)$. Let us denote by I^ρ the Riemann-Liouville integral operator of order $\rho>0$ defined by $I^\rho f(t)=\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}f(s)ds$. It has the property such that $(1-I^\alpha\Delta)=(1\pm I^{\alpha/2}\nabla)(1\mp I^{\alpha/2}\nabla)$, where 1 stands for the identity operator (see Proposition 2 below). Thus the above decomposition of u_α corresponds to the decomposition of the operator $(1-I^\alpha\Delta)$ of $(\mathrm{IDE})_\alpha$ into the product of two operators $(1+I^{\alpha/2}\nabla)$ of $(\mathrm{IDE})^+_{\alpha/2}$ and $(1-I^{\alpha/2}\nabla)$ of $(\mathrm{IDE})^-_{\alpha/2}$.

The present paper is organized as follows. First, using the probability theory, we shall show that $u_\alpha^\pm$ is the unique solution of $(\mathrm{IDE})^\pm_{\alpha/2}$ (Theorem A). Next, using this result and the above decomposition of $(1-I^\alpha\Delta)$, we shall show that u_α is the unique solution of $(\mathrm{IDE})_\alpha$ (Theorem B).

2. Theorems and their proofs

Let $X_\alpha(t)$ $(1\leq\alpha\leq 2)$ be the stable process defined on a probability space $(\Omega,\mathcal{F},\boldsymbol{P})$ such that its characteristic function $\boldsymbol{E}\exp\{is\,X_\alpha(t)\}$ $(s\in\boldsymbol{R},\ t\geq 0)$ is givne by

$$\boldsymbol{E}\exp\{is\,X_\alpha(t)\}=\exp\{-t|s|^{2/\alpha}e^{-(\pi i/2)(2-2/\alpha)\mathrm{sgn}(s)}\}\,.$$

We choose a version such that $X_\alpha(t)$ is right continuous and has left limit. We remark that $X_2(t)=t$ and $X_1(t)$ is a Brownian motion with mean 0 and variance $2t$. Put

$$Y_\alpha(t)=\sup_{0\leq s\leq t}X_\alpha(s).$$

Proposition 1. *Let* $1\leq\alpha\leq 2$.

(I) *With probability* 1, $Y_\alpha(t)$ *is continuous, nondecreasing and nonnegative process with* $Y_\alpha(0)=0$.
(II) $Y_\alpha(t)$ *has the Mittag-Leffler distributions of order* $\alpha/2$:

$$\boldsymbol{E}\exp\{-s\,Y_\alpha(t)\}=\sum_{n=0}^{\infty}\frac{(-s\,t^{\alpha/2})^n}{\Gamma\left(1+\frac{n\alpha}{2}\right)}\quad(s\in\boldsymbol{C},\ t\geq 0)\,. \tag{4}$$

(III) *For every* $A\geq 0$, *there exists some constant* $C(A,\alpha)>0$ *such that*

$$\boldsymbol{E}\exp\{A\,Y_\alpha(t)\}\leq C(A,\alpha)\exp\{A^{2/\alpha}t\}\quad(t\geq 0)\,.$$

Proof. (I) By the definition of $Y_\alpha(t)$, it is nondecreasing. Since $X_\alpha(0)=0$, $Y_\alpha(t)$ is nonnegative process with $Y_\alpha(0)=0$. It remains to prove that $Y_\alpha(t)$ is continuous. The stochastic process $X_\alpha(t)$ has no positive jump, i.e., $X_\alpha(t-)\geq X_\alpha(t)$ for every $t>0$ (cf. pp. 276 of [2]). Thus, we have $Y_\alpha(t-)\geq Y_\alpha(t)$. The opposite inequality $Y_\alpha(t-)\leq Y_\alpha(t)$ is trivial, so that $Y_\alpha(t-)=Y_\alpha(t)$ and $Y_\alpha(t)$ is left continuous. The right continuity of $Y_\alpha(t)$ is obvious. Hence $Y_\alpha(t)$ is continuous a.s..
(II) This is trivial for $\alpha=2$, since $Y_2(t)=t$. Thus, assume that $1\leq\alpha<2$. By Proposition 1 of [2], the equality (4) holds for $s,t\geq 0$. Here we remark that the constant c_1 of §1 (9) of [2] is equal to 1 in our case. Since $\limsup_{n\to\infty}[\Gamma(1+\frac{n\alpha}{2})]^{-1/n}=0$ by Stirling's formula, the right hand side of (4) is an analytic function of $s\in\boldsymbol{C}$ for every $t\geq 0$. On the other hand, by Proposition 3b of [2], there exist some constants $A_\alpha, B_\alpha>0$ such that

