

## ON HYPERSINGULAR INTEGRALS

Dedicated to Professor Tatsuo Fuji'i'e on the  
 occasion of his 60th birthday

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### 1. Introduction

The purposes of this paper are to generalize a result of E.M. Stein concerning hypersingular integrals and to give its application. The same problems for singular difference integrals are discussed in [2]. Let  $R^n$  be the  $n$ -dimensional Euclidean space and for each point  $x=(x_1, \dots, x_n)$  we write  $|x|=(x_1^2+\dots+x_n^2)^{1/2}$ . For a multi-index  $\gamma=(\gamma_1, \dots, \gamma_n)$  we denote  $|\gamma|=\gamma_1+\dots+\gamma_n$ ,  $\gamma!=\gamma_1! \dots \gamma_n!$ ,  $x^\gamma=x_1^{\gamma_1} \dots x_n^{\gamma_n}$  and  $D^\gamma=D_1^{\gamma_1} \dots D_n^{\gamma_n}$ . The Riesz kernel  $\kappa_\alpha$  of order  $\alpha>0$  is defined as

$$\kappa_\alpha(x) = \begin{cases} |x|^{\alpha-n}, & \alpha < n \text{ or } \alpha \geq n, \alpha-n \text{ not even} \\ (\delta_{\alpha,n} - \log|x|)|x|^{\alpha-n}, & \alpha \geq n, \alpha-n \text{ even} \end{cases}$$

where  $\delta_{\alpha,n}$  is the constant (see [1]).

For  $\beta>0$ ,  $\varepsilon>0$  and a positive integer  $l$ , we consider the integral (hypersingular integral)

$$H_\varepsilon^{l,\beta}u(x) = \int_{|t| \geq \varepsilon} \frac{(R_l^t u)(x)}{|t|^{n+\beta}} dt$$

where  $(R_l^t u)(x)$  is the remainder of order  $l$ , namely

$$(R_l^t u)(x) = u(x+t) - \left( \sum_{|\gamma| \leq l-1} D^\gamma u(x) / \gamma! \right) t^\gamma.$$

We denote  $\mu^{l,\alpha,\beta}(x) = H_\varepsilon^{l,\beta} \kappa_\alpha(x)$ . E.M. Stein showed in [3; p. 162] that  $\mu^{1,\alpha,\alpha} \in L^1$  for  $0 < \alpha < 2$ . In § 2 we generalize this result (Theorems 2.3 and 2.14). R.L. Wheeden [4; Theorem 1] established an inequality of the following type for functions  $u$  in Bessel potential spaces:

$$\|H_\varepsilon^{l,\beta}u\|_p \leq C \|u\|_{p,\beta},$$

where  $\|\cdot\|_p$  is the  $L^p$ -norm and  $\|\cdot\|_{p,\beta}$  is the norm in Bessel potential spaces.

In § 3 we give integral estimates of  $H_t^{l,\beta}u$  for Beppo Levi functions  $u$  (Theorem 3.7).

Throughout this paper, the letter  $C$  is used for a generic positive constant whose value may be different at each occurrence.

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## 2. Hypersingular integrals of Riesz kernels

For a positive integer  $l$ ,  $C^l$  denote the space of all  $l$  times continuously differentiable functions on  $R^n$ . For a point  $t$  in  $R^n$  we write  $t'=t/|t|$ . By Taylor's formula we have

**Lemma 2.1.** *If  $\phi \in C^l$ , then*

$$(R_t^l \phi)(x) = l \sum_{|\gamma|=l} \int_0^{|t|} \frac{(|t|-s)^{l-1}}{\gamma!} (t')^\gamma D^\gamma \phi(x+st') ds.$$

For  $x, y \in R^n$ , we put  $L_{x,y} = \{sx + (1-s)y; 0 \leq s \leq 1\}$  and  $L_x = L_{x,0}$ . We denote by  $d(y, L_x)$  the distance between  $y$  and  $L_x$ . By Lemma 2.1 we obtain

**Corollary 2.2.** (i) *If  $\phi \in C^l$ , then*

$$|(R_t^l \phi)(y)| \leq |t|^l \sum_{|\gamma|=l} (1/\gamma!) \max_{z \in L_{y,y+t}} |D^\gamma \phi(z)|.$$

(ii) *If  $l > \alpha - n$ , then for  $d(y, L_{-t}) \geq |t|/2$  we have*

$$|(R_t^l \kappa_\alpha)(y)| \leq C |t|^l |y|^{\alpha-l-n}.$$

*Proof.* (i) By Lemma 2.1 we have

$$\begin{aligned} |(R_t^l \phi)(y)| &\leq l \sum_{|\gamma|=l} \int_0^{|t|} \frac{(|t|-s)^{l-1}}{\gamma!} |D^\gamma \phi(y+st')| ds \\ &\leq l \sum_{|\gamma|=l} (1/\gamma!) \max_{z \in L_{y,y+t}} |D^\gamma \phi(z)| \int_0^{|t|} (|t|-s)^{l-1} ds \\ &= |t|^l \sum_{|\gamma|=l} (1/\gamma!) \max_{z \in L_{y,y+t}} |D^\gamma \phi(z)|. \end{aligned}$$

(ii) We note that if  $d(y, L_{-t}) \geq |t|/2$ , then

$$|y|/3 \leq |y+st'| \leq 3|y|$$

for  $0 \leq s \leq |t|$ . Further, if  $|\gamma|=l$  and  $l > \alpha - n$ , then

$$|D^\gamma \kappa_\alpha(x)| \leq C |x|^{\alpha-l-n}.$$

Hence by Lemma 2.1, for  $d(y, L_{-t}) \geq |t|/2$  we have

$$\begin{aligned} |(R_i^l \kappa_\alpha)(y)| &\leq l \sum_{|\gamma|=l} \int_0^{|t|} \frac{(|t|-s)^{l-1}}{\gamma!} |D^\gamma \kappa_\alpha(st'+y)| ds \\ &\leq C \sum_{|\gamma|=l} \int_0^{|t|} \frac{(|t|-s)^{l-1}}{\gamma!} |st'+y|^{\alpha-l-n} ds \\ &\leq C |y|^{\alpha-l-n} \sum_{|\gamma|=l} \int_0^{|t|} \frac{(|t|-s)^{l-1}}{\gamma!} ds \\ &= C |t|^l |y|^{\alpha-l-n}. \end{aligned}$$

For  $\alpha > 0, \beta > 0$  and a positive integer  $l$ , we set

$$\mu^{l, \alpha, \beta}(y) = \int_{|t| \geq 1} \frac{(R_i^l \kappa_\alpha)(y)}{|t|^{n+\beta}} dt$$

and

$$\mu_*^{l, \alpha, \beta}(y) = \int_{|t| \geq 1} \frac{|(R_i^l \kappa_\alpha)(y)|}{|t|^{n+\beta}} dt.$$

We note that  $\mu_*^{l, \alpha, \beta}(y)$  is finite for  $\beta > \alpha - n, \beta > l - 1$  and  $y \neq 0$ .

