Ikawa, M.
Osaka J. Math.
27 (1990), 281-300

SINGULAR PERTURBATION OF SYMBOLIC FLOWS AND POLES OF THE ZETA FUNCTIONS

Dedicated to Professor Nobuyuki Ikeda on his sixtieth birthday

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(Received February 28, 1989)

1. Introduction. Let L be an integer ≥ 2, and let $A=[A(i,j)]_{i,j=1,2,\cdots,L}$ be a zero-one $L\times L$ matrix. Let $B=[B(i,j)]_{i,j=1,2,\cdots,L}$ be a zero-one $L\times L$ matrix such that

$$B(i,j)=1 \quad \text{implies} \quad A(i,j)=1\,. \tag{1.1}$$

We set

$$K=\max\{i;\, B(i,j)=1 \quad \text{for some } j\}\,.$$

Let C be the $K\times K$ matrix defined by

$$C=[B(i,j)]_{i,j=1,2,\cdots,K}\,.$$

We assume that

$$C^N>0 \quad \text{for some positive integer N}, \tag{1.2}$$

that is, all the entries of matrix C^N are positive.

Following Parry and Pollicott [10] we set

$$\Sigma_A^+=\{\xi=(\xi_1,\xi_2,\cdots,\xi_n,\cdots)\in\prod_{i=1}^{\infty}\{1,2,\cdots,L\}\,;\, A(\xi_i,\xi_{i+1})=1 \quad \text{for all } i\}\,,$$

$$\Sigma_C^+=\{\psi=(\psi_1,\psi_2,\cdots,\psi_n,\cdots)\in\prod_{i=1}^{\infty}\{1,2,\cdots,K\}\,;\, B(\psi_i,\psi_{i+1})=1 \quad \text{for all } i\}.$$

We classify the elements of Σ_A^+ into two groups:

$$\Sigma(1)=\{\xi\in\Sigma_A^+;\, B(l,\xi_1)=1 \quad \text{for some} \quad 1\leq l\leq K\}$$

and

This work was supported by Grant-in-Aid for Scientific Research, The Ministry of Education, Science and Culture.

$$\Sigma(2) = \{\xi \in \Sigma_A^+ ; B(l, \xi_1) = 0 \quad \text{for all} \quad 1 \leq l \leq K\} .$$

We regard Σ_C^+ as a subset of Σ_A^+. Denote by σ the shift transformation defined by

$$(\sigma\xi)_i = \xi_{i+1} ,$$

and let σ_A and σ_C be the restrictions of σ to Σ_A^+ and Σ_C^+ respectively.

Let $\varepsilon_1 > 0$, and let

$$f_\varepsilon, h_\varepsilon \in \mathcal{F}_\theta(\Sigma_A^+) \quad \text{for all} \quad 0 \leq \varepsilon \leq \varepsilon_1 \quad (0 < \theta < 1) .$$

Because we shall use often the Banach space $\mathcal{F}_\theta(\Sigma_A^+)$ and its norm, we recall the definition: For $r \in C(\Sigma_A^+)$ we define $\mathrm{var}_n\, r$ and $\|r\|_\infty$ by

$$\mathrm{var}_n\, r = \sup\{|r(\xi) - r(\psi)| ; \xi, \psi \in \Sigma_A^+ \quad \text{and} \quad \xi_i = \psi_i \quad \text{for} \quad i \leq n\} ,$$
$$\|r\|_\infty = \sup\{|r(\xi)| ; \xi \in \Sigma_A^+\} .$$

We set for $0 < \theta < 1$

$$\|r\|_\theta = \sup_{n \geq 1} \mathrm{var}_n\, r/\theta^n, \; |||r|||_\theta = \max\{\|r\|_\infty, \|r\|_\theta\} ,$$
$$\mathcal{F}_\theta(\Sigma_A^+) = \{r \in C(\Sigma_A^+); |||r|||_\theta < \infty\} .$$

