Fujii, J.
Osaka J. Math.
27 (1990), 739–744

A NOTE ON PIVOTAL MEASURES IN MAJORIZED EXPERIMENTS

JUNJI FUJII

(Received July 20, 1989)

1. Introduction

An *experiment* $\boldsymbol{E}=(X, \boldsymbol{A}, P)$, i.e. a triplet with a family P of probability measures on a measurable space $(X, \boldsymbol{A})$, is said to be majorized by a measure μ equivalent to P (called a *"majorizing"* measure) if each $p \in P$ has a density $dp/d\mu$ with respect to μ. Let $\boldsymbol{S}_0$ be the σ-ring generated by all the $\boldsymbol{E}$-supports $S(p)=\{x \in X; (dp/d\mu)(x)>0\}, p \in P$. Then there exists a *"maximal decomposition"* $\boldsymbol{F}$ such that $\boldsymbol{F} \subset \boldsymbol{S}_0[P]$ (see [4], Lemma 2). A maximal decomposition $\boldsymbol{F}$ is defined as a covering of X of almost disjoint elements each of which is included by an $\boldsymbol{E}$-support $S(p^{(F)})$ of some $p^{(F)}$ in P. For each $F \in \boldsymbol{F}$, we define a dominated sub-experiment $\boldsymbol{E}(F)=(F, \boldsymbol{A} \cap F, P_F)$ by setting $\boldsymbol{A} \cap F=\{A \cap F; A \in \boldsymbol{A}\}$ and $P_F=\{p_F; p \in P, p(F)>0\}$, where $p_F(A \cap F)=p(A \cap F)/p(F)$. The σ-ring generated by $\{dp/d(p+q) \cdot I_{S(p)}; p, q \in P\}$ is called the σ-ring of *pairwise likelihood ratios* and denoted by $\boldsymbol{S}$. The σ-field $\boldsymbol{D}$ generated by $\boldsymbol{S}$ is known to be the smallest PSS (*pairwise sufficient with supports*) subfield, and is equal to $\boldsymbol{S}$ and minimal sufficient when $\boldsymbol{E}$ is dominated (see [3]). A majorizing measure m on $\boldsymbol{A}$ is said to be *pivotal* for $\boldsymbol{E}$ if it holds that for each subfield $\boldsymbol{B}$, $\boldsymbol{B}$ is PSS if and only if each p has a $\boldsymbol{B}$-measurable version of the density dp/dm. A real valued function $f: X \to \boldsymbol{R}$ is said to be $\boldsymbol{S}$-measurable if for each Borel subset B of $\boldsymbol{R}$, $f^{-1}(B) \cap \{x \in X; f(x) \neq 0\} \in \boldsymbol{S}$.

The notion of pivotal measures was devised to obtain a minimal sufficient subfield by Halmos & Savage [6] and Bahadur [1] for the dominated experiment. Ramamoorthi & Yamada [9] generalized it to the majorized experiment. Recently Luschgy, Mussmann & Yamada [8] proved the following characterization theorem of pivotal measures by the method and in the terminology of vector lattices: Every pivotal measure is represented as the sum of a maximal orthogonal system in the minimal $\boldsymbol{L}$-space.

The *minimal* $\boldsymbol{L}$*-space* is the closed vector sublattice generated by P, and a *maximal orthogonal system* is a family of non-zero measures on $\boldsymbol{A}$ such that any two distinct elements in the family are singular with each other and a measure which is singular with all the elements of the family is zero.

In the present note, we show that every pivotal measure in a majorized experiment is expressed as the sum of pivotal measures in the dominated sub-experiments $\boldsymbol{E}(F)$ for some maximal decomposition $\boldsymbol{F}(\subset \boldsymbol{S}_0[P])$ and that each pivotal measure on $\boldsymbol{E}(F)$ has a positive $\boldsymbol{S}$-measurable density with respect to a pivotal measure. Further, pointing out that the minimal $\boldsymbol{L}$-space coincides with the totality of the signed measures which have $\boldsymbol{S}$-measurable and integrable densities with respect to a pivotal measure, we show that the present result is a measure theoretical version of the above chatacterization theorem.

2. Pivotal measures

We consider the relation between a pivotal measure for $\boldsymbol{E}$ and a family of pivotal measures for the dominated sub-experiments $\boldsymbol{E}(F)$.