**2.1. Integrability of  $\mu^{l, \alpha, \beta}$**

For  $1 < q \leq \infty$ , we denote the conjugate exponent to  $q$  by  $q'$ , namely,  $(1/q) + (1/q') = 1$ . At first, we prove

**Theorem 2.3.** Let  $1 < q \leq \infty$  and  $\beta > \alpha - (n/q)$ .

- (i) If  $l - 1 < \alpha - (n - q) < l$ , then  $\mu_*^{l, \alpha, \beta} \in L^{q'}$ .
- (ii) If  $l$  is an odd number, then for  $l - 1 < \alpha - (n/q) < l + 1$  we have  $\mu^{l, \alpha, \beta} \in L^{q'}$ .

Proof. (i) By Minkowski's inequality for integrals we obtain

$$\begin{aligned} I &= \left( \int |\mu_*^{l, \alpha, \beta}(y)|^{q'} dy \right)^{1/q'} \\ &= \left( \int \left( \int_{|t| \geq 1} \frac{|(R_i^l \kappa_\alpha)(y)|}{|t|^{n+\beta}} dt \right)^{q'} dy \right)^{1/q'} \\ &\leq \int_{|t| \geq 1} \frac{dt}{|t|^{n+\beta}} \left( \int |(R_i^l \kappa_\alpha)(y)|^{q'} dy \right)^{1/q'}. \end{aligned}$$

Further, we have

$$\begin{aligned} \left( \int |(R_i^l \kappa_\alpha)(y)|^{q'} dy \right)^{1/q'} &\leq \left( \int_{d(y, L_{-t}) \geq |t|/2} |(R_i^l \kappa_\alpha)(y)|^{q'} dy \right)^{1/q'} \\ &\quad + \left( \int_{d(y, L_{-t}) < |t|/2} |(R_i^l \kappa_\alpha)(y)|^{q'} dy \right)^{1/q'} \\ &= I_1 + I_2. \end{aligned}$$

Since  $l > \alpha - (n/q) > \alpha - n$ , it follows from Corollary 2.2 (ii) that

$$I_1 \leq C |t|^l \left( \int_{|y| \geq |t|/2} |y|^{(\alpha-l-n)q'} dy \right)^{1/q'} = C |t|^{\alpha-(n/q)}.$$

Moreover it follows from  $\alpha - (n/q) > l - 1 \geq 0$  that

$$\begin{aligned} I_2 &\leq \left( \int_{d(y, L_{-l}) < |t|/2} |\kappa_\alpha(y+t)|^{q'} dy \right)^{1/q'} \\ &\quad + \sum_{|\gamma| \leq l-1} (|t|^{|\gamma|} / \gamma!) \left( \int_{d(y, L_{-l}) < |t|/2} |D^\gamma \kappa_\alpha(y)|^{q'} dy \right)^{1/q'} \\ &\leq C \begin{cases} |t|^{\alpha-(n/q)}, & \alpha-n \text{ is not a nonnegative even number} \\ |t|^{\alpha-(n/q)}(1+|\log|t||), & \alpha-n \text{ is a nonnegative even number.} \end{cases} \end{aligned}$$

Thus

$$I \leq C \begin{cases} \int_{|t| \geq 1} \frac{|t|^{\alpha-(n/q)}}{|t|^{n+\beta}} dt, & \alpha-n \text{ is not a nonnegative even number} \\ \int_{|t| \geq 1} \frac{|t|^{\alpha-(n/q)}(1+|\log|t||)}{|t|^{n+\beta}} dt, & \alpha-n \text{ is a nonnegative even number.} \end{cases}$$

By the condition  $\beta > \alpha - (n/q)$ , we obtain  $I < \infty$ .

(ii) Let  $l$  be a n odd number. We have

$$\begin{aligned} J &= \left( \int |\mu^{l, \alpha, \beta}(y)|^{q'} dy \right)^{1/q'} \\ &\leq \left( \int \left| \int_{|t| \geq 1, |t| \geq 2|y|/3} \frac{(R_t^l \kappa_\alpha)(y)}{|t|^{n+\beta}} dt \right|^{q'} dy \right)^{1/q'} \\ &\quad + \left( \int \left| \int_{|t| \geq 1, |t| > 2|y|/3} \frac{(R_t^l \kappa_\alpha)(y)}{|t|^{n+\beta}} dt \right|^{q'} dy \right)^{1/q'} \\ &= J_1 + J_2. \end{aligned}$$

Since  $l$  is an odd number, we see that

$$\int_{|t| \geq 1, |t| \leq 2|y|/3} \sum_{|\gamma| = l} \frac{t^\gamma}{|t|^{n+\beta}} dt = 0.$$

Hence we have

$$\begin{aligned} J_1 &= \left( \int \left| \int_{|t| \geq 1, |t| \leq 2|y|/3} \frac{(R_t^{l+1} \kappa_\alpha)(y)}{|t|^{n+\beta}} dt \right|^{q'} dy \right)^{1/q'} \\ &\leq \int_{|t| \geq 1} \frac{dt}{|t|^{n+\beta}} \left( \int_{|y| \geq 3|t|/2} |R_t^{l+1} \kappa_\alpha(y)|^{q'} dy \right)^{1/q'}. \end{aligned}$$

Note that  $|y| \geq 3|t|/2$  implies  $d(y, L_{-l}) \geq |t|/2$ . Since  $l+1 > \alpha - (n/q) > \alpha - n$  and  $\beta > \alpha - (n/q)$ , we obtain from Corollary 2.2 (ii)

$$\begin{aligned}
 J_1 &\leq C \int_{|t| \geq 1} \frac{dt}{|t|^{n+\beta}} \left( \int_{|y| \geq 3|t|/2} (|t|^{l+1}|y|^{\alpha-l-1-n})^{q'} dy \right)^{1/q'} \\
 &= C \int_{|t| \geq 1} \frac{|t|^{\alpha-(n/q)}}{|t|^{n+\beta}} dt < \infty.
 \end{aligned}$$

In the same way as in (i) we obtain  $J_2 < \infty$  since  $\alpha - (n/q) > l - 1 \geq 0$  and  $\beta > \alpha - (n/q)$ . The proof of the theorem is complete.

**2.2. Integrability of  $\mu^{l, \alpha}$**

By Taylor's formula we have

**Lemma 2.4.** *If  $P$  is a polynomial of order  $m$  and  $l > m$ , then  $(R_l^i P)(x) = 0$ .*

The symbol  $\mathcal{D}$  denotes the  $LF$ -space of all  $C^\infty$ -functions with compact support and  $\mathcal{D}'$  stands for the topological dual of  $\mathcal{D}$ . For  $f \in \mathcal{D}$ , the Riesz potential of order  $\alpha$  of  $f$  is defined as

$$U_\alpha^f(x) = \int \kappa_\alpha(x-y)f(y)dy.$$

We easily see

**Lemma 2.5.** *Let  $f \in \mathcal{D}$ . Then*

- (i)  $U_\alpha^f \in C^\infty$ .
- (ii) For  $|\gamma| < \alpha$

$$D^\gamma U_\alpha^f(x) = \int D^\gamma \kappa_\alpha(x-y)f(y)dy.$$

(iii) If  $|\gamma| = l$  is odd, then

$$\lim_{\epsilon \rightarrow 0} \int_{|x-y| \geq \epsilon} D^\gamma \kappa_l(x-y)f(y)dy$$

exists for all  $x$  and

$$D^\gamma U_l^f(x) = \lim_{\epsilon \rightarrow 0} \int_{|x-y| \geq \epsilon} D^\gamma \kappa_l(x-y)f(y)dy.$$

If  $\alpha - n$  is not a nonnegative even number, then the Riesz kernel  $\kappa_\alpha(x)$  is homogeneous of degree  $\alpha - n$ . Hence

$$(2.1) \quad D^\gamma \kappa_\alpha(\lambda x) = \lambda^{\alpha - |\gamma| - n} D^\gamma \kappa_\alpha(x), \quad \lambda > 0.$$

When  $\alpha - n$  is a nonnegative even number, we have

**Lemma 2.6.** *Let  $\alpha - n$  be a nonnegative even number. Then*

$$D^\gamma \kappa_\alpha(\lambda x) = \lambda^{\alpha - |\gamma| - n} ((-\log \lambda) D^\gamma \kappa_\alpha(x) + D^\gamma \kappa_\alpha(x))$$

with  $\chi_\alpha(x) = |x|^{\alpha-n}$ .