$$\boldsymbol{P}(Y_\alpha(t)\geq x)\sim A_\alpha(xt^{-\alpha/2})^{-\frac{1}{2-\alpha}}\exp\{-B_\alpha(xt^{-\alpha/2})^{-\frac{2}{2-\alpha}}\}\quad(x\to\infty,\ t>0)\,,$$

so that the left hand side of (4) is also an analytic function of $s\in\boldsymbol{C}$ for every $t\geq 0$ (for $t=0$, this is trivial). Therefore the equality (4) holds for $s\in\boldsymbol{C}$ and $t\geq 0$ by the theorem of identity.
(III) By (II), we have for $t\geq 0$

$$\boldsymbol{E}\exp\{A\,Y_\alpha(t)\}=\sum_{n=0}^{\infty}\frac{(A\,t^{\alpha/2})^n}{\Gamma\left(1+\frac{n\alpha}{2}\right)}\,.$$

By (10) of pp. 208 of [4], it holds that

$$\sum_{n=0}^{\infty}\frac{(A\,t^{\alpha/2})^n}{\Gamma\left(1+\frac{n\alpha}{2}\right)}\sim\frac{2}{\alpha}\exp\{A^{2/\alpha}t\}\quad(t\to\infty)\,.$$

Thus the assertion (III) follows easily.
This completes the proof of Proposition 1. □

Now we shall consider the solutions of $(\mathrm{IDE})_\alpha$ and $(\mathrm{IDE})^{\pm}_{\alpha/2}$.

DEFINITION. (I) The function $u=u(t, x)$ on $[0, \infty)\times\boldsymbol{R}$ is said to be a solution of $(\mathrm{IDE})_\alpha$, if u and Δu are continuous on $[0, \infty)\times\boldsymbol{R}$ and u satisfies $(\mathrm{IDE})_\alpha$ for every $(t, x)\in(0, \infty)\times\boldsymbol{R}$.
(II) The function $u=u(t, x)$ on $[0, \infty)\times\boldsymbol{R}$ is said to be a solution of $(\mathrm{IDE})^{\pm}_{\alpha/2}$, if u and ∇u are continuous on $[0, \infty)\times\boldsymbol{R}$ and u satisfies $(\mathrm{IDE})^{\pm}_{\alpha/2}$ for every $(t, x)\in(0, \infty)\times\boldsymbol{R}$.

REMARK. In [6] we defined the solution of $(\mathrm{IDE})_\alpha$, adding the condition that the solution was in $C([0, \infty)\colon \mathcal{S}(\boldsymbol{R}))$ ($\mathcal{S}(\boldsymbol{R})$: the Schwartz class). This condition is not imposed in the present paper.

We shall construct the solutions of $(\mathrm{IDE})_\alpha$ and $(\mathrm{IDE})^{\pm}_{\alpha/2}$ in the following spaces. Let $\mathcal{E}(\boldsymbol{R})$ be the space of the continuous functions f on $\boldsymbol{R}$ such that there exist some constants $A, C>0$ satisfying $|f(x)|\leq C\,e^{A|x|}$ for any $x\in\boldsymbol{R}$, and $\mathcal{E}([0, \infty)\times\boldsymbol{R})$ the space of the continuous functions v on $[0, \infty)\times\boldsymbol{R}$ such that there exist some constants $A', C'>0$ satisfying $|v(t, x)|\leq C'\,e^{A'[t+|x|]}$ for any $(t, x)\in[0, \infty)\times\boldsymbol{R}$. For every positive integer m, define $\mathcal{E}^m(\boldsymbol{R})$ and $\mathcal{E}^{0,m}([0, \infty)\times\boldsymbol{R})$ by $\mathcal{E}^m(\boldsymbol{R})=\{f\colon \nabla^j f\in\mathcal{E}(\boldsymbol{R})$ for $0\leq j\leq m\}$ and $\mathcal{E}^{0,m}([0, \infty)\times\boldsymbol{R})=\{v\colon \nabla^j v\in\mathcal{E}([0, \infty)\times\boldsymbol{R})$ for $0\leq j\leq m\}$ respectively. Throughout this paper, we use the notation $\Psi(x)=\int_0^x \psi(y)\,dy$ (we assume that ψ is always locally integrable on $\boldsymbol{R}$).