Let $k \in \mathcal{F}_\theta(\Sigma_A^+)$ be a real valued function satisfying

$$\begin{cases} k(\xi) = 0 & \text{if} \quad B(\xi_1, \xi_2) = 1 \\ k(\xi) > 0 & \text{if} \quad B(\xi_1, \xi_2) = 0 \end{cases} \tag{1.3}$$

and

$$\inf_{\xi \in \Sigma(2)} k(\xi) \geq \sup_{\xi \in \Sigma(1)} k(\xi) = c_0 > 0 . \tag{1.4}$$

Suppose that

$$|||f_\varepsilon - f_0|||_\theta, \; |||h_\varepsilon - h_0|||_\theta \to 0 \quad \text{as} \quad \varepsilon \to 0 . \tag{1.5}$$

For $0 < \varepsilon \leq \varepsilon_1$ we define $Z_\varepsilon(s)$ by

$$Z_\varepsilon(s) = \exp\{\sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma_A^n \xi = \xi} \exp(S_n r_\varepsilon(\xi, s))\} , \tag{1.6}$$

where

$$\begin{aligned} r_\varepsilon(\xi, s) &= -s f_\varepsilon(\xi) + h_\varepsilon(\xi) + k(\varepsilon) \log \varepsilon , \\ S_n r_\varepsilon(\xi, s) &= r_\varepsilon(\xi, s) + r_\varepsilon(\sigma_A \xi, s) + \cdots + r_\varepsilon(\sigma_A^{n-1} \xi, s) \end{aligned} \tag{1.7}$$

and the summation in (1.6) is taken over all $\xi \in \Sigma_A^+$ such that $\sigma_A^n \xi = \xi$. Note that $Z_\varepsilon(s)$ is nothing but the zeta function $\zeta(r_\varepsilon(\cdot, s))$ in the sense of Parry [9, Section 3]. In this paper we call $Z_\varepsilon(s)$ the zeta function of a symbolic flow (Σ_A^+, σ_A) associated to r_ε.

Our main theorem is the following

Theorem 1. *Suppose that* (1.1)~(1.5) *are satisfied, and that*

(1.8) $$f_0(\xi)>0 \quad \textit{for all} \quad \xi \in \Sigma_A^+ ,$$

(1.9) $$h_0(\xi) \quad \textit{is real for all} \quad \xi \in \Sigma_A^+ \quad \textit{satisfying} \quad B(\xi_1, \xi_2) = 1$$

and

(1.10) $$0 < \mathrm{Im}\, h_0(\xi) < \pi \quad \textit{for all} \quad \xi \in \Sigma_A^+ \quad \textit{satisfying} \quad B(\xi_1, \xi_2) = 0 .$$

Then there exist $s_0 \in \boldsymbol{R}$, D a neighborhood of s_0 in $\boldsymbol{C}$ and $\varepsilon_0>0$ such that, for every $0<\varepsilon\leq\varepsilon_0$, $Z_\varepsilon(s)$ is meromorphic in D and it has a pole s_ε in D with

$$s_\varepsilon \to s_0 \quad as \quad \varepsilon \to 0 .$$

We should mention about the reason why we need the above theorem. In stuides of scattering by an obstacle $\mathcal{O}$ consisting of several strictly convex bodies, we introduced a function $F_D(s)$, which was defined by means of the geometry of periodic rays in the exterior of $\mathcal{O}$. The definition of $F_D(s)$ and its brief explanation will be given in the beginning of Section 4. Theorem 1 of Ikawa [5] says that, if $F_D(s)$ cannot be prolonged anlytically to an entire function, the scattering matrix $\mathcal{S}(z)$ for $\mathcal{O}$ has an infinite number of poles in $\{z;\ \mathrm{Im}\, z<\alpha\}$ for some $\alpha>0$. This means that the modified Lax and Phillips conjecture is valid for $\mathcal{O}$(cf, [8, page 158] and [2]). But it is difficult in general to show the impossibility of F_D of prolongation to an entrire function, and we could verify it only for special examples [5, Theorem 2].

