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THE METHOD OF MOVING FRAMES APPLIED TO KÄHLER SUBMANIFOLDS OF COMPLEX SPACE FORMS[1]

Dedicated to Professor Shingo Murakami on his 60th birthday

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0. Introduction

In this article, we will study Kähler submanifolds of complex space forms by the method of moving frames, which was used originally by E. Cartan in his researches for submanifolds of homogeneous spaces. In the last twenty years, the method itself has been reviewed and discussed in fairly rigorous ways by several authors. In [5], Griffiths pointed out among several problems that it is possible to prove the rigidity theorem of Calabi by the moving frame method for holomorphic curves of complex projective spaces. We would like to show that the method of moving frames works well also for Kähler submanifolds of complex space forms. We note that H.-S. Tai [12] constructed "Frenet frames" for complex submanifolds of complex projective spaces and applied them to solving problems for surfaces in $\boldsymbol{P}^5(\boldsymbol{C})$, but our interest lies in a different place. We will study the rigidity and the homogeneity of Kähler submanifolds of complex space forms.

Let $S_c(N)$ be the N-dimensional complex space form of holomorphic sectional curvature 4c. Let $G_c(N)$ be the group of holomorphic isometries of $S_c(N)$ and $\mathfrak{g}_c(N)$ its Lie algebra. Basically we follow the formulation presented by Sulanke and Švec [10, 11], which interprets E. Cartan's method of moving frames in terms of fibre bundles and Lie algebra valued 1-forms. In **2**, we will introduce the S_c-structure $(\boldsymbol{P}, \omega)$ over a connected complex manifold M (see Definition 2.1), where $\boldsymbol{P}$ is a principal fiber bundle over M and ω is a $\mathfrak{g}_c(N)$-valued 1-form on $\boldsymbol{P}$. This is a kind of G, H-structures in the sense of [10] with some additional characteristic properties and it gives an interpretation of the higher order structure equations of a Kähler immersion of a connected Kähler manifold (M, g) into $S_c(N)$.

In **3**, we will prove the uniqueness of S_c-structures for the Kähler metric

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on M (see Theorem 3.1) and give a proof of the rigidity theorem of Calabi from our stand point of view (see Corollary 3.1). We will show that a Kähler immersion of M into $S_c(N)$ induces an S_c-structure over M and conversely if M is simply connected, every S_c-structure over M is obtained in this way (see Theorem 3.2).

In **4**, we will carry out the reduction of structure group of an S_c-structure in an appropriate way to our case, and define a subbundle RF, associated with the S_c-structure, of the unitary frame bundle over a certain open set of M. Then our final result is, roughly speaking, as follows: a connected complete Kähler submanifold (M, g) of $S_c(N)$ is homogeneous if and only if all the coefficients of every component of ω restricted to RF with respect to a canonically chosen co-frame field on RF are constant. In addition, when this is the case, RF can be regarded as the group of holomorphic isometries of (M, g) imbedded in the bundle of unitary frames of M in a natural manner (see Theorems 4.1 and 4.2).

1. Preliminaries

In this paper we will use the following ranges of indices:

$$1 \leq A, B, C \leq N, \quad 1 \leq i, j, k \leq n, \quad n_{p-1} < r(p), s(p), t(p) \leq n_p,$$

where $\{n_0, n_1, \cdots, n_d\}$ is a sequence of integers with $0 = n_0 < n = n_1 < \cdots < n_d$. Note that the indices $i, j, k, r(1), s(1)$, and $t(1)$ will have the same range. For any repeated indices we will always take the summation over the corresponding ranges. Differential forms on manifolds are assumed to take their values in $\boldsymbol{C}$, the field of complex numbers, unless otherwise stated. For a matrix $(X^\lambda_\mu)_{\lambda=1,\cdots,q,\mu=1,\cdots,m}$ we often denote it by $(X^\lambda_\mu)_{\lambda\mu}$ or (X^λ_μ) if the ranges of indices are obviously understood there. For a Lie group G, its Lie algebra is denoted by $\mathrm{Lie}(G)$ or by the corresponding German lower-case letter. Let $\pi_P: P \to M$ be a principal G-bundle over an m-dimensional manifold M. For any $E \in \mathfrak{g}$ we denote by E^* the vector field on P induced by the infinitesimal right action of E. Suppose that P is a G-structure over M, i.e., G is a Lie subgroup of $GL(m; \boldsymbol{R})$ and P is a principal G-subbundle of the linear frame bundle $L(M)$ of M, $L(M) = \{e: \boldsymbol{R}^m \to T_xM$ linear isomorphism, $x \in M\}$. Then P admits the canonical form $\phi_{\boldsymbol{R}} = (\phi^\lambda_{\boldsymbol{R}})_{\lambda=1,\cdots,m}$, which is an $\boldsymbol{R}^m$-valued 1-form on P defined by $\phi_{\boldsymbol{R}}(X^*) = e^{-1}(\pi_{P*}X^*)$ for $X^* \in T_eP$, the tangent space to P at $e \in P$. If M is an n-dimensional complex manifold with complex structure J, we regard $\phi_{\boldsymbol{R}}$ as a $\boldsymbol{C}^n$-valued form, identifying $\boldsymbol{R}^{2n}$ with $\boldsymbol{C}^n$ by $(x^1, y^1, \cdots, x^n, y^n) \to (x^1 + \sqrt{-1}\, y^1, \cdots, x^n + \sqrt{-1}\, y^n)$. Then we call it the $\boldsymbol{C}^n$-valued canonical form and denote it by $\phi = (\phi^i)_{i=1,\cdots,n}$: $\phi^j = \phi_{\boldsymbol{R}}{}^{2i-1} + \sqrt{-1}\, \phi_{\boldsymbol{R}}{}^{2i}$ $(i = 1, \cdots, n)$. Suppose that the complex manifold M carries a hermitian metric g. We denote by $U(M, g)$ the bundle of unitary frames over M. A frame at $x \in M$ is called a unitary frame, if it is a complex linear isometry of $\boldsymbol{C}^n$ with the standard inner product $\langle z, w\rangle = \bar{z}^i w^i$ $(z, w \in \boldsymbol{C}^n)$ onto $T_x M$ with inner product g_x and complex structure J_x. For a Lie group G with Lie algebra $\mathfrak{g}$, the Maurer-Cartan form Φ of G is the $\mathfrak{g}$-valued 1-form on G defined by $\Phi(X) = L_{a*}^{-1}(X)$ $(X \in T_a G)$ under the usual identification of $\mathfrak{g}$ with the tangent space to G at the identity element $e \in G$, where L_a denotes the left multiplication on G by $a \in G$.

Now let $S_c(N)$ be the N-dimensional simply connected complex space form with complex structure J_c and Kähler metric g_c of constant holomorphic sectional curvature 4c. As a complex manifold, $S_c(N)$ is the N-dimensional complex projective space, the vector space $\boldsymbol{C}^N$ of N-tuples of complex numbers ${}^t(z^1, \cdots, z^N)$, or the unit ball $\boldsymbol{D}^N = \{z \in \boldsymbol{C}^N : \|z\| < 1\}$ according as c is positive, zero, or negative respectively. Let $G_c(N)$ be the group of isometric and holomorphic transformations of $S_c(N)$. In the following, we always consider $S_c(N)$ as the quotient space of the Lie group $G_c(N)$ by an isotropy subgroup H at a point and denote the projection of $G_c(N)$ onto $S_c(N) = G_c(N)/H$ by π_G. Let $\mathfrak{g}_c(N)$ and $\mathfrak{h}$ be the Lie algebras of $G_c(N)$ and H respectively and

$$\mathfrak{g}_c(N) = \mathfrak{h} + \mathfrak{n} \tag{1,1}$$

the canonical decomposition of the symmetric pair $(\mathfrak{g}_c(N), \mathfrak{h})$. If we identify $\mathfrak{g}_c(N)$ with the tangent space $T_e G_c(N)$ to $G_c(N)$ at the identity element e, $\pi_{G*}|_{\mathfrak{n}}$ maps $\mathfrak{n}$ isomorphically onto the tangent space $T_o S_c(N)$ at $o = H$. We fix a linear isomorphism $\mathcal{E}$: $\boldsymbol{C}^N \to \mathfrak{n}$ so that $e_o = \pi_{G*}|_{\mathfrak{n}} \circ \mathcal{E}$: $\boldsymbol{C}^N \to T_o S_c(N)$ becomes a complex liner isometry of $\boldsymbol{C}^N$ with the standard inner product onto $T_o S_c(N)$. The things above may be given as follows.

In the case $c > 0$,

$$G_c(N) = SU(N+1)/C\,,$$
$$H = S(U(1) \times U(N))/C = \left(\begin{bmatrix} U(1) & 0 \\ 0 & U(N) \end{bmatrix} \cap SU(N+1)\right)/C\,,$$

where $U(m)$ is the group of $m \times m$ unitary matrices, $SU(N+1)$ the group of $(N+1) \times (N+1)$ unitary matrices of determinant 1, and C its subgroup of $(N+1) \times (N+1)$ scalar matrices $a 1_{N+1}$ with $a^{N+1} = 1$;

$\mathfrak{g}_c(N) = \mathfrak{su}(N+1)$, the Lie algebra of $(N+1) \times (N+1)$ skew hermitian matrices of null trace;

$$\mathfrak{h} = \left\{\begin{bmatrix} a & 0 \\ 0 & A \end{bmatrix} : a \in \sqrt{-1}\,\boldsymbol{R},\ A \in M_N(\boldsymbol{C}),\ {}^t\bar{A} + A = 0,\ \operatorname{tr} A + a = 0\right\};$$
$$\mathfrak{n} = \left\{\begin{bmatrix} 0 & -{}^t\bar{z} \\ z & 0 \end{bmatrix} : z \in \boldsymbol{C}^N\right\};$$
$$\mathcal{E}(z) = \begin{bmatrix} 0 & -\sqrt{c}\,{}^t\bar{z} \\ \sqrt{c} & 0 \end{bmatrix} \quad (z \in \boldsymbol{C}^N)\,.$$

In the case $c=0$,

$$G_c(N)=\boldsymbol{C}^N\cdot U(N)=\begin{bmatrix}1 & 0\\ \boldsymbol{C}^N & U(N)\end{bmatrix},\ H=U(N)\ ;$$

$$\mathfrak{g}_c(N)=\left\{\begin{bmatrix}0 & 0\\ z & A\end{bmatrix}: z\in\boldsymbol{C}^N,\ A\in M_N(\boldsymbol{C}),\ {}^t\bar{A}+A=0\right\}\ ;$$

$$\mathfrak{h}=\left\{\begin{bmatrix}0 & 0\\ 0 & A\end{bmatrix}: A\in M_N(\boldsymbol{C}),\ {}^t\bar{A}+A=0\right\}\ ;$$

$$\mathfrak{n}=\left\{\begin{bmatrix}0 & 0\\ z & 0\end{bmatrix}: z\in\boldsymbol{C}^N\right\}\ ;$$

$$\mathcal{E}(z)=\begin{bmatrix}0 & 0\\ z & 0\end{bmatrix}\quad (z\in\boldsymbol{C}^N)\,.$$

In the case $c<0$,

$G_c(N)=SU(1,N)/C$, where $SU(1,N)$ is the group of $(N+1)\times(N+1)$ matrices which leave the quadratic form $-|z^0|^2+|z^1|^2+\cdots+|z^N|^2$ on $\boldsymbol{C}^{N+1}$ invariant and C its center;

$H=S(U(1)\times U(N))/C$;

$$\mathfrak{g}_c(N)=\left\{\begin{bmatrix}a & {}^t\bar{z}\\ z & A\end{bmatrix}: a\in\sqrt{-1}\,\boldsymbol{R},\ z\in\boldsymbol{C}^N,\ {}^t\bar{A}+A=0,\ \mathrm{tr}\,A+a=0\right\}\ ;$$

$$\mathfrak{h}=\left\{\begin{bmatrix}a & 0\\ 0 & A\end{bmatrix}\in\mathfrak{g}_c(N)\right\}\ ;$$

$$\mathfrak{n}=\left\{\begin{bmatrix}0 & {}^t\bar{z}\\ z & 0\end{bmatrix}: z\in\boldsymbol{C}^N\right\}\ ;$$

$$\mathcal{E}(z)=\begin{bmatrix}0 & \sqrt{-c}\,{}^t\bar{z}\\ \sqrt{-c}\,z & 0\end{bmatrix}\quad (z\in\boldsymbol{C}^N)\,.$$