Proof. By Leibniz's formula we see that

$$D^\gamma \kappa_\alpha(x) = (\delta_{\alpha,n} - \log|x|)D^\lambda \chi_\alpha(x) + \sum_{0 < \delta \leq \gamma} \gamma C_\delta D^\delta (\delta_{\alpha,n} - \log|x|)D^{\gamma-\delta} \chi_\alpha(x)$$

with  $\gamma C_\delta = \gamma!/\delta!(\gamma-\delta)!$ . For  $\delta > 0$ , we have

$$D^\delta (\delta_{\alpha,n} - \log|x|) = \frac{P_\delta(x)}{|x|^{2|\delta|}}$$

where  $P_\delta(x)$  is a homogeneous polynomial of degree  $|\delta|$ . Hence we have

$$\begin{aligned} D^\gamma \kappa_\alpha(\lambda x) &= (\delta_{\alpha,n} - \log|\lambda x|)D^\gamma \chi_\alpha(\lambda x) + \sum_{0 < \delta \leq \gamma} \gamma C_\delta \frac{P_\delta(\lambda x)}{|\lambda x|^{2|\delta|}} D^{\gamma-\delta} \chi_\alpha(\lambda x) \\ &= (-\log \lambda)\lambda^{\alpha-|\gamma|-n} D^\gamma \chi_\alpha(x) + (\delta_{\alpha,n} - \log|x|)\lambda^{\alpha-|\gamma|-n} D^\gamma \chi_\alpha(x) \\ &\quad + \sum_{0 < \delta \leq \gamma} \gamma C_\delta \lambda^{\alpha-|\gamma|-n} \frac{P_\delta(x)}{|x|^{2|\delta|}} D^{\gamma-\delta} \chi_\alpha(x) \\ &= \lambda^{\alpha-|\gamma|-n} (-\log \lambda) D^\gamma \chi_\alpha(x) + D^\gamma \kappa_\alpha(x). \end{aligned}$$

Thus we obtain the lemma.

**Lemma 2.7.** *Let  $l$  be a positive integer, and moreover we assume that  $l > \alpha - n$  in case  $\alpha - n$  is a nonnegative even number. Then*

- (i)  $(R_l^i \kappa_\alpha)(y) = |t|^{\alpha-n} (R_{l'}^i \kappa_\alpha)(y/|t|)$ ,
- (ii)  $(R_l^i \kappa_\alpha)(y) = |y|^{\alpha-n} (R_{l'}^i \kappa_\alpha)(y')$ .

Proof. (i) If  $\alpha - n$  is not a nonnegative even number, then by (2.1) we have

$$\begin{aligned} (R_l^i \kappa_\alpha)(y) &= |y+t|^{\alpha-n} - \sum_{|\gamma| \leq l-1} (D^\gamma \kappa_\alpha(y)/\gamma!) t^\gamma \\ &= |t|^{\alpha-n} \left( \left| \frac{y}{|t|} + t' \right|^{\alpha-n} - \sum_{|\gamma| \leq l-1} \frac{D^\gamma \kappa_\alpha(y/|t|)}{\gamma!} (t')^\gamma \right) \\ &= |t|^{\alpha-n} (R_{l'}^i \kappa_\alpha)(y/|t|). \end{aligned}$$

If  $\alpha - n$  is a nonnegative even number, then by Lemma 2.6 we see that

$$\begin{aligned} (R_l^i \kappa_\alpha)(y) &= (\delta_{\alpha,n} - \log|y+t|)|y+t|^{\alpha-n} - \sum_{|\gamma| \leq l-1} (D^\gamma \kappa_\alpha(y)/\gamma!) t^\gamma \\ &= |t|^{\alpha-n} \left\{ \left( \delta_{\alpha,n} - \log \left| \frac{y}{|t|} + t' \right| - \log|t| \right) \left| \frac{y}{|t|} + t' \right|^{\alpha-n} \right. \\ &\quad \left. - \sum_{|\gamma| \leq l-1} \frac{D^\gamma \kappa_\alpha(y)}{\gamma! |t|^{\alpha-|\gamma|-n}} (t')^\gamma \right\} \end{aligned}$$

$$\begin{aligned}
 &= |t|^{\alpha-n} \left\{ \left( \delta_{\alpha,n} - \log \left| \frac{y}{|t|} + t' \right| - \log |t| \right) \left| \frac{y}{|t|} + t' \right|^{\alpha-n} \right. \\
 &\quad \left. - \sum_{|\gamma| \leq l-1} (1/\gamma!) ((-\log |t|) D^\gamma \chi_\alpha(y/|t|) + D^\gamma \kappa_\alpha(y/|t|)) (t')^\gamma \right\} \\
 &= |t|^{\alpha-n} (-\log |t|) \left( \left| \frac{y}{|t|} + t' \right|^{\alpha-n} - \sum_{|\gamma| \leq l-1} (D^\gamma \chi_\alpha(y/|t|)/\gamma!) (t')^\gamma \right) \\
 &\quad + |t|^{\alpha-n} \left\{ \left( \delta_{\gamma,n} - \log \left| \frac{y}{|t|} + t' \right| \right) \left| \frac{y}{|t|} + t' \right|^{\gamma-n} - \sum_{|\gamma| \leq l-1} \frac{D^\gamma \kappa_\alpha(y/|t|)}{\gamma!} (t')^\gamma \right\} \\
 &= |t|^{\alpha-n} (-\log |t|) (R_l^i \chi_\alpha)(y/|t|) + |t|^{\alpha-n} (R_l^i \kappa_\alpha)(y/|t|).
 \end{aligned}$$

Since  $l > \alpha - n$ , by Lemma 2.4 we obtain the required equality.

Using (2.1) and Lemma 2.6 we can prove (ii) in the same way as in (i). We complete the proof of the lemma.