The main results of the present paper are the following:

Theorem A. *Let ϕ and Ψ be in $\mathcal{E}^1(\boldsymbol{R})$. Then, for $1\leq\alpha\leq 2$, u^+_α defined by* (2) *and u^-_α defined by* (3) *are the unique solutions of* $(\mathrm{IDE})^+_{\alpha/2}$ *and* $(\mathrm{IDE})^-_{\alpha/2}$ *respectively in* $\mathcal{E}^{0,1}([0, \infty)\times\boldsymbol{R})$.

Theorem B. *Let ϕ and Ψ be in $\mathcal{E}^2(\boldsymbol{R})$. Then, for $1\leq\alpha\leq 2$, u_α defined by* (1) *is the unique solution of* $(\mathrm{IDE})_\alpha$ *in* $\mathcal{E}^{0,2}([0, \infty)\times\boldsymbol{R})$. *Furthermore it holds that* $u_\alpha=u^+_\alpha+u^-_\alpha$.

REMARK. For $\phi\in\mathcal{S}(\boldsymbol{R})$ and $\psi\equiv 0$, the expression (1) and the one obtained in (1.6) of [6] coincide mutually. This is due to the simple properties of the stable processes (see §2 (17) and Proposition 1 (iii) of [2]).

To prove Theorem A, we need a lemma.

Lemma *Let $1\leq\alpha\leq 2$, and A a positive constant such that* $\sup_{x\in\boldsymbol{R}}\{e^{-A|x|}|f(x)|\}<\infty$ *for $f\in\mathcal{E}(\boldsymbol{R})$. Then we have for $\lambda>A^{2/\alpha}$*

$$(5)\qquad \int_0^\infty e^{-\lambda t}\,\boldsymbol{E}[f(Y_\alpha(t))]\,dt=\lambda^{(\alpha/2)-1}\int_0^\infty f(y)\,e^{-y\lambda^{\alpha/2}}\,dy\,.$$

Proof. Since

$$\int_0^\infty e^{-\lambda t}\,\boldsymbol{E}[f(Y_\alpha(t))]\,dt=\int_0^\infty f(y)\,d_y\Big[\int_0^\infty e^{-\lambda t}\,\boldsymbol{P}(Y_\alpha(t)\leq y)\,dt\Big]\,,$$

it is sufficient to show that for every $y \geq 0$

$$\int_0^\infty e^{-\lambda t} \boldsymbol{P}(Y_\alpha(t) \leq y)\, dt = \frac{1}{\lambda}[1-e^{-y\lambda^{\alpha/2}}]. \tag{6}$$

To prove (6), we shall calculate the Laplace-Stieltjes transform of the both sides of (6). For $s \in (0, \lambda^{\alpha/2})$, we have by Proposition 1 and the dominated convergence theorem

$$\begin{aligned}
&\int_0^\infty e^{-sy}\, d_y\Big[\int_0^\infty e^{-\lambda t} \boldsymbol{P}(Y_\alpha(t) \leq y)\, dt\Big] \\
&= \int_0^\infty e^{-\lambda t} \boldsymbol{E} \exp\{-s\, Y_\alpha(t)\}\, dt \\
&= \int_0^\infty e^{-\lambda t} \Big\{\sum_{n=0}^\infty \frac{(-s\, t^{\alpha/2})^n}{\Gamma\left(1+\frac{n\alpha}{2}\right)}\Big\}\, dt \\
&= \sum_{n=0}^\infty \frac{(-s)^n}{\Gamma\left(1+\frac{n\alpha}{2}\right)} \int_0^\infty e^{-\lambda t}\, t^{(\alpha n/2)}\, dt \\
&= \frac{\lambda^{(\alpha/2)-1}}{s+\lambda^{\alpha/2}} \\
&= \int_0^\infty e^{-sy}\, d_y \Big\{\frac{1}{\lambda}[1-e^{-y\lambda^{\alpha/2}}]\Big\}.
\end{aligned}$$