Let Φ be the Maurer-Cartan form of $G_c(N)$. We denote by $\Phi^{\mathfrak{h}}$ and $\Phi^{\mathfrak{n}}$ the $\mathfrak{h}$- and $\mathfrak{n}$-part of Φ with respect to the decomposition (1, 1). They satisfy that $\Phi^{\mathfrak{h}}(E^*)=E$, $R_a^*\Phi^{\mathfrak{h}}=\mathrm{Ad}\,a^{-1}\,\Phi^{\mathfrak{h}}$, $\Phi^{\mathfrak{n}}(E^*)=0$, and $R_a^*\,\Phi^{\mathfrak{n}}=\mathrm{Ad}\,a^{-1}\,\Phi^{\mathfrak{n}}(E\in\mathfrak{h},\ a\in H)$, where R_a denotes the right multiplication on $G_c(N)$ by a. In particular, the $\mathfrak{h}$-valued 1-form $\Phi^{\mathfrak{h}}$ defines a connection in the principal H-bundle $\pi_G\colon G_c(N)\to S_c(N)$. The natural representation ρ of H on $\mathfrak{n}$ induced by the adjoint representation of $G_c(N)$ on $\mathfrak{g}_c(N)$ corresponds to the linear isotropy representation of H on $T_oS_c(N)$ under the isomorphism $\pi_{G*}|_{\mathfrak{n}}$. If we identify $\mathfrak{n}$ with $\boldsymbol{C}^N$ by $\mathcal{E}$, ρ maps H isomorphically onto $U(N)$. In the following, using these $\mathcal{E}$ and ρ, we identify $\mathfrak{h}$ with $\mathfrak{u}(N)$, the Lie algebra of $N\times N$ skew hermitian matrices, and $\mathfrak{n}$ with $\boldsymbol{C}^N$:

$$\mathfrak{g}_c(N)=\mathfrak{u}(N)+\boldsymbol{C}^N\,. \tag{1,2}$$

Here we note that the $U(N)$-action Ad on the right hand side of (1, 2) corresponding to the adjoint action of H on $\mathfrak{g}_c(N)$ is given by

$$\operatorname{Ad} a\cdot((X^A_B)_{A,B=1,\cdots,N}, (Y^A)_{A=1,\cdots,N}) = (a(X^A_B)\, a^{-1}, a(Y^A))$$
$$((X^A_B)\in \mathfrak{u}(N),\ (Y^A)\in \boldsymbol{C}^N,\ a\in U(N))\,.$$

Let $(\varphi^A_B)_{A,B=1,\cdots,N}=\rho_*(\Phi^{\mathfrak{h}})$ be the $\mathfrak{u}(N)$-part and $(\varphi^A)_{A=1,\cdots,N}=\mathcal{E}^{-1}(\Phi^{\mathfrak{n}})$ the $\boldsymbol{C}^N$-part of Φ with respect to the decomposition (1, 2). They are actually given by

$$\varphi^A_B = \begin{cases}\Phi^A_B-\delta^A_B\,\Phi^0_0 & (c\neq 0)\\ \Phi^A_B & (c=0)\end{cases} \tag{1,3}$$

$$\varphi^A = \begin{cases}\dfrac{1}{\sqrt{|c|}}\,\Phi^A_0 & (c\neq 0)\\ \Phi^A_0 & (c=0)\,.\end{cases} \tag{1,4}$$

In the following, to avoid using excessive symbols, for any $X\in\mathfrak{g}_c(N)$ we will always denote by $(X^A_B)_{A,B=1,\cdots,N}$ and $(X^A)_{A=1,\cdots,N}$ the $\mathfrak{u}(N)$- and $\boldsymbol{C}^N$-part of X with respect to the decomposition (1, 2). So the above φ^A_B and φ^A will be denoted by Φ^A_B and Φ^A respectively in the rest of this paper.

Since the tangent bundle $TS_c(N)$ is associated with the principal bundle $\pi_G\colon G_c(N)\to S_c(N)$ by the representation ρ, the $\mathfrak{u}(N)$-valued 1-form $(\Phi^A_B)=\rho_*(\Phi^{\mathfrak{h}})$ defines a hermitian connection in the tangent bundle $TS_c(N)$.

Let $\pi_c\colon U(S_c(N))\to S_c(N)$ be the bundle of unitary frames over $S_c(N)$. Fixing the frame $e_o=\pi_{G^*}|_{\mathfrak{n}}\circ\mathcal{E}$, we define $\iota\colon G_c(N)\to U(S_c(N))$ by $\iota(a)=L_{a*}e_o$ $(a\in G_c(N))$, where L_{a*} denotes the action of a on $U(S_c(N))$. Then we have

$$\iota(ab)=L_{a*}\,\iota(b),\ \iota(ah)=\iota(a)\cdot\rho(h)\quad (a, b\in G_c(N),\ h\in H)$$

and ι is a diffeomorphism. We frequently identify $G_c(N)$ with $U(S_c(N))$ by ι.

Lemma 1.1. (i) *The $\boldsymbol{C}^N$-valued 1-form (Φ^A) on $G_c(N)$ is equal to the $\boldsymbol{C}^K$-valued canonical form on $U(S_c(N))$ under the identification $G_c(N)=U(S_c(N))$, that is,*

$$\Phi^A(X)\,e_A=\pi_{c*}(\iota_*(X))\quad (X\in T_aG_c(N),\ e=(e_1,\cdots,e_N)=\iota(a))\,. \tag{1,5}$$

(ii) *Similarly, the $\mathfrak{u}(N)$-valued 1-form (Φ^A_B) on $G_c(N)$ is the Levi-Civita connection form on $U(S_c(N))$.*
(iii) *The Maurer-Cartan equation $d\Phi+[\Phi,\Phi]=0$ for Φ is equivalent to the structure equations of $S_c(N)$,*

$$d\Phi^A+\Phi^A_B\wedge\Phi^B=0\,, \tag{1,6}$$

$$d\Phi^A_B+\Phi^A_C\wedge\Phi^C_B=c(\Phi^A\wedge\bar{\Phi}^B+\delta^A_B\,\Phi^C\wedge\bar{\Phi}^C)\,. \tag{1,7}$$

Proof. We have

$$\Phi^A(X)\, e_A = e((\Phi^A(X))_A) = L_{a*}\, e_o(\mathcal{E}^{-1}\,\Phi^{\mathfrak{n}}(X)) = L_{a*}\,\pi_{G*}\,\mathcal{E}(\mathcal{E}^{-1}\,\Phi^{\mathfrak{n}}(X))$$
$$= L_{a*}\,\pi_{G^*}(\Phi(X)) = L_{a*}\,\pi_{G^*}(L_{a*}{}^{-1}X) = \pi_{c^*}(\iota(X))\,,$$

showing (i). The equations (1, 6) and (1, 7) can be obtained directly from the Maurer-Cartan equation by using (1, 3) and (1, 4). Then the equation (1, 6) shows that the hermitain connection (Φ^A_B) has no torsion and hence it is the Levi-Civita connection. q.e.d.

The following proposition is well known and plays a basic role in our study. For the proof, see [5].

Proposition 1.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and Φ the Maurer-Cartan form of G.*
(i) *Let F and F' be smooth mappings of a connected smooth manifold P into G. Then $F=a\cdot F'$ for a fixed $a\in G$ if and only if $F^*\Phi=F'^*\Phi$ on P. Moreover if it is the case, such an $a\in G$ is unique.*
(ii) *Let P be a simply connected manifold and φ a $\mathfrak{g}$-valued 1-form on P. In order that for any $e\in P$ and $a\in G$ there exists a unique smooth mapping $F\colon P\to G$ such that $F(e)=a$ and $F^*\Phi=\varphi$, it is necessary and sufficient to hold $d\varphi+[\varphi, \varphi]=0$.*

For later use, we prepare a few lemmas. Let V be an n-dimensional complex vector space. A mapping $\eta\colon V\times V\to \boldsymbol{C}$ is called *sesqui-linear* if $\eta(u, v)$ is complex linear in v and anti-linear in u. A $\boldsymbol{C}$-valued skew symmetric $\boldsymbol{R}$-bilinear form Δ on V is called of type (1, 1), if $\Delta(iu, iv)=\Delta(u, v)$ $(u, v\in V)$. There is a natural bijection h between the set of skew symmetric $\boldsymbol{R}$-bilinear forms on V of type (1, 1), $\Lambda^{1,1} V$, and the set of sesqui-linear forms on V, $\mathcal{S}(V)$. It is explicitly given by

$$\Lambda^{1,1}\, V\ni\Delta=\sqrt{-1}\, a_{i,j}\, \bar{z}^i\wedge z^j \to h(\Delta) = a_{i,j}\, \bar{z}^i\otimes z^j\in\mathcal{S}(V)\,,$$

where $\{z^1, \cdots, z^n\}$ is a basis of complex linear forms on V. Then, Δ is real valued if and only if $h(\Delta)$ is hermitian symmetric. A real form Δ in $\Lambda^{1,1}\, V$ is called *positive of rank r*, if the corresponding $h(\Delta)$ is positive semi-definite of rank r. It is equivalent to that $(a_{i,j})={}^t\bar{B}B$ for an $r\times n$-matrix B of rank r. Moreover, such a B is determined up to the left multiplication by a unitary matrix of order r. Now let $\Delta_{s,t}(s, t=1, \cdots, m)$ be skew symmetric $\boldsymbol{R}$-bilinear forms on V such that $\overline{\sqrt{-1}\,\Delta_{s,t}}=\sqrt{-1}\,\Delta_{t,s}$. The matrix $(\sqrt{-1}\,\Delta_{s,t})_{s,t}$ of $\boldsymbol{R}$-bilinear forms on V is said to be *positive semi-definite of rank r*, if $\overline{\sqrt{-1}\,\Delta_{s,t}}$ $=\sqrt{-1}\,\Delta_{t,s}$ $(s, t=1, \cdots, m)$ and the hermitian form on $V\otimes_{\boldsymbol{C}}\boldsymbol{C}^m$ defined by

$$h(\sqrt{-1}\,\Delta_{s,t})\,(u, v)\,\xi^s\,\eta^t \quad (u, v\in V,\ \xi, \eta\in\boldsymbol{C}^m)$$

is positive semi-definite of rank r. For a matrix $(\omega^r_s)_{r,s}$ $(r=1, \cdots, q,\ s=1, \cdots, m)$ of $\boldsymbol{C}$-valued $\boldsymbol{R}$-linear forms on V, its rank is by definition $\dim_{\boldsymbol{C}}$ Span $\{(\omega^r_s(v)\,z^s)$

$\in C^q$: $v\in V$, $z\in C^m\}$ and denoted by rank (ω_s^r). If all the ω_s^r are complex linear, then rank (ω_s^r) is equal to the rank of the linear mapping $V\otimes_C C^m\to C^q$ naturally defined by (ω_s^r).

Lemma 1.2. (i) *Suppose that* $(\sqrt{-1}\,\Delta_{s,t})_{s,t=1,\cdots,m}$ *is positive semi-definite of rank* q. *Then there exists a matrix* $(\omega_s^r)_{r,s}$ $(r=1,\cdots,q,\ s=1,\cdots,m)$ *of complex linear forms on* V *of rank* q *such that* $\Delta_{s,t}=\bar{\omega}_s^r\wedge\omega_t^r$ $(s,t=1,\cdots,m)$. *Moreover, such a matrix* $(\omega_s^r)_{r,s}$ *is uniquely determined up to the left multiplication by a unitary matrix of order* q.

(ii) *Let* $\omega_s^r(r=1,\cdots,q,s=1,\cdots,m)$ *be complex linear forms on* V. *If we set* $\Delta_{s,t}=\bar{\omega}_s^r\wedge\omega_t^r$ $(s,t=1,\cdots,m)$, *then the matrix* $(\sqrt{-1}\,\Delta_{s,t})_{s,t}$ *is positive semi-definite and* rank $(\sqrt{-1}\,\Delta_{s,t})_{s,t}=$ rank (ω_s^r).

Lemma 1.3. (cf. [6].) (i) (Cartan's lemma) *Let* $\omega^r,\psi^r(r=1,\cdots,q)$ *be* C*-valued* R*-linear forms on* V. *Suppose that* $\omega^1,\cdots,\omega^q$ *are linerly independent over* C *and* $\psi^r\wedge\omega^r=0$. *Then there exist uniquely* $a_{r,s}\in C$ *such that* $\psi^r=a_{r,s}\,\omega^s$ *and* $a_{s,r}=a_{r,s}$.

(ii) *Let* $(\omega_s^r)\,(r=1,\cdots,q,\ s=1,\cdots,m)$ *be a matrix of* C*-valued* R*-linear forms on* V *of rank* q. *Suppose that* C*-valued* R*-linear forms* $\psi_r(r=1,\cdots,q)$ *on* V *satisfy* $\psi_r\wedge\omega_s^r=0$ $(s=1,\cdots,m)$. *Then each* ψ_r *is linearly dependent to the linear forms* ω_s^r.

2. The S_c-structure

Let (M,g) be an n-dimensional connected Kähler manifold with Kähler metric g. Let $f\colon M\to S_c(N)$ be a Kähler immersion, which means a holomorphic isometric immersion of a Kähler manifold. First we define higher order osculating bundles of f. We denote by ∇ the natural connection in the induced bundle $f^*TS_c(N)$ over M induced by f from the Levi-Civita connection in $TS_c(N)$. For $p=1,2,\cdots$, we set

$$O^p(f)=\bigcup_{x\in M}O_x^p(f)\,,$$
$$O_x^p(f)=\mathrm{Span}\,\{(\nabla_{X_1}\nabla_{X_2}\cdots\nabla_{X_{p'-1}}f_*(X_{p'}))_x\colon X_1,\cdots,X_{p'}\in\mathcal{X}(M),\ p'\le p\}\,,$$

where $\mathcal{X}(M)$ denotes the set of smooth vector fields on M. For convenience, we set $O^0(f)=M\times\{0\}$.

If the dimension of $O_x^p(f)$ is constant in $x\in M$, then $O^p(f)$ is a complex vector subbundle[2] of $f^*TS_c(N)$ and called the *p-th osculating bundle* of f.