**Lemma 2.8.** *Let  $f \in \mathcal{D}$ .*

(i) *Let  $l < \alpha + 1$ , and moreover we assume that  $l > \alpha - n$  in case  $\alpha - n$  is a nonnegative even number. Then*

$$(R_l^i U_\alpha^f)(x) = |t|^\alpha \int (R_l^i \kappa_\alpha)(z) f(x - |t|z) dz.$$

(ii) *If  $l$  is an odd number, then*

$$(R_l^{i+1} U_l^f)(x) = |t|^l \lim_{\varepsilon \rightarrow 0} \int_{|z| \geq \varepsilon} (R_l^{i+1} \kappa_\alpha)(z) f(x - |t|z) dz.$$

Proof. (i) Since  $l - 1 < \alpha$ , by Lemma 2.5 (ii) we have

$$\begin{aligned}
 (R_l^i U_\alpha^f)(x) &= U_\alpha^f(x+t) - \sum_{|\gamma| \leq l-1} (D^\gamma U_\alpha^f(x)/\gamma!) t^\gamma \\
 &= \int (\kappa_\alpha(x+t-y) - \sum_{|\gamma| \leq l-1} (D^\gamma \kappa_\alpha(x-y)/\gamma!) t^\gamma) f(y) dy \\
 &= \int (R_l^i \kappa_\alpha)(x-y) f(y) dy \\
 &= \int (R_l^i \kappa_\alpha)(y) f(x-y) dy.
 \end{aligned}$$

Furthermore it follows from Lemma 2.7 (i) that

$$\begin{aligned}
 (R_l^i U_\alpha^f)(x) &= |t|^{\alpha-n} \int (R_l^i \kappa_\alpha)(y/|t|) f(x-y) dy \\
 &= |t|^\alpha \int (R_l^i \kappa_\alpha)(z) f(x - |t|z) dz.
 \end{aligned}$$

(ii) By Lemma 2.5 (ii) and (iii) we have

$$\begin{aligned}
(R_i^{l+1}U_i^f)(x) &= U_i^f(x+t) - \sum_{|\gamma| \leq l} (D^\gamma U_i^f(x)/\gamma!) t^\gamma \\
&= \int \kappa_i(x+t-y)f(y)dy - \sum_{|\gamma| \leq l-1} (t^\gamma/\gamma!) \int D^\gamma \kappa_i(x-y)f(y)dy \\
&\quad - \sum_{|\gamma|=l} (t^\gamma/\gamma!) \lim_{\varepsilon \rightarrow 0} \int_{|x-y| \geq \varepsilon|t|} D^\gamma \kappa_i(x-y)f(y)dy \\
&= \lim_{\varepsilon \rightarrow 0} \int_{|x-y| \geq \varepsilon|t|} (R_i^{l+1}\kappa_i)(x-y)f(y)dy \\
&= \lim_{\varepsilon \rightarrow 0} \int_{|y| \geq \varepsilon|t|} (R_i^{l+1}\kappa_i)(y)f(x-y)dy.
\end{aligned}$$

By Lemma 2.7 (i) we see that

$$\begin{aligned}
(R_i^{l+1}U_i^f)(x) &= \lim_{\varepsilon \rightarrow 0} |t|^{l-n} \int_{|y| \geq \varepsilon|t|} (R_i^{l+1}\kappa_i)(y/|t|)f(x-y)dy \\
&= \lim_{\varepsilon \rightarrow 0} |t|^l \int_{|z| \geq \varepsilon} (R_i^{l+1}\kappa_i)(z)f(x-|t|z)dz.
\end{aligned}$$

Thus we obtain the lemma.

**Corollary 2.9.** (i) *If  $\alpha < l < \alpha + 1$ , then  $(R_i^l \kappa_\alpha)(y)$  is integrable as a function of  $y$  and for all  $t \in \mathbb{R}^n$*

$$\int_{\mathbb{R}^n} (R_i^l \kappa_\alpha)(y)dy = 0.$$

(ii) *If  $l$  is an odd number, then  $(R_i^{l+1} \kappa_i)(y)$  is integrable on  $\{|y| \geq \varepsilon\}$  ( $\varepsilon > 0$ ) and for all  $t \in \mathbb{R}^n$*

$$(2.2) \quad \lim_{\varepsilon \rightarrow 0} \int_{|y| \geq \varepsilon} (R_i^{l+1} \kappa_i)(y)dy = 0.$$

Proof. (i) We easily see that  $(R_i^l \kappa_\alpha)(y)$  is locally integrable for  $l < \alpha + 1$ . It follows from Corollary 2.2 (ii) that

$$|(R_i^l \kappa_\alpha)(y)| \leq C |y|^{\alpha-l-n}$$

for  $|y| \geq |t|/2$ . Hence for  $\alpha < l < \alpha + 1$ ,  $(R_i^l \kappa_\alpha)(y)$  is integrable with respect to  $y$ . Next we shall show

$$\int (R_i^l \kappa_\alpha)(y)dy = 0.$$

Since

$$\int (R_i^l \kappa_\alpha)(y)dy = |t|^\alpha \int (R_i^l \kappa_\alpha)(z)dz$$

by Lemma 2.7 (i), it suffices to show

$$\int (R_{\xi}^l \kappa_{\alpha})(y) dy = 0$$

for all  $|\xi|=1$ . Let  $|\xi|=1$ . We take a function  $f \in \mathcal{D}$  such that  $f(0)=1$ . By Lemma 2.8 (i) we have

$$(R_{r\xi}^l U_{\alpha}^f)(0) = r^{\alpha} \int (R_{\xi}^l \kappa_{\alpha})(z) f(-rz) dz.$$

Since  $l > \alpha$ , it follows from Corollary 2.2 (i) that

$$\frac{(R_{r\xi}^l U_{\alpha}^f)(0)}{r^{\alpha}} \rightarrow 0 \quad (r \rightarrow 0).$$

On the other hand, since  $(R_{\xi}^l \kappa_{\alpha})(z)$  is integrable we see that

$$\int (R_{\xi}^l \kappa_{\alpha})(z) f(-rz) dz \rightarrow \int (R_{\xi}^l \kappa_{\alpha})(z) dz \quad (r \rightarrow 0).$$

Hence we obtain

$$\int (R_{\xi}^l \kappa_{\alpha})(z) dz = 0.$$

(ii) We easily see that  $(R_{\xi}^{l+1} \kappa_l)(y)$  is integrable on  $\{|y| \geq \varepsilon\}$ . For (2.2) it suffices to show

$$\lim_{\varepsilon \rightarrow 0} \int_{|y| \geq \varepsilon} (R_{\xi}^{l+1} \kappa_l)(y) dy = 0$$

for  $|\xi|=1$ . We take a function  $f \in \mathcal{D}$  which satisfies the condition  $f(x)=1$  on  $\{|x| < 1\}$ . By Lemma 2.8 (ii) we have

$$(R_{r\xi}^{l+1} U_l^f)(0) = \lim_{\varepsilon \rightarrow 0} r^l \int_{|z| \geq \varepsilon} (R_{\xi}^{l+1} \kappa_l)(z) f(-rz) dz.$$

It follows from Corollary 2.2 (i) that

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{|z| \geq \varepsilon} (R_{\xi}^{l+1} \kappa_l)(z) f(-rz) dz = 0.$$

We easily see that the left hand side is equal to

$$\lim_{\varepsilon \rightarrow 0} \int_{|z| \geq \varepsilon} (R_{\xi}^{l+1} \kappa_l)(z) dz.$$

Hence we obtain the required property.