An interesting property of F_D is its close relation to the zeta function of a symboic flow on (Σ_A^+, σ_A). Namely, by using the matrix A determined by the configuration of bodies, we define $\zeta(s)$ by

$$\zeta(s) = \exp\ \{\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma_A^n \xi = \xi} \exp\,(S_n(-sf(\xi)+g(\xi)+i\pi))\}\ ,$$

where f and g are appropriate real-valued functions in $\mathcal{F}_\theta(\Sigma_A^+)$ attached to $\mathcal{O}$. Then the singularities of $-\frac{d}{ds}\log \zeta(s)$ in a certain domain coincide with those of $F_D(s)$. Thus, the consideration of singularities of F_D is essentially reduced to that of $\zeta(s)$. But unfortunately it is likewise difficult to find singularities of $\zeta(s)$. The main difficulty comes from the fact that there is no $s\in\boldsymbol{C}$ such that $-sf(\xi)+g(\xi)+i\pi$ is real for all $\xi\in\Sigma_A^+$. In the case that convex bodies consisting $\mathcal{O}$ are small comparing to the distances between each others, we can apply Theorem 1 to find a pole of the zeta function $\zeta(s)$ corresponding to $\mathcal{O}$(Theorem 2). This application will be discussed in Section 4. Thus, by virtue of Theorem 1, we can show that the modified Lax and Phillips conjecture is valid for obstacles of this type.

Next we make some remarks on Theorem 1. Consider the term for an n fixed

$$\sum_{\sigma_A^n \xi=\xi} \exp\left(S_n(-sf_\varepsilon(\xi)+h_\varepsilon(\xi)+k(\xi)\log\varepsilon)\right),$$

and let ε tend to zero. Then it converges to

$$\sum_{\sigma_C^n \xi=\xi} \exp\left(S_n(-sf_0(\xi)+h_0(\xi))\right),$$

because of the effect of the term $k(\xi)\log\varepsilon$. If we set

$$\tilde{Z}_0(s)=\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\sigma_C^n \xi=\xi}\exp\left(S_n(-sf_0(\xi)+h_0(\xi))\right)\right\},$$

$\tilde{Z}_0(s)$ is the zeta function of the symbolic flow on (Σ_C^+, σ_C) associated with $-sf_0+h_0$. It suggests us that $Z_\varepsilon(s)$ is a perturbation of $\tilde{Z}_0(s)$. But if we compare the symbolic flows of which they are the zeta functions, not only the function $-sf_0+h_0$ but also the structure matrix C are perturbed. Thus we should call this perturbation of dynamical system as "*singular perturbation*". Even though Theorem 1 says that a pole of $Z_\varepsilon(s)$ is close to that of $\tilde{Z}_0(s)$ when ε is small, we do not know whether $Z_\varepsilon(s)$ itself is close to $\tilde{Z}_0(s)$ or not. It seems to us that $Z_\varepsilon(s)$ does not converge to $\tilde{Z}_0(s)$ as ε tends to zero.

We would like to express our sincere gratitude to Professor T. Morita for his kind teaching of the ergodic theory of symbolic flows and discussions on the geometry of rays in the exterior of several convex bodies. We also express our hearty thanks to the referee for his suggestions, with which we could correct our original manuscript.

2. On the spectrum of $\mathcal{L}'_{0,s}$.

In order to find a pole of $Z_\varepsilon(s)$ it is important to examine the spectrum of the Ruelle operator attached to $r_\varepsilon(\xi, s)$ of (1.6), which is the operator in $C(\Sigma_A^+)$ defined by

$$\mathcal{L}_{\varepsilon,s}\, u(\xi)=\sum_{\sigma_A \eta=\xi} e^{r_\varepsilon(\eta,s)}\, u(\eta) \quad \text{for} \quad u\in C(\Sigma_A^+) \tag{2.1}$$

where the summation is taken over all $\eta\in\Sigma_A^+$ satisfying $\sigma_A\,\eta=\xi$. Namely, it is crucial to show that $\mathcal{L}_{\varepsilon,s}$ has the property P of Parry [9, Section 2]. Here, we say that an operator $\mathcal{L}$ has the property P when $\mathcal{L}$ has a simple eigenvalue λ and the rest of the spectrum is contained in a disc with radius less than $|\lambda|$. To this end we introduce another operator $\mathcal{L}'_{\varepsilon,s}$ in $C(\Sigma_A^+)$, which is defined as follows: for $u \in C(\Sigma_A^+)$