Theorem 1. *Let $\boldsymbol{E}=(X, \boldsymbol{A}, P)$ be a majorized experiment. If $\boldsymbol{F}(\subset \boldsymbol{S}_0[P])$ is a maximal decomposition and a measure m_F on $\boldsymbol{A}\cap F$ is pivotal for each $F\in \boldsymbol{F}$, then $m(A)=\Sigma_{F\in \boldsymbol{F}} m_F(A\cap F)$ is pivotal for $\boldsymbol{E}$.*

Proof. It is enough to show that each $p\in P$ has an $\boldsymbol{S}$-measurable version of dp/dm. Fix $p\in P$. There exists a countable subfamily $\{F_n; n\geqq 1\}$ of $\boldsymbol{F}$ such that $S(p)\subset \bigcup_{n\geqq 1} F_n[P]$ and $S(p)\cap F_n \neq \phi[P]$ for all n. We may assume that $\{F_n; n\geqq 1\}$ is a disjoint family. As each m_{F_n} is pivotal for $\boldsymbol{E}(F_n)$, there exists an $\boldsymbol{S}\cap F_n$-measurable version g_n of dp_{F_n}/dm_{F_n}. For each $n\geqq 1$, put $f_n(x)=p(F_n)\cdot g_n(x)$ if $x\in F_n$, and $=0$ if $x\in F_n^c$. As each F_n is in $\boldsymbol{S}_0$, f_n is $\boldsymbol{S}$-measurable. It is immediate that $\Sigma_{n\geqq 1} f_n$ is an $\boldsymbol{S}$-measurable version of dp/dm.

Theorem 2. *Let $\boldsymbol{E}=(X, \boldsymbol{A}, P)$ be a majorized experiment. A measure m on $\boldsymbol{A}$ is pivotal for $\boldsymbol{E}$ if and only if for each maximal decomposition $\boldsymbol{F}(\subset \boldsymbol{S}_0[P])$, there exists a family of pivotal measures $\{m_F; F\in \boldsymbol{F}\}$ for the sub-experiments $\boldsymbol{E}(F)$ such that $m(A)=\Sigma_{F\in \boldsymbol{F}} m_F(A\cap F)$.*

Proof. "If" part is Theorem 1 itself, and so we prove "only if" part. Let m be a pivotal measure for $\boldsymbol{E}$ and take a maximal decomposition $\boldsymbol{F}(\subset \boldsymbol{S}_0[P])$. Fix $F\in \boldsymbol{F}$. We define a measure m_F on $\boldsymbol{E}(F)$ by $m_F(A\cap F)=m(A\subset F)$. It is clear that $m_F\equiv P_F$ and $m(A)=\Sigma_{F\in \boldsymbol{F}} m_F(A\cap F)$. Take $p\in P$ with $p(F)>0$. Let f_p be an $\boldsymbol{S}$-measurable version of dp/dm. Then it is easily checked that $f_p\cdot I_F/p(F)$ is an $\boldsymbol{S}$-measurable version of dp_F/dm_F. This implies that m_F is a pivotal measure for $E(F)$ for each $F\in \boldsymbol{F}$.

Next we consider pivotal measures on each dominated sub-experiment.

Let $\boldsymbol{F}(\subset \boldsymbol{S}_0[P])$ be a maximal decomposition and n a pivotal measure for $\boldsymbol{E}$. Each F is included by a support $S(p^{(F)})$, and so $p_F^{(F)}\equiv P$. Fix $F\in \boldsymbol{F}$. Then the restriction n_F of n to $\boldsymbol{A}\cap F$ is pivotal for $\boldsymbol{E}(F)$ as in the proof of Theorem 2. Let u be a pivotal measure for $\boldsymbol{E}(F)$. Notice that $du/dn_F=(dp_F^{(F)}/du)^{-1}\cdot$

$(dp_F^{(F)}/dn_F)^{-1}[P_F]$ as $p_F^{(F)} \equiv n_F \equiv u$. Two densities in the right side have $\boldsymbol{S}$-measurable versions as u and n_F are pivotal. That is, $u(A \cap F) = \int_{F \cap F} f\, dn$ for some $\boldsymbol{S} \cap F$-measurable function f: $F \to \boldsymbol{R}$, with $f > 0\,[P_F]$.

Conversely a measure u of this form is pivotal for $\boldsymbol{E}(F)$. Because, n_F is pivotal and $dp_F/du = (dp_F/dn_F)/f\,[P_F]$ for all $p_F \in P_F$.

Thus we have proved the following

Theorem 3. *Let $\boldsymbol{E}=(X, \boldsymbol{A}, P)$ be a majorized experiment and n a pivotal measure for $\boldsymbol{E}$. Then, a measure m on $\boldsymbol{A}$ is pivotal for $\boldsymbol{E}$ if and only if it is expressed as $m(A) = \Sigma_{F \in \boldsymbol{F}} \int_{A \cap F} f_F\, dn$ for some maximal decomposition $\boldsymbol{F}(\subset \boldsymbol{S}_0[P])$ and some family $\{f_F: F \in \boldsymbol{F}\}$ of $\boldsymbol{S}$-measurable functions such that $f_F > 0\,[P]$ on F and $f_F = 0\,[P]$ on F^c.*

In case $\boldsymbol{E}$ is weakly dominated, we have the following simpler expression.