Thus we have obtained for $s \in (0, \lambda^{\alpha/2})$

$$\int_0^\infty e^{-sy}\, d_y\Big[\int_0^\infty e^{-\lambda t} \boldsymbol{P}(Y_\alpha(t) \leq y)\, dt\Big] = \int_0^\infty e^{-sy}\, d_y \Big\{\frac{1}{\lambda}[1-e^{-y\lambda^{\alpha/2}}]\Big\}. \tag{7}$$

Since the both sides of (7) are analytic functions of s in $\{s \in \boldsymbol{C} : \mathcal{Re}\, s > 0\}$, the equality (7) holds for s, $\mathcal{Re}\, s > 0$ by the theorem of identity. Note that (6) holds for $y=0$. Then the uniqueness of the Laplace-Stieltjes transform leads to (6). This completes the proof of Lemma. □

Proof of Theorem A. We shall prove the case u_α^+ only, since the case u_α^- can be proved similarly. By Proposition 1, it is clear that u_α^+ belongs to $\mathcal{E}^{0,1}([0, \infty) \times \boldsymbol{R})$. Let $A > 0$ be a constant such that $\sup_{x \in \boldsymbol{R}} \{e^{-A|x|}[|\phi(x)| + |\Psi(x)|]\} < \infty$. Applying the Laplace transform to u_α^+ and using Lemma, we have for $\lambda > A^{2/\alpha}$

$$U_\alpha^+(\lambda, x) = \frac{\lambda^{(\alpha/2)-1}}{2} \int_0^\infty [\phi(x-y) - \Psi(x-y)]\, e^{-y\lambda^{\alpha/2}}\, dy \tag{8}$$

where $U_\alpha^+(\lambda, x) = \int_0^\infty e^{-\lambda t} u_\alpha^+(t, x)\, dt$. Using the change of the variable $z = x-y$ in (8), we get

$$(9)\qquad e^{x\lambda^{\alpha/2}}\, U_\alpha^+(\lambda, x) = \frac{\lambda^{(\alpha/2)-1}}{2}\int_{-\infty}^{x} [\phi(z)-\Psi(z)]\, e^{z\lambda^{\alpha/2}}\, dz\,.$$

Differentiating the both sides of (9) with respect to x, we have for $(\lambda, x)\in (A^{2/\alpha}, \infty)\times \boldsymbol{R}$

$$(10)\qquad U_\alpha^+(\lambda, x)+\frac{\nabla U_\alpha^+(\lambda, x)}{\lambda^{\alpha/2}} = \frac{1}{2\lambda}\,[\phi(x)-\Psi(x)]\,.$$

Since $\frac{1}{\lambda^\rho}$ $(\rho>0)$ is the Laplace transform of $\frac{t^{\rho-1}}{\Gamma(\rho)}$, the inverse Laplace transform of (10) shows that u_α^+ satisfies $(\mathrm{IDE})_{\alpha/2}^+$ for every $(t, x)\in(0, \infty)\times\boldsymbol{R}$. Therefore it is a solution of $(\mathrm{IDE})_{\alpha/2}^+$ in $\mathcal{E}^{0,1}([0, \infty)\times\boldsymbol{R})$. It remains to prove the uniqueness. It is sufficient to show that if $v\in\mathcal{E}^{0,1}([0, \infty)\times\boldsymbol{R})$ satisfies

$$(11)\qquad v(t, x)+\frac{1}{\Gamma(\alpha/2)}\int_0^t (t-s)^{(\alpha/2)-1}\, \nabla v(x, s)\, ds = 0$$

for $(t, x)\in(0, \infty)\times\boldsymbol{R}$, then $v\equiv 0$. Applying the Laplace transform to (11), we get

$$(12)\qquad V(\lambda, x)+\lambda^{-\alpha/2}\, \nabla V(\lambda, x) = 0 \quad (\lambda, x)\in(B, \infty)\times\boldsymbol{R}$$

where $V(\lambda, x)=\int_0^\infty e^{-\lambda t}\, v(t, x)\, dt$ and $B>0$ is a constant such that $\sup_{t\geq 0, x\in\boldsymbol{R}}\{e^{-B(t+|x|)}[|v(t, x)|+|\nabla v(t, x)|]\}<\infty$. By (12), the function $V(\lambda, x)\exp[\lambda^{\alpha/2} x]$ depends on only $\lambda\in(B, \infty)$. Put $C(\lambda)=V(\lambda, x)\exp[\lambda^{\alpha/2} x]$. Since there exists a constant C such that $|V(\lambda, x)|\leq\frac{C\, e^{|B|x}}{\lambda-B}$ on $(B, \infty)\times\boldsymbol{R}$, we have for $\lambda>B_0\equiv\max\{B, B^{2/\alpha}\}$