REMARK 2.1. In general, $\dim_C O_x^p(f)$, considered as a function on M, is upper semi-continuous, and hence it is constant on a connected open set M'

2 This means that $O^p(f)$ is a J-invariant subbundle of $f^*TS_c(N)$, where J is the almost complex structure on $f^*TS_c(N)$ induced by f from that on $TS_c(N)$.

of M. Further, because of the analyticity of f, $\dim_{\boldsymbol{C}} O_x^p(f)$ is constant on an open dense subset of M. However, in the following, we always assume that $\dim_{\boldsymbol{C}} O_x^p(f)$ is a constant n_p on M for each p.

By definition, the osculating bundles are increasing

$$O^1(f)=f_*TM\subset O^2(f)\subset\cdots\subset O^p(f)\subset O^{p+1}(f)\subset\cdots,$$

so there exists an integer d such that $n_{d-1}<n_d=n_{d+1}$. We call it the degree of f and denote it by $d(f)$. For $p=1,\cdots,d$, we define the *p-th normal bundle* $\nu^p(f)$ of f to be the orthogonal complement to $O^{p-1}(f)$ in $O^p(f)$ and denote its $\boldsymbol{C}$-rank by q_p $(q_p=n_p-n_{p-1})$. We further define Out (f) to be the orthogonal complement to $O^d(f)$ in $f^*TS_c(N)$, whose $\boldsymbol{C}$-rank q_{d+1} is $N-n_d$. Then we have

$$f^*TS_c(N)=\nu^1(f)\oplus\nu^2(f)\oplus\cdots\oplus\nu^d(f)\oplus\mathrm{Out}\,(f)\,. \tag{2,1}$$

Next, for $p=1,\cdots,d$, let $\pi_p\colon P_p(f)\to M$ be the principal $U(q_1)\times\cdots\times U(q_p)$-bundle associated with the vector bundle $\nu^1(f)\oplus\cdots\oplus\nu^p(f)$:

(2,2) $P_p(f)=\{(x,(e_1,\cdots,e_{n_p}))\colon \{e_{r(p')}\colon n_{p'-1}<r(p')\leq n_{p'}\}$ forms a unitary basis of $\nu_x^{p'}(f)$ for $p'=1,\cdots,p,\ x\in M\}$.

Furthermore, let $\pi_{OP}\colon OP(f)\to M$ be the principal $U(q_1)\times\cdots\times U(q_{d+1})$—bundle over M associated with the decomposition (2,1):

(2.3) $OP(f)=\{(x,(e_1,\cdots,e_N))\colon \{e_r\colon n_{p'-1}<r\leq n_{p'}\}$ forms a unitary basis of $\nu_x^{p'}(f)$ for $p'=1,\cdots,d$, and $\{e_\alpha\colon \alpha=n_{d+1},\cdots,N\}$ forms a unitay basis of $\mathrm{Out}_x(f)$, $x\in M\}$.

Before going on, let us further prepare a few terminologies. Let $\pi_P\colon P\to M$ be a principal fibre bundle over a complex manifold M with almost complex structure J. In the following, a $\boldsymbol{C}$-valued 1-form φ on P will be called of *type* $(1,0)^h$, if it vanishes in the directions of fibres of π_P and if, for any $e\in P$, φ_e can be written as $\varphi_e=\psi_x\circ\pi_{P*}$, where ψ_x is a $\boldsymbol{C}$-valued linear form on $T_xM(x=\pi_P(e))$ such that $\psi_x\circ J=\sqrt{-1}\,\psi_x$. If P is equipped with a conneation, we can define a linear endomorphism J^h of T_eP for any $e\in P$ such that $\pi_{P*}J^h(X^*)=J(\pi_{P*}X^*)$ $(X^*\in TP)$, $(J^h)^2=-1$ on $\mathcal{H}_e$, and $J^h=0$ on $\mathcal{V}_e$, where $\mathcal{H}_e$ is the horizontal subspace of T_eP with respect to the connection and $\mathcal{V}_e$ the vertical one. Then

(2,4) *a* $\boldsymbol{C}$*-valued* 1*-form* φ *on* P *is of type* $(1,0)^h$, *if and only if* $\varphi\circ J^h=\sqrt{-1}\,\varphi$.

Now let F denote the natural immersion $F((x,(e_1,\cdots,e_N)))=(e_1,\cdots,e_N)$ of $OP(f)$ into $U(S_c(N))$. We set $\tilde{\omega}=F^*\Phi$, where Φ denotes the Maurer-Cartan form of $U(S_c(N))=G_c(N)$. Then we have the following

Proposition 2.1. (i) $\tilde{\omega}(E^*)=E$ $(E\in\mathrm{Lie}\,(U(q_1)\times\cdots\times U(q_{d+1}))$;

(ii) $R_a^* \tilde{\omega} = \mathrm{Ad}\, a^{-1} \tilde{\omega} \quad (a \in U(q_1) \times \cdots \times U(q_{d+1}))$;
(iii) $d\tilde{\omega} + [\tilde{\omega}, \tilde{\omega}] = 0$;
(iv) $\tilde{\omega}^r = 0 \quad (n < r \leq N)$,
$\tilde{\omega}^{r(p)}{}_{s(p')} = 0 \quad (1 \leq p, p' \leq d,\ |p - p'| \geq 2)$;
(v) *the* $\tilde{\omega}^i$ *are of type* $(1, 0)^h$ *and linearly independent over* $\boldsymbol{C}$ *at every point of* $OP(f)$;
(vi) $\mathrm{rank}\, (\tilde{\omega}^{r(p)}{}_{s(p-1)})_{r(p), s(p-1)} = q_p$ *at any point of* $OP(f)$ *for* $p = 2, \cdots, d$;
(vii) $\tilde{\omega}^r{}_\alpha = 0 \quad (r = 1, \cdots, n_d,\ \alpha = n_d + 1, \cdots, N)$.

Proof. We will show (v). For any $X^* \in T_e OP(f)$ $(e = (x, (e_1, \cdots, e_N)),$ $(e_1, \cdots, e_N) \in U_{f(x)}(S_c(N)))$, we have, by Lemma 1.1,

$$\tilde{\omega}^A(X^*)\, e_A = \Phi^A(F_* X^*)\, e_A = \pi_{c*} F_* X^* = f_* \pi_{OP*} X^* . \tag{2,5}$$

Recall that by definition, $(e_1, \cdots, e_n)$ is a unitary frame of $f_* T_x M$ and the other e_r are orthogonal to $f_* T_x M$, and it follows from (2, 5) that $\tilde{\omega}^r = 0$ $(r = n+1, \cdots, N)$ and the $\tilde{\omega}^i$ are linearly independent. At the same time, because f is holomorphic, the $\tilde{\omega}^i$ are of type $(1, 0)^h$. The rest would be easily obtained from the definition of $OP(f)$ and the fact that $(\tilde{\omega}^A_B)$ is just the restriction to $OP(f)$ of the connection form of the connection in $f^* TS_c(N)$ induced from Levi-Civita's one in $TS_c(N)$. q.e.d.

Now we consider the $\mathfrak{g}_c(n_d)$-part of $\tilde{\omega}$, which we denote by φ for a moment. Here and in the following, the $\mathfrak{g}_c(n')$-part of $X \in \mathfrak{g}_c(N)$ $(n' < N)$ means the element $((X^\lambda_\mu)_{\lambda, \mu = 1, \cdots, n'}, (X^\lambda)_{\lambda = 1, \cdots, n'})$ of $\mathfrak{g}_c(n')$ defined by the identification $\mathfrak{g}_c(n') = \mathfrak{u}(n') + \boldsymbol{C}^{n'}$. So φ is by definition $((\tilde{\omega}^\lambda_\mu)_{\lambda, \mu = 1, \cdots, n_d}, (\tilde{\omega}^\lambda)_{= 1, \cdots, n_d})$. It is $U(q_{d+1})$-invariant, because the $U(q_{d+1})$-action on $\mathfrak{g}_c(n_d)$ is trivial. And it vanishes in the directions of fibres of the natural projection β: $OP(f) \to P_d(f)$ by Proposition 2.1 (i), (ii), and by its definition. Hence there exists a unique $\mathfrak{g}_c(n_d)$-valued 1-form ω on $P_d(f)$ such that $\beta^* \omega = \varphi$. The pair $(P_d(f), \omega)$ thus obtained is a model of S_c-structures over M of type $(n_1, \cdots, n_d)$ we so call, the meaning of which is given by the following

DEFINITION 2.1. Let M be a connected complex manifold of complex dimension n. A pair $(\boldsymbol{P}, \omega)$ is called an *S_c-structure* over M of type $(n_1, \cdots, n_d)$ if it fulfills the following conditions:
(A) $(n_1, \cdots, n_d)$ is a sequence of increasing integers with $n_1 = n$, and if we set $n_0 = 0$ and $q_p = n_p - n_{p-1}$ $(p = 1, \cdots, d)$, then P is a principal $U(q_1) \times \cdots \times U(q_d)$-bundle over M;
(B) ω is a $\mathfrak{g}_c(n_d)$-valued 1-form on $\boldsymbol{P}$ such that
$(SC, 1)$ $\omega(E^*) = E \quad (E \in \mathrm{Lie}\,(U(q_1) \times \cdots \times U(q_d)))$;
$(SC, 2)$ $R_a^* \omega = \mathrm{Ad}\, a^{-1} \omega \quad (a \in U(q_1) \times \cdots \times U(q_d))$;
$(SC, 3)$ $d\omega + [\omega, \omega] = 0$;

(*SC*, 4) $\omega^r = 0 \qquad (n<r\leq n_d)$,

$\omega^{r(p)}{}_{s(p')} = 0 \qquad (1\leq p, p'\leq d,\ |p-p'|\geq 2)$;

(*SC*, 5) the ω^i are of type $(1, 0)^h$ and linearly independent over $\boldsymbol{C}$ at every point of $\boldsymbol{P}$;

(*SC*, 6) rank $(\omega^{r(p+1)}{}_{s(p)})_{r(p+1),s(p)} = q_{p+1}$ for $p = 1, \cdots, d-1$ at every point of $\boldsymbol{P}$.

The preceding $(\boldsymbol{P}_d(f), \omega)$ will be called the *S_c-structure induced by* Kähler immersion f and often denoted by $(\boldsymbol{P}(f), \omega)$.

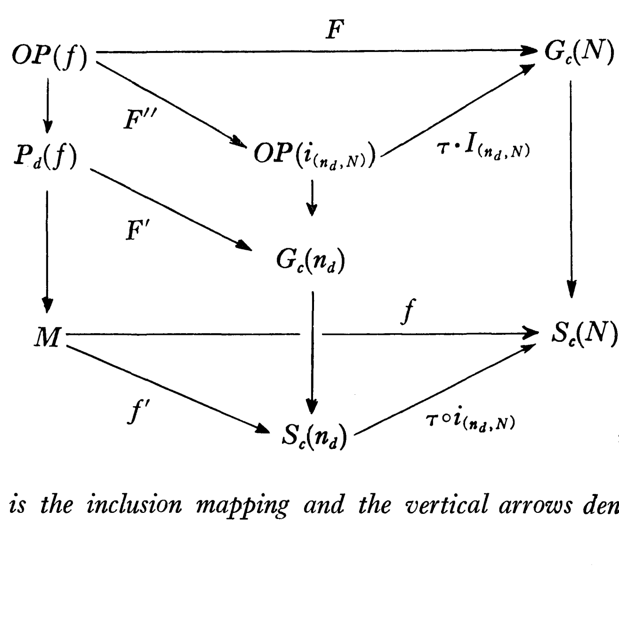
EXAMPLE 2.1. Let us consider the natural totally geodesic Kähler imbedding $i_{(N',N)}$: $S_c(N')\to S_c(N)$ $(N'<N)$. Let Φ and Φ' denote the Maurer-Cartan forms of $G_c(N)$ and $G_c(N')$ respectively. In this case, it is not difficult to see that $\boldsymbol{OP}(i_{(N',N)})$ coincides with the connected Lie subgroup $G_c(N')\cdot U(N-N')$ of $G_c(N)$ which is the maximal integral manifold through the identity element of $G_c(N)$ of the involutive differential system on $G_c(N)$:

$$\begin{cases} \Phi^\alpha = 0 & (N'<\alpha\leq N), \\ \Phi^\alpha_r = 0 & (N'<\alpha\leq N,\ 1\leq r\leq N'). \end{cases} \tag{2,6}$$

Moreover, the S_c-structure $(\boldsymbol{P}(i_{(N',N)}), \omega)$ is just the pair $(G_c(N'), \Phi')$.