**Lemma 2.10.** *Let  $\beta > \alpha - n$ ,  $\beta < l - 1$ , and moreover we assume that  $l > \alpha - n$  in case  $\alpha - n$  is a nonnegative even number. Then*

$$\mu^{l,\alpha,\beta}(y) = \frac{1}{|y|^{n+\beta-\alpha}} \int_{|z| \geq 1/|y|} \frac{(R_z^l \kappa_\alpha)(y')}{|z|^{n+\beta}} dz.$$

Proof. By Lemma 2.7 (ii) and the change of variables  $t/|y|=z$ , we have

$$\begin{aligned} \mu^{l,\alpha,\beta}(y) &= \int_{|t| \geq 1} \frac{(R_t^l \kappa_\alpha)(y)}{|t|^{n+\beta}} dt \\ &= \int_{|t| \geq 1} \frac{|y|^{\alpha-n} (R_{t/|y|}^l \kappa_\alpha)(y')}{|t|^{n-\beta}} dt \\ &= \frac{1}{|y|^{n+\beta-\alpha}} \int_{|z| \geq 1/|y|} \frac{(R_z^l \kappa_\alpha)(y')}{|z|^{n+\beta}} dz. \end{aligned}$$

REMARK 2.11. Let

$$\mu_\varepsilon^{l,\alpha,\beta}(y) = \int_{|t| \geq \varepsilon} \frac{(R_t^l \kappa_\alpha)(y)}{|t|^{n+\beta}} dt.$$

Then

$$\mu_\varepsilon^{l,\alpha,\beta}(y) = \frac{1}{|y|^{n+\beta-\alpha}} \int_{|z| \geq \varepsilon/|y|} \frac{(R_z^l \kappa_\alpha)(y')}{|z|^{n+\beta}} dz.$$

**Lemma 2.12.** *Let  $K_m(x, y) = \sum_{|\gamma|=m} (D^\gamma \kappa_\alpha(x)/\gamma!) y^\gamma$ . Then for  $|x|=|y|$  we have  $K_m(x, y) = K_m(y, x)$ .*

Proof. Let  $|x|=|y|$ . We set  $F_{x,y}(s) = \kappa_\alpha(x+sy)$  for  $-1 < s < 1$ . We easily see that  $F_{x,y}(s) = F_{y,x}(s)$ . Hence  $(d^m/ds^m)F_{x,y}(0) = (d^m/ds^m)F_{y,x}(0)$ . Since

$$(d^m/ds^m)F_{x,y}(0) = \sum_{|\gamma|=m} (m!/\gamma!) D^\gamma \kappa_\alpha(x) y^\gamma,$$

we obtain the required property.

**Lemma 2.13.** *Let  $\beta > \alpha - n$ ,  $\beta > l - 1$ , and moreover we assume that  $l < \alpha - n$  in case  $\alpha - n$  is a nonnegative even number. Then*

$$\mu^{l,\alpha,\beta}(y) = \frac{1}{|y|^{n+\beta-\alpha}} \int_{|v| \leq |y|} \frac{(R_{y'}^l \kappa_\alpha)(v)}{|v|^{\alpha-\beta}} dv.$$

Proof. By Lemma 2.10 we have

$$\mu^{l,\alpha,\beta}(y) = \frac{1}{|y|^{n+\beta-\alpha}} \int_{|u| \geq 1/|y|} \frac{(R_u^l \kappa_\alpha)(y')}{|u|^{n+\beta}} du.$$

Let  $\alpha - n$  be not a nonnegative even number. By the inversion  $u = v/|v|^2$ , we see that

$$\mu^{l,\alpha,\beta}(y) = \frac{1}{|y|^{n+\beta-\alpha}} \int_{|v| \leq |y|} |v|^{\beta-\alpha} \left( \left| |v|y' + \frac{v}{|v|} \right|^{\alpha-n} - \sum_{|\gamma| \leq l-1} (D^\gamma \kappa_\alpha(y')/\gamma!) |v|^{\alpha-2|\gamma|-n} v^\gamma \right) dv.$$

By Lemma 2.12 we have

$$\begin{aligned} & \sum_{|\gamma| \leq l-1} (D^\gamma \kappa_\alpha(y')/\gamma!) |v|^{\alpha-2|\gamma|-n} v^\gamma \\ &= \sum_{m=0}^{l-1} |v|^{\alpha-m-n} \sum_{|\gamma|=m} (D^\gamma \kappa_\alpha(y')/\gamma!) (v/|v|)^\gamma \\ &= \sum_{m=0}^{l-1} |v|^{\alpha-m-n} \sum_{|\gamma|=m} (D^\gamma \kappa_\alpha(v/|v|)/\gamma!) (y')^\gamma \\ &= \sum_{m=0}^{l-1} \sum_{|\gamma|=m} (D^\gamma \kappa_\alpha(v)/\gamma!) (y')^\gamma \\ &= \sum_{|\gamma| \leq l-1} (D^\gamma \kappa_\alpha(v)/\gamma!) (y')^\gamma. \end{aligned}$$

Further, since  $||v|y' + (v/|v|)| = |v+y'|$  by an easy argument, we have

$$\begin{aligned} \mu^{l,\alpha,\beta}(y) &= \frac{1}{|y|^{n+\beta-\alpha}} \int_{|v| \leq |y|} |v|^{\beta-\alpha} \left( |v+y'|^{\alpha-n} - \sum_{|\gamma| \leq l-1} \frac{D^\gamma \kappa_\alpha(v)}{\gamma!} (y')^\gamma \right) dv \\ &= \frac{1}{|y|^{n+\beta-\alpha}} \int_{|v| \leq |y|} \frac{(R_{y'}^l \kappa_\alpha)(v)}{|v|^{\alpha-\beta}} dv. \end{aligned}$$

Next, let  $\alpha-n$  be a nonnegative even number. By the inversion  $u=v/|v|^2$ , we see that

$$\begin{aligned} \mu^{l,\alpha,\beta}(y) &= \frac{1}{|y|^{n+\beta-\alpha}} \int_{|v| \leq |y|} |v|^{\beta-\alpha} \left( (\delta_{\alpha,n} - \log | |v|y' + \frac{v}{|v|} | + \log |v| ) \right. \\ & \quad \left. \times \left| |v|y' + \frac{v}{|v|} \right|^{\alpha-n} - \sum_{|\gamma| \leq l-1} \frac{D^\gamma \kappa_\alpha(y')}{\gamma!} |v|^{\alpha-2|\gamma|-n} v^\gamma \right) dv. \end{aligned}$$

By Lemmas 2.12 and 2.6 we have

$$\begin{aligned} & \sum_{|\gamma| \leq l-1} (D^\gamma \kappa_\alpha(y')/\gamma!) |v|^{\alpha-2|\gamma|-n} v^\gamma \\ &= \sum_{|\gamma| \leq l-1} (D^\gamma \kappa_\alpha(v)/\gamma!) (y')^\gamma + \log |v| \sum_{|\gamma| \leq l-1} (D^\gamma \chi_\alpha(v)/\gamma!) (y')^\gamma. \end{aligned}$$

Therefore, by Lemma 2.4 we obtain

$$\begin{aligned} \mu^{l,\alpha,\beta}(y) &= \frac{1}{|y|^{n+\beta-\alpha}} \int_{|v| \leq |y|} |v|^{\beta-\alpha} \left( (\delta_{\alpha,n} - \log |v+y'|) |v+y'|^{\alpha-n} \right. \\ & \quad \left. - \sum_{|\gamma| \leq l-1} \frac{D^\gamma \kappa_\alpha(v)}{\gamma!} (y')^\gamma + (\log |v|) \left( |v+y'|^{\alpha-n} - \sum_{|\gamma| \leq l-1} \frac{D^\gamma \chi_\alpha(v)}{\gamma!} (y')^\gamma \right) \right) dv \\ &= \frac{1}{|y|^{n+\beta-\alpha}} \int_{|v| \leq |y|} |v|^{\beta-\alpha} \left( (R_{y'}^l \kappa_\alpha)(v) + (\log |v|) (R_{y'}^l \chi_\alpha)(v) \right) dv \end{aligned}$$

$$= \frac{1}{|y|^{n+\beta-\alpha}} \int_{|v| \leq |y|} \frac{(R'_{y'}\kappa_\alpha)(v)}{|v|^{\alpha-\beta}} dv.$$

The proof of the lemma is completed.