Corollary. *Let $\boldsymbol{E}=(X, \boldsymbol{A}, P)$ be a weakly dominated experiment and n a pivotal measure for $\boldsymbol{E}$. A measure m on $\boldsymbol{A}$ is pivotal for $\boldsymbol{E}$ if and only if $m(A) = \int_A f\, dn$ for some $\boldsymbol{B}$-measurable function f with $f > 0\,[P]$, where $\boldsymbol{B}$ is a minimal sufficient subfield.*

Proof. Let $\boldsymbol{B}$ be a minimal sufficient subfield. It follows from Theorem 1.1 in [7] that $(X, \boldsymbol{B}, P|\boldsymbol{B})$ is weakly dominated. Hence for the family of $\boldsymbol{S}$-measurable functions $\{f_F;\ F \in \boldsymbol{F}\}$ in Theorem 3, there exists a $\boldsymbol{B}$-measurable function f such that $f \cdot I_F = f_F\,[P]$ for all $F \in \boldsymbol{F}$.

REMARK. In Theorem 3, "if" part remains true with "any" maximal decomposition and "any" family $\{f_F;\ F \in \boldsymbol{F}\}$. "Only if" part also holds true with "any" maximal decomposition and some family $\{f_F;\ F \in \boldsymbol{F}\}$. Gooßen [5] attained to a similar characterization by making use of a pivotal measure $\Sigma_{F \in \boldsymbol{F}}\, p^{(F)}(A \cap F)$, which was obtained by Diepenbrock.

In the following Examples 1 to 4, we observe that the totality of pivotal measures is fairly large.

EXAMPLE 1 (N-th product normal family with a location parameter). $X = \boldsymbol{R}^N$, $\boldsymbol{A}$ = the Borel σ-field on X. $P = \{p_\xi;\ \xi \in \boldsymbol{R}\}$. $(dp_\xi/d\mu)(x) = (2\pi)^{-N/2} \exp \{(-1/2) \Sigma_{i=1}^N (x_i - \xi)^2\}$, where μ is the Lebesgue measure on X. The statistic $t(x) = \Sigma_{i=1}^N x_i$ is minimal sufficient and the σ-ring $\boldsymbol{S}$ is a minimal sufficient subfield, which is induced by t. Each p_ξ is pivotal for $\boldsymbol{E}$ as $p_\xi \equiv P$ and $\boldsymbol{E}$ is dominated. According to Corollary, a measure m on $\boldsymbol{A}$ is pivotal for $\boldsymbol{E}$ if and only if $m(A) = \int_A f\, dp_0$ for some $\boldsymbol{S}$-measurable function f with $f > 0\,[P]$. Thus the totality of pivotal measures coincides with all the measures whose density with respect to μ are of the form $\exp \{(-1/2) \cdot \Sigma_{i=1}^N x_i^2 \cdot f(\Sigma_{i=1}^N x_i)\}$ for some

$f>0[\mu]$. In case $N=1$, all the measures which are equivalent to μ are pivotal, and in particular so is μ. In case $N\geqq 2$, μ is not pivotal.

Example 2. $X=\boldsymbol{R}^3$. $\boldsymbol{F}$=all the planes parallel to a coordinate plane. $\boldsymbol{A}=\{A\subset X;\ A\cap F\in\boldsymbol{B}_{\boldsymbol{R}^2}$ for all $F\in\boldsymbol{F}\}$, where $\boldsymbol{B}_{\boldsymbol{R}^2}$ denotes the Borel σ-field on $\boldsymbol{R}^2$. On each plane $F\in\boldsymbol{F}$, consider the same normal family as in Example 1 and extend each element of the family to $\boldsymbol{A}$ in an obvious way that it vanishes outside F. P=the union of such families. Define a measure μ by $\mu(A)=\Sigma_{F\in\boldsymbol{F}}\,\nu(A\cap F)$, where ν is the Lebesgue measure on $\boldsymbol{R}^2$.