$$|C(\lambda)|\leq\frac{C}{\lambda-B}\exp\{\lambda^{\alpha/2} x+B|x|\}\to 0 \quad (x\to-\infty)\,,$$

so that $V(\lambda, x)\equiv 0$ on $(B_0, \infty)\times\boldsymbol{R}$. The uniqueness of the Laplace transform leads to $v(t, x)\equiv 0$ on $[0, \infty)\times\boldsymbol{R}$. This completes the proof. □

Next we shall prove Theorem B. For $f\in\mathcal{E}([0, \infty)\times\boldsymbol{R})$, define the Riemann-Liouville integral operator $I^\rho f(\rho>0)$ by

$$I^\rho f(t, x) = \begin{cases} \dfrac{1}{\Gamma(\rho)}\displaystyle\int_0^t (t-s)^{\rho-1} f(s, x)\, ds & (t>0) \\ 0 & (t=0)\,. \end{cases}$$

The following proposition is crucial to prove Theorem B.

Proposition 2. *Let* $f\in\mathcal{E}^{0,2}([0, \infty)\times\boldsymbol{R})$. *Then*

$$(1-I^\alpha\,\Delta) f(t, x) = (1\pm I^{\alpha/2}\,\nabla)\,(1\mp I^{\alpha/2}\,\nabla) f(t, x)$$

for any $(t, x)\in[0, \infty)\times\boldsymbol{R}$, *where* 1 *stands for the identity operator.*

Since the proof is obvious, we omit it (cf. [3]).

Proof of Theorem B. Since ϕ and Ψ belong to $\mathcal{E}^2(\boldsymbol{R})$, both u_α^+ and u_α^- belong to $\mathcal{E}^{0,2}([0, \infty)\times\boldsymbol{R})$ by Proposition 1. By Theorem A, they satisfy for $(t, x)\in(0, \infty)\times\boldsymbol{R}$

$$(\text{IDE})_{\alpha/2}^{\pm} \qquad (1\pm I^{\alpha/2}\nabla)\, u_\alpha^{\pm}(t, x) = \frac{1}{2}\,[\phi(x)\mp\Psi(x)]\,.$$

By Proposition 2, we have on $(0, \infty)\times\boldsymbol{R}$

$$\begin{aligned}
&(1-I^\alpha \Delta)\,(u_\alpha^+ + u_\alpha^-)\\
&= (1-I^{\alpha/2}\nabla)\,[(1+I^{\alpha/2}\nabla)\,u_\alpha^+]+(1+I^{\alpha/2}\nabla)\,[(1-I^{\alpha/2}\nabla)\,u_\alpha^-]\\
&= \frac{1}{2}\,(1-I^{\alpha/2}\nabla)\,[\phi(x)-\Psi(x)]+\frac{1}{2}\,(1+I^{\alpha/2}\nabla)\,[\phi(x)+\Psi(x)]\\
&= \phi(x)+I^{\alpha/2}\,\psi(x)\\
&= \phi(x)+\frac{t^{\alpha/2}}{\Gamma\left(1+\frac{\alpha}{2}\right)}\psi(x)\,.
\end{aligned}$$

Since $u_\alpha=u_\alpha^+ + u_\alpha^-$, the function u_α is a solution of $(\text{IDE})_\alpha$ in $\mathcal{E}^{0,2}([0, \infty)\times\boldsymbol{R})$. It remains to prove the uniqueness. It is sufficient to show that if $v\in\mathcal{E}^{0,2}([0, \infty)\times\boldsymbol{R})$ satisfies $(1-I^\alpha \Delta)\, v=0$ on $(0, \infty)\times\boldsymbol{R}$, then $v\equiv 0$. Since $(1+I^{\alpha/2}\nabla)\, v$ belongs to $\mathcal{E}^{0,1}([0, \infty)\times\boldsymbol{R})$, Theorem A and Proposition 2 lead to $(1+I^{\alpha/2}\nabla)\, v\equiv 0$ on $[0, \infty)\times\boldsymbol{R}$. Theorem A also leads to $v\equiv 0$ on $[0, \infty)\times\boldsymbol{R}$. This completes the proof. □

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