Now we prove

**Theorem 2.14.** *Let  $1 < q \leq \infty$ . If  $l-1 < \alpha - (n/q) < i$ , then  $\mu^{l,\alpha,\alpha} \in L^{q'}$ . Moreover, when  $l$  is an odd number,  $\mu^{l,\alpha,\alpha} \in L^{q'}$  still holds for  $l-1 < \alpha - (n/q) < l+1$ .*

Proof. In case  $1 < q < \infty$ , the conclusion follows from Theorem 2.3 since  $\alpha > \alpha - (n/q)$ . Hence it is sufficient to show that if  $l-1 < \alpha < l$ , then  $\mu^{l,\alpha,\alpha} \in L^1$ , and that if  $l$  is odd and  $l-1 < \alpha < l+1$ , then  $\mu^{l,\alpha,\alpha} \in L^1$ .

First let  $l-1 < \alpha < l$ . By Lemma 2.13 and  $\alpha > l-1$ , we see

$$\int_{|y| \leq 1/2} |\mu^{l,\alpha,\alpha}(y)| dy \leq \int_{|y| \leq 1/2} \frac{1}{|y|^n} \int_{|v| \leq |y|} |(R'_{y'}\kappa_\alpha)(v)| dv dy < \infty.$$

Since  $\alpha < l < \alpha + 1$ , by Lemma 2.13 and Corollary 2.9 (i) we have

$$\begin{aligned} \mu^{l,\alpha,\alpha}(y) &= \frac{1}{|y|^n} \int_{|v| \leq |y|} (R'_{y'}\kappa_\alpha)(v) dv \\ &= \frac{1}{|y|^n} \int_{|v| > |y|} (R'_{y'}\kappa_\alpha)(v) dv. \end{aligned}$$

Hence by Corollary 2.2 (ii) we obtain

$$\begin{aligned} \int_{|y| > 1/2} |\mu^{l,\alpha,\alpha}(y)| dy &= \int \frac{1}{|y|^n} \left| \int_{|v| > |y|} (R'_{y'}\kappa_\alpha)(v) dv \right| dy \\ &\leq C \int_{|y| > 1/2} \frac{1}{|y|^n} \int_{|v| > |y|} |v|^{\alpha-l-n} dv dy \\ &= C \int_{|y| > 1/2} |y|^{\alpha-l-n} dy < \infty \end{aligned}$$

since  $l < \alpha$ . Thus  $\mu^{l,\alpha,\alpha} \in L^1$  for  $l-1 < \alpha < l$ .

Next let  $l$  be an odd number and  $l < \alpha < l+1$ . Since  $\alpha > l > l-1$ , we see

$$\int_{|y| \geq 1/2} |\mu^{l,\alpha,\alpha}(y)| dy < \infty$$

by the same reason as above. Since  $l$  is an odd number and  $l < \alpha$ , for  $|\gamma| = l$  we have

$$\int_{|v| \leq |y|} D^\gamma \kappa_\alpha(v) dv = 0.$$

Hence by Lemma 2.13 we have

$$\mu^{l,\alpha,\alpha}(y) = \frac{1}{|y|^n} \int_{|v| \leq |y|} (R'_y{}^l \kappa_\alpha)(v) dv = \frac{1}{|y|^n} \int_{|v| \leq |y|} (R_{y'}^{l+1} \kappa_\alpha)(v) dv .$$

Furthermore, because of  $\alpha < l + 1 < \alpha + 1$  by Corollary 2.9 (i) we obtain

$$\mu^{l,\alpha,\alpha}(y) = \frac{1}{|y|^n} \int_{|v| > |y|} (R_{y'}^{l+1} \kappa_\alpha)(v) dv .$$

Therefore by Corollary 2.2 (ii) we have

$$\begin{aligned} \int_{|y| > 1/2} |\mu^{l,\alpha,\alpha}(y)| dy &\leq C \int_{|y| > 1/2} \frac{1}{|y|^n} \int_{|v| > |y|} |v|^{\alpha-l-1-n} dv dy \\ &= C \int_{|y| > 1/2} |y|^{\alpha-l-1-n} dy < \infty \end{aligned}$$

since  $\alpha < l + 1$ .

Finally let  $l$  be an odd number and  $\alpha = l$ . We easily see

$$\int_{|y| \leq 1/2} |\mu^{l,l,l}(y)| dy < \infty .$$

Since  $l$  is an odd number, for  $|y| = l$  we obtain

$$\int_{\varepsilon < |v| \leq |y|} D^y \kappa_l(v) dv = 0, \quad \varepsilon > 0 .$$

Hence by Lemma 2.13 and Corollary 2.9 (ii) we have

$$\begin{aligned} \mu^{l,l,l}(y) &= \frac{1}{|y|^n} \int_{|v| \leq |y|} (R'_y{}^l \kappa_l)(v) dv \\ &= \frac{1}{|y|^n} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |v| \leq |y|} (R_{y'}^{l+1} \kappa_l)(v) dv \\ &= \frac{1}{|y|^n} \int_{|v| > |y|} (R_{y'}^{l+1} \kappa_l)(v) dv . \end{aligned}$$

Therefore by Corollary 2.2 (ii) we obtain

$$\begin{aligned} \int_{|y| > 1/2} |\mu^{l,l,l}(y)| dy &\leq C \int_{|y| > 1/2} \frac{1}{|y|^n} \int_{|v| > |y|} |v|^{-1-n} dv dy \\ &= C \int_{|y| > 1/2} |y|^{-1-n} dy < \infty . \end{aligned}$$

The proof of the theorem is completed.