Proposition 2.2. *Let f: $M\to S_c(N)$ be a Kähler immersion of a connected Kähler manifold (M, g) into $S_c(N)$ and $(\boldsymbol{P}_d(f), \omega)$ the S_c-structure of type $(n_1, \cdots, n_d)$ induced by f.*

(i) *There exist a Kähler immersion f': $M\to S_c(n_d)$ and $\tau\in G_c(N)$ such that $f=\tau\circ i_{(n_d,N)}\circ f'$, where $i_{(n_d,N)}$ denotes the standard imbedding of $S_c(n_d)$ into $S_c(N)$. Furthermore there exist smooth mappings F': $\boldsymbol{P}_d(f)\to G_c(n_d)$ and F'': $\boldsymbol{OP}(f)\to \boldsymbol{OP}(i_{(n_d,N)})$ which make the following diagram commutative:*



,

where $I_{(n_d,N)}$ is the inclusion mapping and the vertical arrows denote the obvious *projections.*

(ii) *The above immersion f' is ful, that is, its image does not lie in any proper totally geodesic complex submanifold of $S_c(n_d)$.*

Proof. We choose $\sigma \in G_c(N)$ so that $\sigma \cdot F(OP(f))$ contains the identity element e_o of $G_c(N)$. It follows $(\sigma \cdot F)^*\Phi = F^*\Phi = \tilde{\omega}$. By Proposition 2.1, we have $\tilde{\omega}^{\alpha}=0$ and $\tilde{\omega}^{\alpha}_r=0$ for any $\alpha=n_d+1, \cdots, N$ and $r=1, \cdots, n_d$. This means that $\sigma \cdot F$ satisfies the differential system (2, 6), where we must replace N' with n_d. By the assumption that M is connected, $OP(f)$ is connected. Hence $\sigma \cdot F(OP(f))$ is contained in the maximal integral manifold of (2, 6) through $e_o \in G_c(N)$ which is nothing but $OP(i_{(n_d, N)})$. In particular, $\sigma \circ f$ has its image in $S_c(n_d)$ and gives rise to a holomorphic isometric immersion f'' of M into $S_c(n_d)$, where σ is regarded as an isometry of $S_c(N)$. If we set $\tau = \sigma^{-1}$ and $f' = \tau \circ f''$, then τ and f' have the desired properties. The rest of Proposition 2.2. (i) would be now obvious.

Next, we will prove (ii). From the definition of n_d, the dimension of a totally geodesic complex submanifold of $S_c(n_d)$ containing $f(M)$ is not less than n_d. Hence we have (ii). q.e.d.

Corollary 2.1. *The integer n_d equals to the minimum dimension of totally geodesic complex submanifolds of $S_c(N)$ containing $f(M)$. In particular, f is full if and only if $OP(f)=P(f)$, in other words, $N=n_d$.*

Proof. It is immediate from Proposition 2.2. q.e.d.

REMARK 2.2. Corollary 2.1 is known in the case $c>0$ (cf. [13] and it is still valid without the assumption in Remark 2.1 because of the analyticitiy of f.

For later use, we rewrite $(SC, 3)$ in detail. First we set

$$\Delta_{r(p),s(p)} = d\omega^{r(p)}_{s(p)} + \omega^{r(p)}_{t(p)} \wedge \omega^{t(p)}_{s(p)} + \overline{\omega^{s(p)}_{t(p-1)}} \wedge \omega^{r(p)}_{t(p-1)} + \begin{cases} c\delta^{r(p)}_{s(p)}\, \omega^k \wedge \overline{\omega^k} & (p>1), \\ c(\omega^{r(1)} \wedge \overline{\omega^{s(1)}} + \delta^{r(1)}_{s(1)}\, \omega^k \wedge \overline{\omega^k}) & (p=1). \end{cases} \tag{2,7}$$

Then the equation $(SC, 3)$ is rewritten in component wise as follows:

$$d\omega^i + \omega^i_j \wedge \omega^j = 0\,; \tag{2,8}$$

$$\omega^{r(2)}_j \wedge \omega^j = 0\,; \tag{2,9}$$

$$\overline{\omega^{t(p+1)}_{r(p)}} \wedge \omega^{t(p+1)}_{s(p)} = \Delta_{r(p),s(p)}\,; \tag{2,10}$$

$$d\omega^{r(p+1)}_{s(p)} + \omega^{r(p+1)}_{t(p+1)} \wedge \omega^{t(p+1)}_{s(p)} + \omega^{r(p+1)}_{t(p)} \wedge \omega^{t(p)}_{s(p)} = 0\,; \tag{2,11}$$

$$\omega^{r(p+2)}{}_{t(p+1)} \wedge \omega^{t(p+1)}{}_{s(p)} = 0\,. \tag{2,12}$$

REMARK 2.3. In the case where (P, ω) is the S_c-structure induced by a full Kähler immersion f, the geometrical meanings of (2, 8)-(2, 12) would be more or less obvious, since (ω_B^A) is just the restriction to $P=OP(f)$ of the connection form of the connection in $f^*TS_c(N)$. The equations (2,8) and (2,10) for $p=1$ correspond to the first and second structure equations of (M, g) respectively. The (2,10) for $p>1$ and (2,11) may be regarded as the generalized Gauss-Coddazi's and the Ricci-Minardi's equations. The equations (2,9) and (2,12) have the following geometrical consequence: Let α^p be the p-th fundamental form of f, defined by

$$\alpha^p(X_1, \cdots, X_p) = \text{the } \nu^p(f)\text{-part of } \nabla_{X_1}\cdots\nabla_{X_{p-1}} f_* X_p \text{ with respect to the decomposition (2,1) } (X_1, \cdots, X_p \in \mathfrak{X}(M)).$$

Then α^p is symmetric in $X_1, \cdots, X_p$ and satisfies $\alpha^p(X_1, \cdots, JX_r, \cdots, X_p)=J\alpha^p(X_1, \cdots, X_p)$ $(r=1, \cdots, p)$. Indeed it follows from (2,9) that

$$\omega_j^{r(2)}(X^*)\,\omega^j(Y^*) = \omega_j^{r(2)}(Y^*)\,\omega^j(X^*) \quad (X^*, Y^* \in TP).$$

Since $\omega_j^{r(2)}$ can be expressed by a linear combination of ω^j by (2,9) and Cartan's lemma, Proposition 2.1 (v) implies that it is of type $(1,0)^h$. Using (2,12) Proposition 2.1 (vi), and Lemma 1.3 inductively, we see that the other ${\omega^{r(p+1)}}_{s(p)}$ are also of type $(1,0)^h$. Moreover we have immediately from (2,12) that

$$\omega^{r(p)}{}_{s(p-1)}(X^*)\,\omega^{s(p-1)}{}_{s(p-2)}(Y^*) = \omega^{r(p)}{}_{s(p-1)}(Y^*)\,\omega^{s(p-1)}{}_{s(p-2)}(X^*) \quad (X^*, Y^* \in TP).$$

On the other hand, the relation between α^p and ω_B^A is given by

$$\alpha^p(X_1, \cdots, X_p) = \omega^{r(p)}{}_{s(p-1)}(X_1^*)\,\omega^{s(p-1)}{}_{s(p-2)}(X_2^*)\cdots\omega_j^{s(2)}(X_{p-1}^*)\,\omega^j(X_p^*)\,e_{r(p)}$$

for any $X_1, \cdots, X_p \in T_xM$ and $X_1^*, \cdots, X_p^* \in T_eP$ such that $\pi_P(e)=x$ and $(\pi_P)_*(X_r^*)=X_r$ $(r=1, \cdots, p)$. Hence we obtain the desired consequence.

REMARK 2.4. As we have seen above, each ${\omega^{r(p+1)}}_{s(p)}$ is linearly dependent to $\omega^1, \cdots, \omega^n$ and of type $(1,0)^h$. By Lemma 1.2 (ii), the identity (2,10) together with $(SC, 6)$ shows that $q_{p+1}=\text{rank}\,(\Delta_{r(p),s(p)})$.

3. Basic properties of S_c-structures

Now let (P, ω) be an S_c-structure over a connected complex manifold M of type $(n_1, \cdots, n_d)$. We will observe it in more detail. Set $P_p=P/U(q_{p+1})\times\cdots\times U(q_d)$ for $p=1, \cdots, d$. In the case where (P, ω) is induced by a Kähler immersion $f\colon M\to S_c(n_d)$, P_p is canonically isomorphic to $P_p(f)$ which has been defined in (2,2). We have the following natural projections between P, P_p, and M:

$$\beta_p\colon P\to P_p,\ \beta_{p',p}\colon P_{p'}\to P_p \quad (p<p'),\ \pi_p\colon P_p\to M,$$

which are all principal fibrations with obvious structure groups.

Having these fibrations in mind, we will consider the $\mathfrak{g}_c(n_p)$-part of ω, $((\omega^\lambda_\mu)_{\lambda,\mu=1,\dots,n_p}, (\omega^\lambda)_{\lambda=1,\dots,n_p})$. By (SC, 1) and (SC, 2), it is invariant under the right action of $U(q_{p+1})\times\cdots\times U(q_d)$ and vanishes in the directions of fibres of $\beta_p\colon P\to P_p$. Hence it comes from a unique $\mathfrak{g}_c(n_p)$-valued 1-form $\omega^{(p)}$ on P_p: $\beta_p^*\omega^{(p)}=$the $\mathfrak{g}_c(n_p)$-part of ω. It is obvious that for $p<p'$,

$$\beta_{p',p}{}^*\omega^{(p)} = \text{the } \mathfrak{g}_c(n_p)\text{-part of } \omega^{(p')} . \tag{3,1}$$

If we denote the (i,j)- and the (i)-component of $\omega^{(1)}$ on P_1 by $\underline{\omega}^i_j$ and $\underline{\omega}^i$ respectively, then we have

$$\begin{cases} \omega^i = \beta_1^*\,\underline{\omega}^i , \\ \omega^i_j = \beta_1^*\,\underline{\omega}^i_j . \end{cases} \tag{3,2}$$

By definition, the 1-forms $\underline{\omega}^i$ and $\underline{\omega}^i_j$ on P_1 have the following properties:

(3,3) the $\underline{\omega}^i$ are linearly independent and of type $(1,0)^h$;

(3,4) $\underline{\omega}^i(E^*) = 0 \quad (E\in\mathfrak{u}(n))$;

(3,5) $(\underline{\omega}^i_j(E^*))_{i,j=1,\dots,n} = E \quad (E\in\mathfrak{u}(n))$;

(3,6) $\overline{\underline{\omega}^i_j}+\underline{\omega}^j_i = 0$;

(3,7) $R_a^*(\underline{\omega}^i)_{i=1,\dots,n} = a^{-1}(\underline{\omega}^i)_{i=1,\dots,n} \quad (a\in U(n))$;

(3,8) $R_a^*(\underline{\omega}^i_j)_{i,j=1,\dots,n} = \mathrm{Ad}\, a^{-1}(\underline{\omega}^i_j)_{i,j=1,\dots,n} \quad (a\in U(n))$;

(3,9) $d\underline{\omega}^i+\underline{\omega}^i_j\wedge\underline{\omega}^j = 0$.

In fact, these are direct consequences of (SC, 1), (SC, 2), (SC, 5), (3,2) and (2,8).

Using the properties (3,3), (3,4) and (3,7), we can define a hermitian metric g on M by

$$\pi_1^*\, g = \overline{\underline{\omega}^i}\otimes\underline{\omega}^i+\underline{\omega}^i\otimes\overline{\underline{\omega}^i} . \tag{3,10}$$

Proposition 3.1. *Let P_1, $\underline{\omega}^i$, $\underline{\omega}^i_j$, and g be as above. Then g is a Kähler metric on M and P_1 is naturally isomorphic to $U(M,g)$, the bundle of unitary frames of (M,g). Moreover, under the isomorphism, $(\underline{\omega}^i)_{i=1,\dots,n}$ and $(\underline{\omega}^i_j)_{i,j=1,\dots,n}$ correspond to the $\mathbf{C}$-valued canonidal form and the Levi-Civita connection form on $U(M,g)$ respectively.*

We will call g the *Kähler metric induced by* S_c-structure (P, ω).

For the proof, we need the following lemma. Let M be an n-dimensional smooth manifold and $\phi_{\mathbf{R}}$ the $\mathbf{R}^n$-valued canonical form on the linear frame bundle $L(M)$.

Lemma 3.1. (i) *Let G be a Lie subgroup of $GL(n;\mathbf{R})$ with Lie algebra $\mathfrak{g}$ and $\pi\colon Q\to M$ be a principal G-bundle over an n-dimensional smooth manifold M. Suppose that Q admits an $\mathbf{R}^n$-valued* 1*-form ω on Q such that* (a) $\omega(E^*)=0$ $(E\in\mathfrak{g})$,

(b) $R_a^*\omega = a^{-1}\omega\,(a\in G)$, *and* (c) *the linear mapping* $\omega_e\colon T_eQ\to \boldsymbol{R}^n$ *is surjective at every point* $e\in Q$. *Then* Q *can be naturally considered as a G-structure over M, that is, there exists uniquely a G-subbundle inclusion* $\iota\colon Q\to L(M)$ *such that* $\iota^*\phi_{\boldsymbol{R}}=\omega$.

(ii) *Let* $\pi'\colon Q'\to M'$ *be another principal G-bundle over an n-dimensional manifold* M' *and* ω' *an* $\boldsymbol{R}^n$*-valued* 1*-form on it with the above properties* (*a*), (*b*), *and* (*c*). *Let* $F\colon Q\to Q'$ *be a principal bundle isomorphism such that* $F^*\omega'=\omega$. *Then the following diagram commutes*

$$\begin{array}{ccc} Q & \xrightarrow{\ F\ } & Q' \\ \downarrow\iota & & \downarrow\iota' \\ L(M) & \xrightarrow{\ f_*\ } & L(M')\,, \end{array}$$

where f denotes the diffeomorphism of M onto M′ induced by F and ι' *the canonical G-bundle inclusion* $Q'\to L(M')$ *as in* (*i*).