The experiment $\boldsymbol{E}=(X,\boldsymbol{A},P)$ is majorized by μ, and $\boldsymbol{F}$ is a maximal decomposition such that $\boldsymbol{F}\subset\boldsymbol{S}_0[P]$. In each sub-experiment, the totality of pivotal measures is described in Example 1. By Theorem 2, a pivotal measure on $\boldsymbol{E}$ is expressed as the sum of pivotal measures on $\boldsymbol{E}(F)$. The measure μ is not pivotal for $\boldsymbol{E}$ as it is pointed out in Example 1 that ν is not pivotal on $\boldsymbol{E}(F)$.

Example 3. $X=\boldsymbol{R}^2$. $\boldsymbol{F}$=all horizontal and vertical lines in X. $\boldsymbol{A}=\{A\subset X;\ A\cap F\in\boldsymbol{B}_{\boldsymbol{R}}$ for all $F\in\boldsymbol{F}\}$, where $\boldsymbol{B}_{\boldsymbol{R}}$ denotes the Borel σ-field on $\boldsymbol{R}$. On each $F\in\boldsymbol{F}$, consider the same normal family as in Example 1 and extend each element of the family to $\boldsymbol{A}$ as in Example 2. P=the union of such families. Define a measure μ by $\mu(A)=\Sigma_{F\in\boldsymbol{F}}\,\nu(A\cap F)$, where ν is the Lebesgue measure on $\boldsymbol{R}$.

The experiment $\boldsymbol{E}=(X,\boldsymbol{A},P)$ is majorized by μ, and $\boldsymbol{F}$ is a maxiaml decomposition such that $\boldsymbol{F}\subset\boldsymbol{S}_0[P]$. As in Example 1, every measure on F which is equivalent to ν is pivotal for $\boldsymbol{E}(F)$. By Theorem 2, all the measures which are the sums of those measures are pivotal. In particular, μ is pivotal unlike in Example 2.

Example 4. $X=\boldsymbol{R}_+^N\cdot\boldsymbol{A}$=the Borel σ-field of X. For each $\theta>0$, p_θ denotes the N-th product probability measure of the uniform distribution on $(0,\theta]$, i.e. $(dp_\theta/d\mu)(x)=\theta^{-N}\cdot\Pi_{k=1}^N I_{(0,\theta]}(x_k)$, where μ is the Lebesgue measure on X. The statistic $t(x)=\max_{1\leqq i\leqq N}x_i$ is minimal sufficient, and the σ-ring $\boldsymbol{S}$ is induced by t. We construct a pivotal measure. Take $p_k(k\geqq 1)$ from P and put $F_1=S(p_1)=(0,1]^N$ and $F_k=S(p_k)-\cup_{j=1}^{k-1}F_j=(0,k]^N-(0,k-1]^N$ for $k\geqq 2$. Then it is easily seen that $\boldsymbol{F}=\{F_k;\ k\geqq 1\}$ is a maximal decomposition with $\boldsymbol{F}\subset\boldsymbol{S}_0[P]$. For each $k\geqq 1$, $p_k(A\cap F_k)$ is pivotal for $\boldsymbol{E}(F_k)$ as it is equivalent to P_F. By Theorem 1, the measure n defined by $n(A)=\Sigma_{k\geqq 1}\,p_k(A\cap F_k)$ is pivotal for $\boldsymbol{E}$. This is the same pivotal measure as that obtained by Diepenbrock (see Remark). A pivotal measure m is of the form $m(A)=\int_A f\,dn$ for some $\boldsymbol{S}$-measurable function f with $f>0[P]$. Hence $m(A)=\int_A f\,\Sigma_{k\geqq 1}\,dp_k\,|\,F_k=\Sigma_{k\geqq 1}\int_{A\cap F_k}f\cdot(dp_k/d\mu)\cdot d\mu$, and so $dm/d\mu=k^{-N}\cdot f(\max_{1\leqq i\leqq N}x_i)$ on each F_k, $k\geqq 1$. Therefore the totality of pivotal measures coicides with all the measures whose densities with respect to μ are functions through t.

3. Relationship between Theorem 3 and the Luschgy, Mussmann & Yamada-characterization theorem

In this section, we show that the characterization theorem in [8] can be derived from Theorem 3.