### 2.3. $\mu_*^{l,\alpha,\beta}$ -potentials

For a locally integrable function  $f$  we set

$$V_{l,\alpha,\beta}^f(x) = \int \mu_*^{l,\alpha,\beta}(y) f(x-y) dy .$$

**Theorem 2.15.** *Let  $f$  be a nonnegative  $L^p$ -function,  $\beta > \alpha - (n/p)$  and  $\beta > l - 1$ . If  $\alpha - (n/p) < l < \alpha + 1$ , then  $V_{l,\alpha,\beta}^f(x) < \infty$  for almost every  $x$ . In particular, if  $\alpha - (n/p) < l < \alpha - (n/p) + 1$ , then  $V_{l,\alpha,\beta}^f(x) < \infty$  for all  $x$ .*

Proof. We have

$$\begin{aligned} V_{l,\alpha,\beta}^f(x) &= \int_{|t| \geq 1} \frac{dt}{|t|^{n+\beta}} \int_{d(y, L-t) \geq |t|/2} |(R_t^l \kappa_\alpha)(y)| f(x-y) dy \\ &\quad + \int f(x-y) dy \int_{|t| \geq 1, d(y, L-t) < |t|/2} \frac{|(R_t^l \kappa_\alpha)(y)|}{|t|^{n+\beta}} dt \\ &= V_1(x) + V_2(x) . \end{aligned}$$

It follows from Corollary 2.2 (ii), Hölder's inequality,  $\alpha - (n/p) < l$  and  $\beta > \alpha - (n/p)$  that

$$\begin{aligned} V_1(x) &\leq C \int_{|t| \geq 1} \frac{dt}{|t|^{n+\beta}} \int_{|y| \geq |t|/2} |t|^l |y|^{\alpha-l-n} f(x-y) dy \\ &\leq C \int_{|t| \geq 1} \frac{|t|^l}{|t|^{n+\beta}} dt \left( \int_{|y| \geq |t|/2} |y|^{(\alpha-l-n)p'} dy \right)^{1/p'} \|f\|_p \\ &= C \|f\|_p \int_{|t| \geq 1} \frac{|t|^{\alpha-(n/p)}}{|t|^{n+\beta}} dt < \infty . \end{aligned}$$

For  $V_2(x)$  we see that

$$\begin{aligned} V_2(x) &\leq \int f(x-y) dy \int_{|t| \geq 1, d(y, L-t) < |t|/2} \frac{|\kappa_\alpha(y+t)|}{|t|^{n+\beta}} dt \\ &\quad + C \sum_{|\gamma| \leq l-1} \int |D^\gamma \kappa_\alpha(y)| f(x-y) dy \int_{|t| \geq 1, d(y, L-t) < |t|/2} \frac{|t|^{|\gamma|}}{|t|^{n+\beta}} dt \\ &= V_{21}(x) + V_{22}(x) . \end{aligned}$$

Let  $\alpha - n$  be not a nonnegative even number. Since

$$\int_{|t| \geq 1, d(y, L-t) < |t|/2} \frac{|y+t|^{\alpha-n}}{|t|^{n+\beta}} dt \leq C(1+|y|)^{\alpha-\beta-n} ,$$

we obtain

$$V_{21}(x) \leq C \int (1+|y|)^{\alpha-\beta-n} f(x-y) dy < \infty$$

because of  $\beta > \alpha - (n/p)$ . By  $\beta > l - 1$  we have

$$\begin{aligned}
 V_{22}(x) &\leq C \sum_{|\gamma| \leq l-1} \int |y|^{\alpha-|\gamma|-n} f(x-y) dy \int_{|t| \geq 1, d(y, L-t) < |t|/2} \frac{|t|^{|\gamma|}}{|t|^{n+\beta}} dt \\
 &\leq C \sum_{|\gamma| \leq l-1} \int |y|^{\alpha-|\gamma|-n} (1+|y|)^{-\beta+|\gamma|} f(x-y) dy \\
 &\leq C \sum_{|\gamma| \leq l-1} \int_{|y| < 1} |y|^{\alpha-|\gamma|-n} f(x-y) dy + C \sum_{|\gamma| \leq l-1} \int_{|y| \geq 1} |y|^{\alpha-\beta-n} f(x-y) dy \\
 &= V_{221}(x) + V_{222}(x).
 \end{aligned}$$

It follows from  $\beta > \alpha - (n/p)$  that  $V_{222}(x) < \infty$  for all  $x$ . Moreover, by  $l-1 < \alpha$  and Young's inequality we obtain  $V_{221}(x) < \infty$  for almost every  $x$ . In particular, if  $l-1 < \alpha - (n/p)$ , then by Hölder's inequality we have  $V_{221}(x) < \infty$  for all  $x$ . In case  $\alpha - n$  is a nonnegative even number, the proof is similar. Therefore we obtain the theorem.

### 3. Hypersingular integrals of Beppo Levi functions

For an integer  $k < \alpha$ , we set

$$\kappa_{\alpha,k}(x, y) = \begin{cases} (R_x^{k+1} \kappa_{\alpha})(-y), & 0 \leq k < \alpha, \\ \kappa_{\alpha}(x-y), & k \leq -1. \end{cases}$$

Further, for a locally integrable function  $f$  we set

$$U_{\alpha,k}^f(x) = \int \kappa_{\alpha,k}(x, y) f(y) dy.$$

We denote by  $[r]$  the integral part of a real number  $r$  and by  $f|_E$  the restriction of a function  $f$  to a set  $E$ .

**Lemma 3.1.** ([1]) *Let  $k = [\alpha - (n/p)]$  and  $f \in L^p$ . If  $\alpha - (n/p)$  is not a nonnegative integer, then  $U_{\alpha,k}^f$  exists, and if  $\alpha - (n/p)$  is a nonnegative integer, then  $U_{\alpha,k-1}^{f_1}$  and  $U_{\alpha,k}^{f_2}$  exist where  $f_1 = f|_{B_1}$ ,  $f_2 = f - f_1$  and  $B_1 = \{|x| < 1\}$ .*

**Lemma 3.2.** *Let  $k = [\alpha - (n/p)]$  and  $f \in L^p$ .*

(i) *If  $\alpha - (n/p)$  is not a nonnegative integer, then for  $\alpha - (n/p) < l < \alpha + 1$  we have*

$$(R_l^! U_{\alpha,k}^f)(x) = \int (R_l^! \kappa_{\alpha})(y) f(x-y) dy.$$

(ii) *If  $\alpha - (n/p)$  is a nonnegative integer, then for  $\alpha - (n/p) < l < \alpha + 1$  we have*

$$(R_l^! (U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}))(x) = \int (R_l^! \kappa_{\alpha})(y) f(x-y) dy$$

where  $f_1$  and  $f_2$  are as in Lemma 3.1.

Proof. By  $l-1 < \alpha$ ,  $l > \alpha - (n/p)$  and Lemma 2.4 we have

$$\begin{aligned}
 (R_l^i U_{\alpha,k}^f)(x) &= U_{\alpha,k}^f(x+t) - \sum_{|\gamma| \leq l-1} (D^\gamma U_{\alpha,k}^f(x)/\gamma!) t^\gamma \\
 &= \int (\kappa_\alpha(x+t-y) - \sum_{|\delta| \leq k} ((x+t)^\delta/\delta!) D^\delta \kappa_\alpha(-y)) f(y) dy \\
 &\quad - \sum_{|\gamma| \leq l-1} (t^\gamma/\gamma!) D_x^\gamma \int (\kappa_\alpha(x-y) - \sum_{|\delta| \leq k} (x^\delta/\delta!) D^\delta \kappa_\alpha(-y)) f(y) dy \\
 &= \int (\kappa_\alpha(x+t-y) - \sum_{|\gamma| \leq l-1} (t^\gamma/\gamma!) D_x^\gamma \kappa_\alpha(x-y)) f(y) dy \\
 &\quad - \sum_{|\delta| \leq k} \int (((x+t)^\delta/\delta!) - \sum_{|\gamma| \leq l-1} (t^\gamma/\gamma!) D^\gamma (x^\delta/\delta!)) D^\delta \kappa_\alpha(-y) f(y) dy \\
 &= \int (R_l^i \kappa_\alpha)(x-y) f(y) dy - \sum_{|\delta| \leq k} \int (R_l^i (x^\delta/\delta!)) D^\gamma \kappa_\alpha(-y) f(y) dy \\
 &= \int (R_l^i \kappa_\alpha)(y) f(x-y) dy.
 \end{aligned}$$

Since the proof of (ii) is similar, we obtain the lemma.