Proof. It is clear that the conditions (a) and (c) are equivalent to $\mathrm{Ker}\,\omega = \mathrm{Ker}\,\pi_*$. So, at any point e in Q, we can choose $E_1(e), \cdots, E_n(e)$ in T_eQ such that $\omega^i(E_j(e))=\delta^i_j$. Each $E_j(e)$ is determined uniquely modulo $\mathrm{Ker}\,\pi_*$. If we set $e_j=\pi_*(E_j(e))$ $(j=1, \cdots, n)$, then each e_j depends only on ω and the point e in Q. As is easily seen, the e_j are linearly independent, in other words, $(e_1, \cdots, e_n)$ is a frame of T_xM at $x=\pi(e)$. We can choose $E_j(e)$ to depend smoothly on e at least locally, so the mapping $\iota\colon Q\to L(M)$, defined by $\iota(e)=(e_1, \cdots, e_n)$, is smooth.

We will show that this ι is the desired one. First, we prove that $\iota(ea)=\iota(e)a$ $(e\in Q,\ a\in G)$. From (b), $\omega^i_{ea}(a^k_j R_{a*} E_k(e))=\delta^i_j$. If we set $(e_1', \cdots, e_n')=\iota(ea)$, it follows

$$e_j' = \pi_*(a^k_j R_{a*} E_k(e)) = a^k_j \pi_*(R_{a*} E_k(e)) = a^k_j \pi_*(E_k(e)) = e_k a^k_j\,.$$

Thus we have $\iota(ea)=\iota(e)a$.

Next, to show $\iota^*\phi_{\boldsymbol{R}}=\omega$, we will prove

$$\omega^i(X)\, e_i = \pi_*(X) \quad (X\in T_eQ,\ (e_1, \cdots, e_n) = \iota(e))\,. \tag{3,12}$$

First, we write $X=\lambda_j E_j(e)+E$ $(\lambda_j\in\boldsymbol{R},\ E\in\mathrm{Ker}\,\pi_*)$. Then it follows from (a) that

$$\omega^i(X)\, e_i = \lambda_j\, \omega^i(E_j(e))\, e_i = \lambda_i\, e_i = \lambda_i\, \pi_*(E_i(e)) = \pi_*(X)\,.$$

Hence we have

$$\omega(X) = \iota(e)^{-1}\pi_*(X) = (\iota^*\phi_{\boldsymbol{R}})(X)\,, \tag{3,13}$$

showing $\iota^*\phi_{\boldsymbol{R}}=\omega$.

The uniqueness of such an ι is now clear by (3,13). The statement (ii) can be verified easily. q.e.d.

Proof of Proposition 3.1. By (3,4), (3,7), (3,3), and Lemma 3.1 (i), P_1 can be considered as a $U(n)$-structure over M, regarding $(\underline{\omega}^i)_{i=1,\cdots,n}$ as the C^n-valued canonical form on P_1. And $(\underline{\omega}^i_j)$ can be considered as a connection form on the $U(n)$-bundle P_1 by (3,5), (3,6), and (3,8). Then the relation (3,9) shows that the connection has no torsion. Hence the hermitian metric g on M associated with the $U(n)$-structure turns out to be Kählerian. q.e.d.

The following lemma constitutes a crucial step in the proof of Theorem 3.1 which deeply relates to the rigidity of Kähler submanifolds of complex space forms.

Lemma 3.2. *Let (P, ω) and (P', ω') be two S_c-structures of type $(n_1, \cdots, n_d)$ and of type $(n'_1, \cdots, n'_{d'})$ over connected complex manifolds M and M' respectively. For an integer p with $p\geq 1$, suppose we are given an isomorphism f_p: $P_p \to P'_p$ of the principal bundles over M and M' such that $f^*_p\, \omega'^{(p)}=\omega^{(p)}$. Then $n_{p+1}=n'_{p+1}$ and there exists uniquely an isomorphism f_{p+1} of the principal $U(q_{p+1})$-bundle P_{p+1} over P_p onto P'_{p+1} over P'_p which covers f_p and such that $f_{p+1}{}^*\omega'^{(p+1)}=\omega^{(p+1)}$.*

Proof. We denote the (r, s)-component and the (i)-component of $\omega^{(p)}$ by $(\omega^{(p)})^r_s$ and $(\omega^{(p)})^i$ respectively $(r, s=1, \cdots, n_p, i=1, \cdots, n)$. In view of (2,7), there exist uniquely 2-forms $\Delta_{r(p),s(p)}$ on P_p such that $\Delta_{r(p),s(p)}=\beta^*_p\underline{\Delta}_{r(p),s(p)}$ on P. Similarly, we define the 2-forms $\underline{\Delta}'_{s(p),t(p)}$ on P'_p for (P', ω'). In fact both the $\underline{\Delta}_{r(p),s(p)}$ and $\underline{\Delta}'_{r(p),s(p)}$ can be given in terms of components of $\omega^{(p)}$ and $\omega'^{(p)}$ in the same way as in (2,7). Hence from the assumption $f^*_p\omega'^{(p)}=\omega^{(p)}$, we have

$$f^*_p\underline{\Delta}'_{s(p),t(p)} = \underline{\Delta}_{s(p),t(p)}\,. \tag{3,14}$$

From this and Remark 2.4, we must have $q_{p+1}=q'_{p+1}$ and hence $n_{p+1}=n'_{p+1}$.

By the way, $(\omega^{(p+1)})^{r(p+1)}{}_{s(p)}$ vanishes in the directions of fibres of $\beta_{p+1,p}$: $P_{p+1}\to P_p$ by (SC, 1). So for any $e\in P_{p+1}$, there exists uniquely an element of $T^*_{\underline{e}}P_p\otimes C(\underline{e}=\beta_{p+1,p}(e))$, which we denote by $\underline{\omega}^{r(p+1)}{}_{s(p)}(e)$, such that

$$\beta_{p+1,p}{}^*\underline{\omega}^{r(p+1)}{}_{s(p)}(e) = (\omega^{(p)})^{r(p+1)}{}_{s(p),e}\,. \tag{3,15}$$

By (SC, 6) and Remark 2.4, it is of type $(1, 0)^h$. Similarly, for (P', ω') we can define $\underline{\omega}'^{r(+1)}{}_{s(p)}(e')\in T^*_{\underline{e}'}P'_p\otimes C(\underline{e}'=\beta'_{p+1,p}(e'), e'\in P'_{p+1})$ so that

$$\beta'_{p+1,p}{}^*\underline{\omega}'^{r(p+1)}{}_{s(p)}(e') = (\omega'^{(p)})^{r(p+1)}{}_{s(p),e'}\,. \tag{3,16}$$

Then, using (2,10), (3,15), (3,14), and (3,16), we have

$$\overline{\underline{\omega}^{r(p+1)}_{s(p)}}\,(e)\wedge\underline{\omega}^{r(p+1)}_{t(p)}\,(e) = \underline{\Delta}_{s(p),t(p),\underline{e}} \tag{3,17}$$

$$\overline{\underline{\omega}'^{r(p+1)}_{s(p)}}\,(e')\wedge\underline{\omega}'^{r(p+1)}_{t(p)}\,(e') = \underline{\Delta}'_{s(p),t(p),\underline{e}'}\,. \tag{3,18}$$

Since $\beta_{p+1,p}$: $P_{p+1}\to P_p$ is a $U(q_{p+1})$-bundle, we see that the set $\{\underline{\omega}^{r(p+1)}{}_{s(p)}\,(e)$:

$e\in\beta_{p+1,p}{}^{-1}(\underline{e})\}$ exhausts the set of $(1,0)^h$-forms $\omega^{r(p+1)}{}_{s(p)}$ such that $\overline{\omega^{r(p+1)}_{s(p)}}\wedge\omega^{r(p+1)}_{t(p)}=\underline{\Delta}_{s(p),t(p),\underline{e}}$ by Lemma 1.2 (i) and it is in one-to-one correspondence with the fibre $\beta_{p+1,p}{}^{-1}(\underline{e})$. It is alsov valid for $\{\underline{\omega}'^{r(p+1)}{}_{s(p)}(e')\colon e'\in\beta'_{p+1,p}{}^{-1}(\underline{e}')\}$. Hence from (3,14), (3,17), and (3,18), there exists uniquely a mapping $f_{p+1}\colon P_{p+1}\to P'_{p+1}$ such that

$$\underline{\omega}^{r(p+1)}{}_{s(p)}(e)=f_p^*\,(\underline{\omega}'^{r(p+1)}{}_{s(p)}(f_{p+1}(e)))\quad(e\in P_{p+1})\,. \tag{3,19}$$

We will show that this f_{p+1} is the desired one. It is clear by definition of f_{p+1} that $\beta'_{p+1,p}\circ f_{p+1}=f_p\circ\beta_{p+1,p}$. As is easily seen, f_{p+1} commutes with the right $U(q_{p+1})$-actions on P_{p+1} and P'_{p+1}. It remains to us to prove $f_{p+1}{}^*\omega'^{(p+1)}=\omega^{(p+1)}$ and the uniqueness of such an f_{p+1}.

First, we will show that

$$f_{p+1}{}^*(\omega'^{(p+1)})^r_s=(\omega^{(p+1)})^r_s\quad(1\le r,s\le n_p)\,. \tag{3,20}$$

In fact, it follows from (3,1) and the assumption $f_p^*\omega'^{(p)}=\omega^{(p)}$ that

$$\begin{aligned} f_{p+1}{}^*(\omega'^{(p+1)})^r_s &= f_{p+1}{}^*\,\beta'_{p+1,p}{}^*(\omega'^{(p)})^r_s=\beta_{p+1,p}{}^*\,f_p^*(\omega'^{(p)})^r_s\\ &= \beta_{p+1,p}{}^*(\omega^{(p)})^r_s=(\omega^{(p+1)})^r_s\,. \end{aligned}$$

So in order to verify $f_{p+1}{}^*\,\omega'^{(p+1)}=\omega^{(p+1)}$, we have only to show $f_{p+1}{}^*(\omega'^{(p+1)})^{r(p+1)}{}_{s(p)}=(\omega^{(p+1)})^{r(p+1)}{}_{s(p)}$ and $f_{p+1}{}^*(\omega'^{(p+1)})^{r(p+1)}{}_{s(p+1)}=(\omega^{(p+1)})^{r(p+1)}{}_{s(p+1)}$. From(3,15), (3,16), and (3,19) it follows that for any $X\in T_e\,P_{p+1}$,

$$\begin{aligned} (f_{p+1}{}^*(\omega'^{(p+1)})^{r(p+1)}{}_{s(p)})_e(X) &= (\omega'^{(p+1)})^{r(p+1)}{}_{s(p)}\,(f_{p+1}{}_*X)\\ &= \underline{\omega}'^{r(p+1)}{}_{s(p)}(f_{p+1}(e))\,(\beta'_{p+1,p}{}_*f_{p+1}{}_*X)\\ &= \underline{\omega}'^{r(p+1)}{}_{s(p)}(f_{p+1}(e))\,(f_p{}_*\beta_{p+1,p}{}_*X)\\ &= \underline{\omega}^{r(p+1)}{}_{s(p)}(e)\,(\beta_{p+1,p}{}_*X)\\ &= (\omega^{(p+1)})^{r(p+1)}{}_{s(p),e}\,(X)\,, \end{aligned} \tag{3,21}$$

as desired. The rest is easily obtained by using (2,11), (3,20), (3,21), and Lemma 1.3 (ii).

The uniqueness of f_{p+1} is now obvious, because such an f_{p+1} must satisfy (3,19). q.e.d.

We are now in a position to give a few basic results concerning our S_c-structures. Let (P,ω) and (P',ω') be S_c-structures over connected complex manifolds M and M' respectively. We say that a diffeomorphism $F\colon P\to P'$ is an *isomorphism* of (P,ω) onto (P',ω'), if F is a principal bundle isomorphism of P onto P' such that $\omega=F^*\,\omega'$.

Theorem 3.1. *Let (P,ω) be an S_c-structure over an n-dimensional connected complex manifold M and (P',ω') an S_c-structure over a connected complex* *manifold M'. Let g and g' denote the Kähler metrics on M and M' induced by those S_c-structures respectively.*

(i) *Let F: $P\to P'$ be an isomorphism of (P, ω) onto (P', ω'). Then the base mapping f induced by F of M onto M' is a holomorphic isometry.*

(ii) *Conversely, any holomorphic isometry f of (M, g) onto (M', g') gives rise to a unique isomorphism $f_\sharp$ of (P, ω) onto (P', ω') such that $\pi_{P'}\circ f_\sharp = f\circ\pi_P$, where π_P and $\pi_{P'}$ denote the projections $P\to M$ and $P'\to M'$ respectively. In particular, if $M=M'$ and $g=g'$, then (P, ω) is isomorphic to (P', ω').*

Proof. By Proposition 3.1, P_1 (resp. P_1') can be identified with $U(M, g)$ (resp. $U(M', g')$). On the other hand, F in (i) induces an isomorphism F_1 of P_1 onto P_1' which covers the base mapping f of F. By Lemma 3.1 (ii), F_1 is now regarded as f_*. In other words, f preserves the $U(n)$-structures of (M, g) and (M', g'). Hence (i) follows.

If f is a holomorphic isometry of (M, g) onto (M', g'), then it induces an isomorphism $f_{(1)}$: $P_1\to P_1'$ corresponding to f_*: $U(M, g)\to U(M', g')$ under the identification mentioned above. Then, by Lemma 3.2, it follows by induction that $f_{(1)}$ gives rise to a unique isomorphism $f_\sharp$: $P\to P'$ such that $f_\sharp^*\,\omega'=\omega$ and $\pi_{P'}\circ f_\sharp = f\circ\pi_P$. q.e.d.