We first describe the minimal $\boldsymbol{L}$-space through the σ-ring $\boldsymbol{S}$. Let $\boldsymbol{E}=(X, \boldsymbol{A}, P)$ be a majorized experiment. The minimal $\boldsymbol{L}$-space is the closed vector sublacttice (of the vector lattice of all bounded signed measures on $\boldsymbol{A}$) generated by P, where the topology of the vector lattice is induced by the total variation norm. Let n be a pivotal measure and $\boldsymbol{L}^1(\boldsymbol{S}, n)$ be the set of all $\boldsymbol{S}$-measurable functions which are integrable with respect to n. For each $f \in \boldsymbol{L}^1(\boldsymbol{S}, n)$, $u_f(A)=\int_A f\, dn$ is a signed measure. Notice that the total variation norm of u_f coincides with the usual $\boldsymbol{L}^1$-norm of f with respect to n. As n is a pivotal measure, for each $p \in P$, there is an $\boldsymbol{S}$-measurable density f_p, i.e. $p(A)=\int_A f_p\, dn$. This correspondence between p and f_p implies that the minimal $\boldsymbol{L}$-space is isomorphic to $\boldsymbol{L}^1(\boldsymbol{S}, n)$ as an $\boldsymbol{L}$-space. Hence the minimal $\boldsymbol{L}$-space coincides with the totality of all the signed measures whose densities with respect to n belong to $\boldsymbol{L}^1(\boldsymbol{S}, n)$.

Next we show that Theorem 3 implies the characterization theorem in [8]. Let m be a pivotal measure. Then m is expressed as in Theorem 3. The $\boldsymbol{S}$-measurable function f_F in Theorem 3 is not necessarily integrable with respect to n for every $F \in \boldsymbol{F} (\subset \boldsymbol{S}_0[P])$. However for each $F \in \boldsymbol{F}$, there exists a countable partition of F consisting of $\boldsymbol{S}$-measurable sets such that the restriction of f_F to each element of the partition is integrable with respect to n (see [2], Lemma 3.1), and hence f_F is the sum of such integrable functions. As the union of all such partitions forms a maximal decomposition consisting of $\boldsymbol{S}$-measurable sets, we can replace $\boldsymbol{F}$ by this maximal decomposition and denote the latter by $\boldsymbol{F}$ again. Consequently, m is expressed as $m(A)=\Sigma_{F \in \boldsymbol{F}} \int_{A \cap F} f_F\, dn$ for a family $\{f_F; F \in \boldsymbol{F}\}$ of $\boldsymbol{S}$-measurable functions such that each f_F is integrable with respect to n. Put $m_F(A)=\int_{A \cap F} f_F\, dn$.. Then it follows that $f_F \geqq 0[P]$ and $f_F \neq 0[P]$ for all $F \in \boldsymbol{F}$, $\min(f_F, f_G)=0[P]$ for all $F \neq G \in \boldsymbol{F}$ and for each $f \in \boldsymbol{L}^1(\boldsymbol{S}, n)$, $\min(f, f_F)=0[P]$ for all $F \in \boldsymbol{F}$ implies $f=0[P]$. This implies that $\{m_F; F \in \boldsymbol{F}\}$ is a maximal orthogonal system, and hence m is the sum of the maximal orthogonal system $\{m_F; F \in \boldsymbol{F}\}$.

Conversely, suppose that a measure m is the sum of a maximal orthogonal system. Take a maximal decomposition $\boldsymbol{F}(\subset \boldsymbol{S}_0[P])$ and fix $F \in \boldsymbol{F}$. Adding up the densities of the maximal orthogonal system which are positive $[P]$ on F, we have the function f_F required in Theorem 3. Hence Theorem 3 implies that m is a pivotal measure.

Acknowledgement.

The author thanks the referee for his valuable comments.

References

[1] R.R. Bahadur: *Sufficiency and statistical decision functions*, Ann. Math. Statist. **25** (1954), 423–462.

[2] F.R. Diepenbrock: *Charakterisierung einer allgemeinen Bedingung als Dominiertheit mit Hilfe von localisierbaren Massen*, Thesis, University of Munster, 1971.

[3] J. Fujii: *On the smallest pairwise sufficient subfield in the majorized statistical experiment*, Osaka J. Math., **26** (1989), 429–446.

[4] J. Fujii and H. Morimoto: *Sufficiency and pairwise sufficiency in majorized experiments*, Sankhya (Ser. A) **48** (1986), 315–330.

[5] K. Gooßen: *Partitions and pivotal measures in experiments*, Statistics & Decisions **6** (1988), 283–291,

[6] P.R. Halmos and L.J. Savage: *Applications of the Radon-Nikodym theorem to the theory of sufficient statistics*, Ann. Math. Statist. **20** (1949), 225–241.

[7] T. Kusama and S. Yamada: *On compactness of the statistical structure and sufficiency*, Osaka J. Math., **9** (1972), 11–18.

[8] H. Luschgy, D. Mussmann and S. Yamada: *Minimal L-space and Halmos-Savage criterion for majorized experiments*, Osaka J. Math., **25** (1988), 795–803.

[9] R.V. Ramamoorthi and S. Yamada: *Neyman factorization for experiments admitting densities*, Sankhya (Series A) **45** (1983), 168–180.

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558, Japan