REMARK 3.3. Let  $f \in L^p$ . Then for  $\alpha - (n/p) < l < \alpha + 1$  we have

$$\int |(R_l^i \kappa_\alpha)(y) f(x-y)| dy < \infty$$

for almost every  $x$ . In particular, if  $\alpha - (n/p) < l < \alpha - (n/p) + 1$ , then

$$\int |(R_l^i \kappa_\alpha)(y) f(x-y)| dy < \infty$$

for all  $x$ .

For a positive integer  $l$  and  $\beta > 0$ , we set

$$H_\varepsilon^{l,\beta} u(x) = \int_{|t| \geq \varepsilon} \frac{(R_l^i u)(x)}{|t|^{n+\beta}} dt, \quad \varepsilon > 0.$$

Recall that in Remark 2.11 we defined  $\mu^{l,\alpha,\beta}$  as

$$\mu_\varepsilon^{l,\alpha,\beta}(y) = \int_{|t| \geq \varepsilon} \frac{(R_l^i \kappa_\alpha)(y)}{|t|^{n+\beta}} dt.$$

**Proposition 3.4.** Let  $\alpha - (n/p) < l < \alpha + 1$ ,  $\beta > \alpha - (n/p)$ ,  $\beta > l - 1$ ,  $k = [\alpha - (n/p)]$  and  $f \in L^p$ .

(i) If  $\alpha - (n/p)$  is not a nonnegative integer, then

$$H_\varepsilon^{l,\beta} U_{\alpha,k}^f(x) = \int \mu_\varepsilon^{l,\alpha,\beta}(y) f(x-y) dy.$$

(ii) If  $\alpha - (n/p)$  is a nonnegative integer, then

$$H_{\alpha}^{l,\beta}(U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2})(x) = \int \mu_{\varepsilon}^{l,\alpha,\beta}(y)f(x-y)dy .$$

Proof. (i) By Lemma 3.2, Theorem 2.15 and Fibini's theorem we have

$$\begin{aligned} H_{\varepsilon}^{l,\beta} U_{\alpha,k}^f(x) &= \int_{|t|\geq\varepsilon} \frac{(R_t^l U_{\alpha,k}^f)(x)}{|t|^{n+\beta}} dt \\ &= \int_{|t|\geq\varepsilon} \frac{dt}{|t|^{n+\beta}} \int (R_t^l \kappa_{\alpha})(y)f(x-y)dy \\ &= \int f(x-y)dy \int_{|t|\geq\varepsilon} \frac{(R_t^l \kappa_{\alpha})(y)}{|t|^{n+\beta}} dt \\ &= \int \mu_{\varepsilon}^{l,\alpha,\beta}(y)f(x-y)dy . \end{aligned}$$

The proof of (ii) is similar. Hence the proposition is proved.

**Lemma 3.5.** *Let  $\beta > \alpha - n$ ,  $\beta > l - 1$ , and moreover we assume that  $l > \alpha - n$  in case  $\alpha - n$  is a nonnegative even number. Then*

$$\frac{1}{\varepsilon^{n+\beta-\alpha}} \mu_{\varepsilon}^{l,\alpha,\beta}\left(\frac{y}{\varepsilon}\right) = \mu_{\varepsilon}^{l,\alpha,\beta}(y) .$$

Proof. This lemma follows from Remark 2.11.

Let  $R_j$  ( $j=1, \dots, n$ ) be the Riesz transforms, namely

$$R_j f(x) = \lim_{\varepsilon \rightarrow 0} c_n \int_{|x-y|\geq\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy$$

with  $c_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}$  for  $f \in L^p$ . For a multi-index  $\gamma = (\gamma_1, \dots, \gamma_n)$  we set

$$R^{\gamma} = R_1^{\gamma_1} \dots R_n^{\gamma_n} .$$

For a positive integer  $m$  and  $p > 1$ , the Beppo Levi space  $\mathcal{L}_m^p$  is defined as follows:

$$\mathcal{L}_m^p = \{u \in \mathcal{D}'; |u|_{m,p} = \sum_{|\gamma|=m} \|D^{\gamma}u\|_p < \infty\} .$$

**Lemma 3.6.** ([1]) *Let  $k = [m - (n/p)]$  and  $u \in \mathcal{L}_m^p$ .*

(i) *If  $m - (n/p)$  is not a nonnegative integer, then  $u$  can be represented as*

$$u(x) = \sum_{|\delta| \leq m-1} a_{\delta} x^{\delta} + U_{m,k}^f(x)$$

where  $f = c_{m,n} \sum_{|\gamma|=m} (m!/\gamma!) R^{\gamma} D^{\gamma} u$  and  $c_{m,n}$  is the constant.

(ii) *If  $m - (n/p)$  is a nonnegative integer, then  $u$  can be represented as*

$$u(x) = \sum_{|\delta| \leq m-1} a_{\delta} x^{\delta} + U_{m,k-1}^{f_1}(x) + U_{m,k}^{f_2}(x)$$

where  $f_1=f|_{B_1}$ ,  $f_2=f-f_1$  and  $f$  is as in (i).

**Theorem 3.7.** Let  $u \in \mathcal{L}_m^p$ .

(i) If  $n < q < \infty$ ,  $p \leq q$ ,  $\beta > m - (n/q)$  and  $(1/r) = (1/p) - (1/q)$ , then

$$\|H_\varepsilon^{m,\beta} u\|_r \leq C \varepsilon^{m-\beta-(n/q)} \|u\|_{m,p}.$$

(ii) If  $\beta \geq m$ , then

$$\|H_\varepsilon^{m,\beta} u\|_p \leq C \varepsilon^{m-\beta} \|u\|_{m,p}.$$

Proof. Let  $m - (n/p)$  be not a nonnegative integer. By Lemma 3.6 (i) we have

$$u(x) = \sum_{|\delta| \leq m-1} a_\delta x^\delta + U_{m,k}^f(x).$$

By Lemma 2.4 we see that  $H_\varepsilon^{m,\beta} u = H_\varepsilon^{m,\beta} U_{m,k}^f$ . Therefore by Proposition 3.4 we obtain

$$H_\varepsilon^{m,\beta} u(x) = \int \mu_\varepsilon^{m,m,\beta}(y) f(x-y) dy.$$

Hence by Young's inequality we have

$$\|H_\varepsilon^{m,\beta} u\|_r \leq \|\mu_\varepsilon^{m,m,\beta}\|_{q'} \|f\|_p.$$

Consequently, (i) and (ii) follows from Theorems 2.3, 2.14, Lemma 3.5 and  $\|f\|_p \leq C \|u\|_{m,p}$ . In case  $m - (n/p)$  is a nonnegative integer, the proof is similar. Thus we obtain the theorem.

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