Corollary 3.1. (The Rigidity Theorem of E. Calabi) *Let f and f' be two holomorphic isometric immersions of a connected Kähler manifold (M, g) into $S_c(N)$. Suppose that f is full. Then f' is also full and there exists a unique automorphism τ of $S_c(N)$ which transforms f into f'.*

Proof. Let $(P(f), \omega)$ and $(P(f'), \omega')$ be the S_c-structures over M induced by f and f' respectively. They are isomorphic to each other by Theorem 3.1 (ii) and in particular they are of the same type. By the assumption that f is full and by Corollary 2.1, we have $OP(f)=P(f)$, and hence $OP(f')=P(f')$, i.e., f' is also full. Moreover, if we denote by F and F' the immersions of $P(f)=OP(f)$ and $P(f')=OP(f')$ into $G_c(N)$ respectively, we see by Theorem 3.1 (ii) that there exists uniquely an isomorphism I of $P(f)$ onto $P(f')$ such that it covers the identity mapping of M and satisfies $\omega=I^*\omega'$. Hence $F^*\Phi=(F'\circ I)^*\Phi$, where Φ denotes the Maurer-Cartan form of $G_c(N)$. Then it follows from Proposition 1.1 that there exists uniquely $\tau\in G_c(N)$ such that $\tau\cdot F=F'\circ I$, which implies $\tau\circ f=f'$. q.e.d.

REMARK 3.1. Corollary 3.1 is still valid without the assumption in Remark 2.1 owing to the analyticity of f and f'.

Theorem 3.2. *Let M be a simply connected complex manifold and (P, ω) an S_c-structure over M of type $(n_1, \cdots, n_d)$. Let g be the Kähler metric on M induced by (P, ω).*

(i) *For any* $e_0 \in P$ *and* $u_0 \in G_c(n_d)$, *there exist uniquely immersions* F: $P \to G_c(n_d)$ *and* f: $M \to S_c(n_d)$ *with the properties* (a) $F(e_0)=u_0$, (b) $\omega=F^*\Phi$, *and* (c) $\pi_c \circ F = f \circ \pi$, *where* π_c *and* π *denote the projections* $G_c(n_d) \to S_c(n_d)$ *and* $P \to M$ *respectively and* Φ *denotes the Maurer-Cartan form of* $G_c(n_d)$.

(ii) f: $M \to S_c(n_d)$ *is a holomorphic isometric immersion.*

(iii) (P, ω) *is isomorphic to the* S_c*-structure induced by* f.

Proof. Our S_c-structure (P, ω) is a $G_c(n_d)$, $U(q_1)\times\cdots\times U(q_d)$-structure in the sense of [10]. Although (i) seems to be substantially contained in the results of [10] and [11], we will give a detailed proof of it, because the situation is slightly different from there.

We set $\boldsymbol{U}=U(q_1)\times\cdots\times U(q_d)$. By Satz 3.2 in [11] or Theorem 1.1 in [10], there exist uniquely immersions F: $P \to G_c(n_d)$ and f': $M \to G_c(n_d)/\boldsymbol{U}$ such that (a) $F(e_0)=u_0$, (b) $\omega=F^*\Phi$, and (c) $\pi' \circ F = f' \circ \pi$, where π' is the projection $G_c(n_d) \to G_c(n_d)/\boldsymbol{U}$. Define f: $M \to S_c(n_d)$ by $f=\pi'' \circ f'$, where π'' denotes the natural projection $G_c(n_d)/\boldsymbol{U} \to S_c(n_d)$. Clearly, $\pi_c \circ F = f \circ \pi$.

We will show that f is an immersion. For any $X \in T_x M$, we choose $X^* \in T_e P$ such that $\pi_* X^* = X$. By the identification $G_c(n_d)=U(S_c(n_d))$, we consider F as a mapping of P into $U(S_c(n_d))$, so that we set $F(e)=(e_1, \cdots, e_{n_d})$ $(e \in P)$. We will first show

$$\omega^i(X^*)\, e_i = f_* X\,. \tag{3,22}$$

By (SC, 4) we have $\omega^r = F^*\, \Phi^r = 0$ $(r=n+1, \cdots, n_d)$. This together with (c) and Lemma 1.1 implies that

$$\omega^i(X^*)\, e_i = \Phi^i(F_* X^*) e_i = \Phi^A(F_* X^*) e_A = \pi_{c*} F_* X_* = f_* X\,.$$

We see firstly that $f_* X$ is zero if and only if all the $\omega^i(X^*)$ vanish. Secondly, the ω^i are linearly independent by (SC, 5). Hence we see that $f_* X=0$ if and only if $X=0$. Thus f is an immersion.

We will next prove (ii). By (SC, 1) and (SC, 2), the Lie$(U(q_1)\times\cdots\times U(q_d))$-part of ω can be considered as a connection form on P. Let J^h be the horizontal almost complex structure on P with respect to the connection. Then from (2,4), (SC, 5), and (3,22), it follows that

$$f_* JX = \omega^i(J^h X^*)\, e_i = \sqrt{-1}\, \omega^i(X^*)\, e_i = J f_* X\,,$$

which shows that f is holomorphic.

By (3,10), and (3,22), we have

$$g(X, X) = 2\, \overline{\omega^i(X^*)} \cdot \omega^i(X^*) = g_c(f_* X, f_* X)\,,$$

where g_c denotes the Kähler metric of $S_c(n_d)$. Thus f is isometric.

The statement (iii) follows from Theorem 3.1 (ii). q.e.d.

4. Reduction of the structure group of S_c-structure

Before describing the reduction procedure by which we define the bundle of reduced frames RF mentioned in the introduction, we will first recall some basic facts known in the theory of transformation groups. For detail, we refer to [7] and [2]. Let H be a compact Lie group acting on a smooth manifold W. For our purpose, we may assume that W is a vector space and the H-action is linear. For any $w\in W$, we denote by H_w the isotropy subgroup of H at w and by (H_w) the conjugate class of H_w, $(H_w)=\{a^{-1}H_w a: a\in H\}$. Further we denote by $[H, W]$ the set of conjugate classes of all the isotropy subgroups of H, $[H, W]=\{(H_w): w\in W\}$. There is a naturally defined partial ordering in $[H, W]$, that is, by definition $(H_w)\leq(H_{w'})$ if and only if $a^{-1}H_w a$ is a subgroup of $H_{w'}$ for some $a\in H$.

Proposition 4.1. (i) *The set* $[H, W]$ *is finite.*

(ii) *The mapping* $W\ni w\rightarrow(H_w)\in[H, W]$ *is lower semi-continuous, i.e., for any* $w_0\in W$*, there exists a neighborhood* U *of* w_0 *such that* $(H_w)\leq(H_{w_0})$ *for any* $w\in U$.

For an isotropy subgroup L, we denote by $W_{(L)}$ the set of points w such that $H_w\in(L)$.

Proposition 4.2. (i) *For any isotropy subgroup* L*, the set* $W_{(L)}$ *is an* H*-invariant submanifold of* W.

(ii) *For any* $w_0\in W_{(L)}$ *such that* $H_{w_0}=L$*, there exists a submanifold* Γ_L *of* $W_{(L)}$ *with the following properties*:

(a) Γ_L *meets each* H*-orbit of* $W_{(L)}$ *at most once and crosses transversally those* H*-orbits it intersects*;

(b) $w_0\in\Gamma_L$ *and* $H_w=L$ *for any* $w\in\Gamma_L$;

(c) *the image of* Γ_L *by the natural projection* $W_{(L)}\rightarrow W_{(L)}/H$ *is open.*

We will call the above Γ_L a *normal form* of $W_{(L)}$. Now let $\pi: P\rightarrow M$ be a principal H-bundle over an n-dimensional complex manifold M. Let V be a finite dimensional H-module over $\boldsymbol{C}$. A function $T: P\rightarrow V$ is said to be *tensorial* if it satisfies $T(ea)=a^{-1}T(e)$ $(e\in P,\ a\in H)$. Suppose that $\boldsymbol{R}^{2n}$ is an H-module and there is an H-equivariant reduction $\iota: P\rightarrow L(M)$. Let φ be a V-valued tensorial 1-form on P, i.e., it vanishes in the directions of fibres of P and satisfies $R_a^*\varphi=a^{-1}\varphi$ for any $a\in H$. Then the *tensorial function* T *of* φ is by definition the $\mathrm{Hom}(\boldsymbol{R}^{2n}, V)$-valued tensorial function on P determined by the relation

$$T(e)\left(\phi_{\boldsymbol{R}}(\iota_* X^*)\right)=\varphi_e(X^*)\quad (X^*\in T_eP,\ e\in P)\,, \tag{4,1}$$

where $\phi_{\boldsymbol{R}}$ denotes the $\boldsymbol{R}^{2n}$-valued canonical form on $L(M)$ and $\mathrm{Hom}(\boldsymbol{R}^{2n}, V)$ is now naturally considered as an H-module. Denoting by $\{\varepsilon_1, \cdots, \varepsilon_{2n}\}$ the standard basis of $\boldsymbol{R}^{2n}$ and setting $T_k=\frac{1}{2}(T(e)(\varepsilon_{2k-1})-\sqrt{-1}\,T(e)(\varepsilon_{2k}))$, $T_{\bar{k}}=\frac{1}{2}(T(e)(\varepsilon_{2k-1})+\sqrt{-1}\,T(e)(\varepsilon_{2k}))$ for $k=1, \cdots, n$, we can express φ as

$$\varphi = T_k\, \iota^*\phi^k + T_{\bar{k}}\, \iota^*\bar{\phi}^k\,, \tag{4,2}$$

where $(\phi^k)_{k=1,\cdots,n}$ is the $\boldsymbol{C}^n$-valued canonical form on $L(M)$.

Let W be a finite dimensional H-module over $\boldsymbol{C}$. We will now consider a W-valued tensorial function T on a principal H-bundle P over M. By Proposition 4.1 (i), the set $\{(H_{T(e)}): e\in P\}$ is finite and hence there exists an isotropy subgroup L of H such that its conjugate class is minimum there. We will call such an L a *principal stabilizer* of T. Choosing a normal form Γ_L of $W_{(L)}$, we set $P_L=T^{-1}(\Gamma_L)$ and $M_L=\pi(P_L)$. We will show that M_L is an open set of M and P_L is a principal L-subbundle of $P|_{M_L}$. If we set $P'=T^{-1}(H\cdot\Gamma_L)$, then $M_L=\pi(P')$ and $P'=P|_{M_L}$. By Proposition 4.1 (ii) and Proposition 4.2 (ii) (c), $W_{(L)}$ is an open set of W and $H\cdot\Gamma_L$ is open in $W_{(L)}$. Hence M_L is an open set of M. By Proposition 4.2 (ii) (a) and (b), we see easily that P_L is a principal L-subbundle of $P|_{M_L}$. We call this P_L the *reduction of P with respect to* (T, Γ_L) or (φ, Γ_L), or *the reduction of P with respect to T or φ* for short, if T is the tensorial function of a W-valued 1-form φ on P. By definition, $T|_{P_L}$ takes its values in Γ_L and is constant along any fibre of $P_L\to M_L$.

Now let (P, ω) be an S_c-structure of type $(n_1, \cdots, n_d)$ over a connected complex manifold M. We denote simply by $\boldsymbol{U}$ the structure group $U(q_1)\times\cdots\times U(q_d)$ of P. Let B^p be the vector space $\mathrm{Hom}(\boldsymbol{C}^{q_{p-1}}\otimes\boldsymbol{C}^n, \boldsymbol{C}^{q_p})$ with the obvious $\boldsymbol{U}$-action and b^p the B^p-valued tensorial function of the $\mathrm{Hom}(\boldsymbol{C}^{q_{p-1}}, \boldsymbol{C}^{q_p})$-valued tensorial 1-form $(\omega^{r(p)}{}_{s(p-1)})_{r(p),s(p-1)}$ of type $(1,0)^h$ on P. Denoting the components of b^p with respect to the natural basis of B^p by $b^{r(p)}{}_{s(p-1),k}$, we have

$$\omega^{r(p)}{}_{s(p-1)} = b^{r(p)}{}_{s(p-1),k}\, \omega^k\,. \tag{4,3}$$

We are now going to reduce the structure group of P by these b^p. Let H_2 be a principal stabilizer of b^2. Choosing a normal form Γ_{H_2} of $B^2_{(H_2)}$, we make the reduction RP_2 of P with respect to (b^2, Γ_{H_2}), which is a principal H_2-bundle over a certain open set of M. Then the b^2 restricted to RP_2 takes its values in Γ_{H_2} and is constant along each fibre of RP_2. Next, by means of b^3 restricted to RP_2, we make the reduction RP_3 of RP_2 over a certain open set of M with respect to (b^3, Γ_{H_3}). We continue this reduction process for all $b^4, \cdots, b^d$, and we get the reduction RP_d of RP_{d-1} over a certain open set $M_d=\pi(RP_d)$ with respect to (b^d, Γ_{H_d}), where H_d and Γ_{H_d} denote a principal stabilizer of b^d and a normal form of $B^d_{(H_d)}$ respectively. We will now observe the structure group of RP_d more closely. Let χ_1: $\boldsymbol{U}\to U(n_1)$ be the natural projection and K_1 the image of H_d by χ_1. We have the following

Lemma 4.1. *There exist uniquely homomorphisms* $\rho_p: K_1\to U(q_p)$ $(p=$

$1, \cdots, d$) *with the following properties*:

(a) $\rho_1(a_1) = a_1 \quad (a_1 \in K_1)$;

(b) $\rho_p(a_1)\, b^p(e) = \rho_{p-1}(a_1)^{-1}\, a_1^{-1}\, b^p(e) \quad (a_1 \in K_1,\ e \in RP_d,\ p > 1)$;

(c) *let* $\rho(a_1) = (\rho_1(a_1), \cdots, \rho_d(a_1))\ (a_1 \in K_1)$, *then* ρ *is the inverse mapping of* χ_1: $H_d \to K_1$ *and in particular it is an isomorphism of* K_1 *onto* H_d.

Proof. Let e be a point of RP_d. Then for any $a = (a_1, \cdots, a_d)$ in H_d, we have

$$b^p(e) = b^p(ea) = a^{-1}\, b^p(e) = a_p^{-1}\, a_{p-1}^{-1}\, a_1^{-1}\, b^p(e)$$

and hence

$$a_p(b^p(e)\,(u \otimes v)) = b^p(e)\,(a_{p-1}\, u \otimes a_1\, v) \quad (u \otimes v \in \boldsymbol{C}^{q_{p-1}} \otimes \boldsymbol{C}^n)\,. \tag{4,4}$$

Since the linear mapping $b^p(e)$: $\boldsymbol{C}^{q_{p-1}} \otimes \boldsymbol{C}^n \to \boldsymbol{C}^{q_p}\ (p > 1)$ is surjective by the condition (SC,6), a_p is uniquely determined by a_{p-1}, a_1, and $b^p(e)$. Then by induction, it turns out that a_p depends only on a_1 and $b^p(e)$. Hence we can define mappings ρ_p: $K_1 \to U(q_p)\ (p = 2, \cdots, d)$ such that

$$\rho_p(a_1)\,(b^p(e)\,(u \otimes v)) = b^p(e)\,(\rho_{p-1}(a_1)\, u \otimes a_1\, v) \tag{4,5}$$

for any $u \otimes v \in \boldsymbol{C}^{q_{p-1}} \otimes \boldsymbol{C}^n$ and $a_1 \in K_1$. Moreover, every element $a \in H_d$ is then expressed by $a = (a_1, \rho_2(a_1), \cdots, \rho_d(a_1))$, where $a_1 = \chi_1(a)$. We will show that these ρ_p do not depend on e. For any $e' \in RP_d$, $b^p(e')$ belongs to Γ_{H_p}, because RP_d is a subset of RP_p. Since H_d is a subgroup of H_p, every element of H_d leaves $b^p(e')$ invariant. Hence the equation (4,4) holds for any $a \in H_d$ even if we replace e by e', and the relation between a_p, a_{p-1}, and a_1 remains unchanged. It is easy to verify that each ρ_p is a homomorphism. From (4,5), we see (b). The statement (c) is clear. q.e.d.

Using the projection β_1: $P \to P_1$, we set $RF_1 = \beta_1(RP_d)$. Then without difficulties, we have the following

Proposition 4.3. (i) $\beta_1|_{RP_d}$ *is bijective. Its inverse mapping* γ: $RF_1 \to RP_d$ *is a principal bundle isomorphism, i.e.,*

$$\gamma(ea) = \gamma(e)\,\rho(a) \quad (e \in RF_1,\ a \in K_1)\,, \tag{4,6}$$

where ρ: $K_1 \to H_d$ *is the isomorphism defined in Lemma* 4.1.

(ii) *The* $\mathfrak{g}_c(n_d)$*-valued* 1*-form* $\vartheta = \gamma^*\omega$ *on* RF_1 *have the following properties corresponding to* (*SC*, 1) *and* (*SC*, 2):

(a) $\vartheta(E^*) = \rho_*(E) \quad (E \in \mathfrak{k}_1)$;

(b) $R_a^*\,\vartheta = \mathrm{Ad}\,\rho(a)^{-1}\,\vartheta \quad (a \in K_1)$.

Moreover, the properties (*SC*, 3)-(*SC*, 6) *are hereditarily brought to* ϑ:

(c) $d\vartheta + [\vartheta, \vartheta] = 0$;

(d) $\vartheta^r = 0 \quad (n<r\leq n_d)$,
$\vartheta^{r(p)}{}_{s(p')} = 0 \quad (1\leq p, p'\leq d, \ |p-p'|\geq 2)$;
(e) *the* ϑ^i *are of type* $(1,0)^h$ *and linearly independent over* $\mathbf{C}$ *at every point of* RF_1;
(f) $\mathrm{rank}(\vartheta^{r(p+1)}{}_{s(p)})_{r(p+1),s(p)}$ *is constant* q_{p+1} *for* $p=1, \cdots, d-1$ *at every point of* RF_1.
(g) *the coefficient functions* $\gamma^* b^{r(p+1)}{}_{s(p),k}$ *of* $\vartheta^{r(p+1)}{}_{s(p)}$ *with respect to* ϑ^i *take constant values along each fibre of* $RF_1 \to M_d$.

As we have seen, the S_c-structure P is now reduced to the subbundle RF_1 of $U(M, g)|_{M_d}$ identified with $P_1|_{M_d}$, where g is the Kähler metric induced by (P, ω). From now on, we denote M_d by M_1 anew. We will further perform the reduction procedure for the structure group of RF_1. In the following, for any $\mathfrak{g}_c(n_d)$-valued 1-form φ and $p=1, \cdots, d$, we will denote by $\varphi^{[p]}$ the $\mathfrak{u}(n_p)$-part of it:

$$\varphi^{[p]} = (\varphi^{r(p)}{}_{s(p)})_{r(p),s(p)=n_{p-1}+1,\cdots,n_p}\,.$$

Now we make a decomposition

$$\mathfrak{u}(n) = \mathfrak{k}_1 + \mathfrak{p}_1 \quad \text{such that} \quad \mathrm{Ad}\, K_1 \cdot \mathfrak{p}_1 \subseteq \mathfrak{p}_1\,, \tag{4,7}$$

which is possible because K_1 is compact and denote by θ_1 the $\mathfrak{k}_1$-part of $\vartheta^{[1]}$. By Lemma 4.1 (a), it follows

$$\begin{cases} R_a^* \theta_1 = \mathrm{Ad}\, a^{-1} \theta_1 & (a\in K_1)\,, \\ \theta_1(E^*) = E & (E\in \mathfrak{k}_1)\,. \end{cases} \tag{4,8}$$

Lemma 4.2. *We set* $\eta^{[p]} = \vartheta^{[p]} - \rho_{p*}(\theta_1)$ *for* $p=1, \cdots, d$. *Then the* $\mathfrak{u}(n_p)$-*valued* 1-*form* $\eta^{[p]}$ *is a tensorial form on* RF_1, *i.e.*,
(i) $R_a^* \eta^{[p]} = \mathrm{Ad}\, \rho_p(a)^{-1} \eta^{[p]} \quad (a\in K_1)$;
(ii) $\eta^{[p]}$ *vanishes in the directions of fibres of* $RF_1 \to M_1$.

Proof. From Proposition 4.3 (ii) (b) it follows $R_a^* \vartheta^{[p]} = \mathrm{Ad}\, \rho_p(a)^{-1} \vartheta^{[p]}$ $(a\in K_1)$. At the same time, by (4,8), we have $R_a^* \rho_{p*}(\theta_1) = \mathrm{Ad}\, \rho_p(a)^{-1} \rho_{p*}(\theta_1)$ $(a\in K_1)$. Hence we obtain (i). By Lemma 4.1, $\rho_*(E) = E + \rho_{2*}(E) + \cdots + \rho_{d*}(E)$ $(E\in \mathfrak{k}_1)$. From this together with (4,6) and (4,8), we have

$$\vartheta^{[p]}(E^*) = \omega^{[p]}(\rho_*(E)^*) = \rho_{p*}(E) = \rho_{p*}(\theta_1(E^*)) \quad (E\in \mathfrak{k}_1)\,,$$

which implies (ii). q.e.d.

We perform successive reductions of the structure group of RF_1 by means of the tensorial functions $T^{[p]}$ of $\eta^{[p]}$ for $p=1, \cdots, d$, choosing respective principal stabilizers $K_{1,p}$ and normal forms $\Gamma_{K_{1,p}}$. So let RF_2 be the last reduction of RF_1 over a certain open set M_2 of M_1 and K_2 its structure group. In the following, when we restrict any form on a manifold to a submanifold of it, we will denote it by the same symbol: so we denote $\vartheta|_{RF_2}$ by ϑ and $\theta_1|_{RF_2}$ by θ_1. We make a decomposition $\mathfrak{k}_1=\mathfrak{k}_2+\mathfrak{p}_2$ such that $\mathrm{Ad}\,K_2\,\mathfrak{p}_2\subseteq\mathfrak{p}_2$. We denote by θ_2 the $\mathfrak{k}_2$-part of θ_1 and set $\tau_2=\theta_1-\theta_2$. Then θ_2 satisfies that $R_a^*\,\theta_2=\mathrm{Ad}\,a^{-1}\,\theta_2$ and $\theta_2(E^*)=E(a\in K_2,\ E\in\mathfrak{k}_2)$, i.e., it defines a connection in RF_2. And the $\mathfrak{u}(n)$-valued 1-form τ_2 is a tensorial 1-form on RF_2. Let T_2 be the tensorial function of τ_2.

Furthermore for $\nu=2, 3, \cdots$, we define inductively $RF_{\nu+1}, K_{\nu+1}, M_{\nu+1}, \theta_{\nu+1}, \tau_{\nu+1}$, and $T_{\nu+1}$ as follows: Suppose that we have defined $RF_\nu, K_\nu, M_\nu, \theta_\nu$, and τ_ν. If the tensorial function T_ν of τ_ν is constant along any fibre of RF_ν, we stop making further $RF_{\nu+1}$ etc.. Otherwise, let $RF_{\nu+1}$ be a reduction of RF_ν with respect to T_ν. Let $K_{\nu+1}$ and $M_{\nu+1}$ be the structure group and the base manifold of it. Then by definition, T_ν is constant along any fibre of $RF_{\nu+1}$. Next, making a decomposition $\mathfrak{k}_\nu=\mathfrak{k}_{\nu+1}+\mathfrak{p}_{\nu+1}$ such that $\mathrm{Ad}\,K_{\nu+1}\,\mathfrak{p}_{\nu+1}\subseteq\mathfrak{p}_{\nu+1}$, let $\theta_{\nu+1}$ and $\tau_{\nu+1}$ be the $\mathfrak{k}_{\nu+1}$-part and $\mathfrak{p}_{\nu+1}$-part of θ_ν respectively. Then we see easily that $\theta_{\nu+1}$ is a connection form on $RF_{\nu+1}$ and the $\mathfrak{u}(n)$-valued 1-form τ_ν is a tensorial 1-form on $RF_{\nu+1}$. Let $T_{\nu+1}$ be the tensorial function of $\tau_{\nu+1}$.

At each step in the above procedure of making $RF_{\nu+1}$ etc., since K_1 is compact, either the dimension or the number of connected components of K_ν must decrease. In any case the above procedure terminates after finite steps. Thus, let RF_ν, K_ν, M_ν, θ_ν, and τ_ν be the last ones. By definition, T_ν is constant along any fibre of RF_ν.

Choosing a basis $\{E_\lambda\}$ of $\mathfrak{k}_\nu$, we set $\theta_\nu=\theta_\nu^\lambda\otimes E_\lambda$. Then the set $\{\theta_\nu^\lambda, \vartheta^k, \overline{\vartheta^k}\}$ forms a basis of the space of $\boldsymbol{C}$-valued 1-forms at any point of RF_ν. Using this basis, we will now write down the non-zero components of the $\mathfrak{g}_c(n_d)$-valued 1-form ϑ on RF_ν. First we express $\eta^{[p]}$ and τ_μ in the way as in (4,2) with ι inclusion:

$$\eta^{[p]} = T_k^{[p]}\,\vartheta^k+T_{\bar k}^{[p]}\,\overline{\vartheta^k} \quad (p = 1, \cdots, d)\,,$$
$$\tau_\mu = T_{\mu,k}\,\vartheta^k+T_{\mu,\bar k}\,\overline{\vartheta^k} \quad (\mu = 2, \cdots, \nu)\,.$$

Here all the $T_k^{[p]}$, $T_{\bar k}^{[p]}$, $T_{\mu,k}$, and $T_{\mu,\bar k}$ are constant along any fibre of RF_ν. On the other hand, from $\vartheta^{[p]}=\rho_{p^*}(\theta_1)+\eta^{[p]}$ and $\theta_1=\theta_\nu+\tau_\nu+\cdots+\tau_2$,

$$\vartheta^{[p]} = \rho_{p^*}(\theta_\nu)+\rho_{p^*}(\tau_\nu)+\rho_{p^*}(\tau_{\nu-1})+\cdots+\rho_{p^*}(\tau_2)+\eta^{[p]} \tag{4,9}$$

on RF_ν for $p=1, \cdots, d$. Hence we have that on RF_ν,

$$\vartheta^i = \vartheta^i\,, \tag{4,10}$$

$$\vartheta^{[1]} = \theta_\nu+\sum_{\mu=2}^{\nu}(T_{\mu,k}\,\vartheta^k+T_{\mu,\bar k}\,\overline{\vartheta^k})+T_k^{[1]}\,\vartheta^k+T_{\bar k}^{[1]}\,\overline{\vartheta^k}\,, \tag{4,11}$$

$$\vartheta^{[p]} = \rho_{p^*}(\theta_\nu)+\sum_{\mu=2}^{\nu}(\rho_{p^*}(T_{\mu,k})\,\vartheta^k+\rho_{p^*}(T_{\mu,\bar k})\,\overline{\vartheta^k})+T_k^{[p]}\,\vartheta^k+T_{\bar k}^{[p]}\,\overline{\vartheta^k}\,, \tag{4,12}$$

$$\vartheta^{r(p+1)}{}_{s(p)} = b^{r(s+1)}{}_{s(p),k}\, \vartheta^k . \tag{4,13}$$

From (4,10)-(4,13), all the coefficients of the components of the $\mathfrak{g}_c(n_d)$-valued 1-form ϑ on RF_ν with respect to the basis $\{\theta^\lambda_\nu, \vartheta^k, \overline{\vartheta^k}\}$ become constant along any fibre of RF_ν, because ρ_{p*} is a linear representation of $\mathfrak{k}_1$.

From now on, we will denote these RF_ν, K_ν, M_ν and θ_ν simply by RF, K, M_{red} and θ respectively. The pair (RF, ϑ) will be called a *reduced S_c-structure* of (P, ω) over M_{red}, while $RF \to M_{\text{red}}$ is a principal K-subbundle of $U(M, g)|_{M_{\text{red}}}$. Here we should remark that as is obvious by its definition, (RF, ϑ) does not uniquely determined by the Kähler metric g on M.

We say that ϑ is of *constant coefficients* if and only if all the coefficients of the components of ϑ with respect to the basis $\{\theta^\lambda, \vartheta^k, \overline{\vartheta^k}\}$ are constant functions on RF, where θ^λ is defined by $\theta = \theta^\lambda \otimes E_\lambda$ for a basis $\{E_\lambda\}$ of $\mathfrak{k}$.

Proposition 4.4. *Let (P, ω) be an S_c-structure over M and g the Kähler metric induced by (P, ω). Let f be a holomorphic and isometric transformation of (M, g). Then RF and ϑ are invariant under the mapping f_*: $U(M, g) \to U(M, g)$, and in particular $f(M_{\text{rep}}) \subseteq M_{\text{red}}$.*

Proof. By Theorem 3.1 (ii), f gives rise to $f_\sharp$: $P \to P$ which satisfies $(f_\sharp)^*\omega = \omega$. Then in particular f leaves each component of ω invariant and hence the tensorial functions b^p are $f_\sharp$-invariant in view of (4,3). Since $RP_2 = (b^2)^{-1}\Gamma_{H_2}$, RP_2 is invariant under $f_\sharp$ and so is $\omega|_{PR_2}$. Similarly we see that $f_\sharp$ preserves $RP_3, \cdots, RP_d$ and ω restricted to any of them. By Proposition 4.3 (i), β_1: $P \to P_1$ maps RP_d onto RF_1 isomorphically. Using the identification $P_1 = U(M, g)$ by Proposition 3.1, we see that $\beta_1 \circ f_\sharp = f_* \circ \beta_1$. Hence RF_1 and ϑ are both f_*-invariant.

We will now examine the reduction procedure after that. To define RF_2 from RF_1, we have made the decomposition (4,7) and defined θ_1 by means of it. We see easily that θ_1 and $\vartheta^{[p]}$ are f_*-invariant. Therefore the tensorial functions $T^{[p]}$ of $\eta^{[p]} = \vartheta^{[p]} - \rho_{p*}(\theta_1)$ are f_*-invariant. Hence RF_2 is f_*-invariant. Recall that the remaining $RF_3, \cdots, RF_\nu$ have been defined by means of $\tau_2, \cdots, \tau_{\nu-1}$. As is easily seen, each of $\tau_2, \cdots, \tau_\nu$ is f_*-invariant and hence $RF_3, \cdots, RF_\nu$ are also f_*-invariant. On the other hand, ϑ has already been seen being f_*-invariant on RF_1. q.e.d.

We will give below a few results concerning the homogeneity of Kähler submanifolds of complex space forms.

Proposition 4.5. *Let (M, g) be an n-dimensional connected Kähler submanifold of $S_c(N)$ and (P, ω) the S_c-structure induced by the inclusion f. Let (RF, ϑ) be a reduced S_c-structure of (P, ω) over M_{red} with structure group K. Suppose that ϑ is of constant coefficients. Then*

(i) *each connected component of RF is identified with an open subset of a certain right coset* $\tau\cdot G$, *where G is a connected Lie subgroup of* $G_c(N)$ *and* $\tau\in G_c(N)$;
(ii) *each connected component of* M_{red} *is an open set of an orbit in* $S_c(N)$ *of the above Lie subgroup G of* $G_c(N)$.

The method of proof, given below, of this proposition is due to Sulanke.

Proof. We may assume that the submanifold (M, g) is full in $S_c(N)$. We denote by F the imbedding $RF\rightarrow G_c(N)$ which is defined by the composition of the canonical injection $RF\rightarrow P$ and the inclusion mapping $P\rightarrow G_c(N)$. Then we have $F^*\Phi=\vartheta$, where Φ denotes the Maurer-Cartan form of $G_c(N)$. Choosing a basis $\{E_\lambda\}$ of $\mathfrak{k}$, let $\{\theta^\lambda\}$ be the components of the reduced connection form θ of RF with respect to this basis. Then by (4,10)-(4,13) each component of ϑ can be expressed in the form

$$(4,14)\qquad \begin{cases} \vartheta^i = \vartheta^i\,, & \\ \vartheta^A_B = c^A_{B,\lambda}\,\theta^\lambda + c^A_{B,k}\,\vartheta^k + c^A_{B,\bar{k}}\,\overline{\vartheta^k} & (A, B = 1, \cdots, N)\,, \\ \vartheta^r = 0 & (n<r\leq N)\,, \end{cases}$$

where the coefficients c's are certain $\boldsymbol{C}$-valued functions on RF. By the assumption on ϑ the c's are constant. Let $\Psi=\Psi^\lambda\otimes E_\lambda$ be the $\mathfrak{k}$-part of Φ.

Now we consider the differential system

$$(4,15)\qquad \begin{cases} \Phi^i = \Phi^i & (i = 1, \cdots, n)\,, \\ \Phi^A_B = c^A_{B,\lambda}\,\Psi^\lambda + c^A_{B,k}\,\Phi^k + c^A_{B,\bar{k}}\,\overline{\Phi^k} & (A, B = 1, \cdots, N)\,, \\ \Phi^r = 0 & (n<r\leq N)\,. \end{cases}$$

Since each term of (4,15) is a left invariant 1-form on $G_c(N)$, the distribution defined by (4,15) is $G_c(N)$-invariant. Moreover, since $\{\Psi^\lambda, \Phi^k, \overline{\Phi^k}\}_{\lambda,k}$ are linearly independent, the dimension of our distribution amounts to $\dim\mathfrak{k}+2n$, which coincides with $\dim RF$.

Let M_0 be a connected component of M_{red} and RF^0 a connected component of $RF|_{M_0}$. Then (4,14) shows that $F(RF^0)$ is an integral manifold of (4,15). Therefore it turns out that the differential system (4,15) has a maximal dimensional solution and it is involutive. Since the equation (4,15) is of constant coefficients, the maximal integral manifold of (4,15) through the identity element of $G_c(N)$ becomes a connected Lie subgroup of $G_c(N)$, which we denote by G, and any other maximal integral manifold of (4,15) is a right coset $\tau\cdot G(\tau\in G_c(N))$. Hence (i) follows.

At the same time, we see that M_0 is an open set of a G-orbit in $S_c(N)$, because it is the image of RF^0 under the projection $G_c(N)\rightarrow S_c(N)$ with which the G-actions commute. Thus we have proved (ii). q.e.d.

Theorem 4.1. *In addition to the condition of Proposition* 4.5, *suppose that*

(M, g) is complete. Then (M, g) is a homogeneous Kähler submanifold of $S_c(N)$.

Proof. By Proposition 4.5 (ii), there is a non-empty open subset M' of M which is, at the same time, an open subset of a G-orbit in $S_c(N)$, G being a Lie subgroup of $G_c(N)$. Since the submanifold M is connected and analytic, M is necessarily contained in that G-orbit. By the assumption that (M, g) is complete, M must coincide with that orbit. Then we see that each element of G induces a holomorphic isometry of M, because it is an element of $G_c(N)$ and preserves M. This implies that M is homogeneous. q.e.d.

REMARK 4.1. Theorem 4.1 is valid without the assumption in Remark 2.1.

Conversely we have the following

Theorem 4.2. *Let (M, g) be a connected homogeneous Kähler submanifold of $S_c(N)$ and* Aut (M, g) *the Lie group of holomorphic isometric transformations of M.*

(i) *The S_c-structure (P, ω) induced by the inclusion $f: M \to S_c(N)$ is defined over the whole M* (cf. Remark 2.1).

(ii) *Let (RF, ϑ) be a reduced S_c-structure of (P, ω). Then*

(a) *RF is defined over the whole M, i.e., $M_{\text{red}} = M$;*

(b) *ϑ is of constant coefficients;*

(c) *the* Aut (M, g)*-action on RF is simply transitive, i.e.,* Aut $(M, g)\, e = RF$ $(e \in RF)$.

Proof. We set $G = \text{Aut}\,(M, g)$. We may assume that f is full. By the rigidity theorem of Calabi, any $\tau \in G$ extends to a unique holomorphic isometric transformation $\rho(\tau)$ of $S_c(N)$, and we have an injective homomorphism $\rho: G \to G_c(N)$. Hence we see that all the things used to define $O^p(f)$ for $p = 1, \cdots, d(f)$ are invariant under τ and $\rho(\tau)$. This together with our assumption that G is transitive on M implies that the rank of $O^p(f)$ is constant on M for $p = 1, \cdots, d(f)$. Thus we get (i).

Next, we will prove (ii), By Proposition 4.4, G acts on RF and G is transitive on M. Hence (a) follows. Let $F: RF \to G_c(N)$ be the imbedding defined in the proof of Proposition 4.5. Then we have $\vartheta^A = F^*\Phi^A$ and $\vartheta^A_B = F^*\Phi^A_B$ for any $A, B = 1, \cdots, N$ and moreover $\theta^\lambda = F^*\Psi^\lambda$ for any λ. Since F commutes with both the left G-actions on RF and $G_c(N)$, all the ϑ^A, ϑ^A_B, and θ^λ are G-invariant, because the corresponding components of Φ are left invariant. Hence the coefficients in (4,14) are G-invariant. Since they are constant along any fibre of RF and G is transitive on M, we see that they are constant on RF. Thus we have shown (b).

Finally, we will prove (c). In the following, we identify RF with its image

$F(RF)$. We may assume that RF contains the identity element e_0 of $G_c(N)$. Otherwise, we multiply a certain element of $G_c(N)$ to both F and M. Since $e_0 \in RF$, it follows from Proposition 4.4 that G is contained in RF. Denote by RF^0 the connected component of RF through $e_0 \in G_c(N)$ and by G^0 the connected component of G containing e_0.

First, we claim that $RF^0 = G^0$. From Proposition 4.5 (i) and the above (b), it follows that there exists a connected Lie subgroup G_1 of $G_c(N)$ which is the maximal integral manifold of (4,15) through e_0 and includes RF^0 as its open set, because RF^0 contains e_0 of G. As we have seen in the proof of Theorem 4.1, G_1 acts on M as holomorphic and isometric transformations. Hence G_1 is a subgroup of G^0 and in particular RF^0 is a subset of G^0. On the other hand, since G^0 leaves RF^0 invariant by virtue of the connectivity of them and the latter contains e_0, we see $G^0 \subset RF^0$. Therefore we have $G_1 = G^0 = RF^0$ and in particular RF^0 is the maximal integral manifold of (4,15) through e_0.

Next, let K denote the structure group of our principal bundle $\pi: RF \to M$. Then the image of the fibre $\pi^{-1}(\pi(e_0))$ by F is just the subgroup K of $G_c(N)$, because F commutes with both the right K-actions on RF and $G_c(N)$. For any $a \in K$, we denote by $RF(a)$ the connected component of RF containing a.

We will show that $RF(a) = a \cdot RF^0$. For that purpose, we will first show that $RF(a)$ is the maximal integral manifold through a of (4,15). If we replace our f by $f' = a^{-1} \cdot f$, then $F' = a^{-1} \cdot F$ becomes the corresponding imbedding of RF into $G_c(N)$. Moreover $F'(RF(a)) = a^{-1} \cdot F(RF(a))$ turns out to be the connected component of $F'(RF)$ containing e_0 and hence it is the maximal integral manifold through e_0 of (4,15) as we have seen above. This implies that the integral manifold $RF(a)$ is maximal, because (4,15) is left invariant. On the other hand, by the same reason, $a \cdot RF^0$ is also its maximal integral manifold through a. Hence we must have $RF(a) = a \cdot RF^0$.

From this we have $RF = K \cdot RF^0$. The left multiplication by each element of K preserves RF and also its base manifold M. This shows that K is a subgroup of G. As we have seen above, RF^0 is also a subgroup of G. From these facts, we see that RF is contained in G. Thus we have completed the proof of Theorem 4.2. q.e.d.

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