$$(2.2) \qquad \mathcal{L}'_{\varepsilon,s}\, u(\xi) = \begin{cases} \sum\limits_{\sigma_B \eta = \xi} e^{-sf_\varepsilon(\eta)+h_\varepsilon(\eta)}\, u(\eta) & \text{for} \quad \xi \in \Sigma(1) \\ 0 & \text{for} \quad \xi \in \Sigma(2)\,, \end{cases}$$

where $\sigma_B\, \eta = \xi$ means that $\sigma_A\, \eta = \xi$ and $B(\eta_1, \eta_2) = 1$, and the summation is taken over all η satisfying $\sigma_B\, \eta = \xi$. We introduce another operator $\tilde{\mathcal{L}}_s$ in $C(\Sigma_C^+)$ defined by

$$(2.3) \qquad \tilde{\mathcal{L}}_s\, v(\psi) = \sum_{\sigma_C \nu = \psi} e^{-sf_0(\nu)+h_0(\nu)}\, v(\nu) \quad \text{for} \quad v \in C(\Sigma_C^+)\,.$$

Because of (1.2), (1.8) and (1.9) the Ruelle-Perron-Frobenious theorem [9, Propostion 1] can be applied to $\tilde{\mathcal{L}}_s$, and we have the following proposition

Proposition 2.1. *There exist $s_0 \in \boldsymbol{R}$, a neighborhood D of s_0 in $\boldsymbol{C}$ and a positive constant δ such that, for every $s \in D$, $\tilde{\mathcal{L}}_s$ is decomposed as follows:*

$$\tilde{\mathcal{L}}_s = \tilde{\lambda}_s\, \tilde{E}_s + \tilde{S}_s\,.$$

(i) $\tilde{E}_s\, \tilde{E}_s = \tilde{E}_s$, $\tilde{\mathcal{L}}_s\, \tilde{E}_s = \tilde{\lambda}_s\, \tilde{E}_s$ *and* $\tilde{E}_s\, v = \chi_s(v)\, p_s$ *for* $v \in \mathcal{F}_\theta(\Sigma_C^+)$, *where* $\chi \in \mathcal{F}_\theta(\Sigma_C^+)^*$, *and* $p_s \in \mathcal{F}_\theta(\Sigma_C^+)$ *is an eigenfunction of* $\tilde{\mathcal{L}}_s$, *namely*

$$\tilde{\mathcal{L}}_s\, p_s = \tilde{\lambda}_s\, p_s\,.$$

(ii) $\tilde{\lambda}_s$ *is a holomorphic function in D such that*

$$|\tilde{\lambda}_s - 1| < \delta \quad \textit{for all} \quad s \in D\,,$$

$$\tilde{\lambda}_{s_0} = 1\,, \quad \frac{d}{ds}\tilde{\lambda}_s \Big|_{s=s_0} \neq 0\,.$$

(iii) $\tilde{E}_s\, \tilde{S}_s = \tilde{S}_s\, \tilde{E}_s = 0$ *and*
the spectral radius of $\tilde{\lambda}_s^{-1}\, \tilde{S}_s \,|\, \mathcal{F}_\theta(\Sigma_C^+) < 1 - 2\delta$.

(iv) $|p_s(\psi)| \geq c > 0$ *for all* $\psi \in \Sigma_C^+$, *and*
$p_s(\psi) \geq c > 0$ *for all* $\psi \in \Sigma_C^+$ *when* $s \in \boldsymbol{R} \cap D$.

REMARK. If $f \in \mathcal{F}_\theta(\Sigma_C^+)$ is real valued, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \Big(\sum_{\sigma_C^n \nu = \psi} \exp\,(S_n f(\nu)) \Big)$$

exists and is independent of ψ. We denote it by $\tilde{P}(f)$. It is known that we have for all n

$$(2.5) \qquad C_1\, e^{n\tilde{P}(f)} \leq \sum_{\sigma_C^n \nu = \psi} \exp\,(S_n f(\nu)) \leq C_2\, e^{n\tilde{P}(f)} \quad \text{for all} \quad \psi \in \Sigma_C^+\,.$$

By using the right hand side inequality of (2.5) we have for complex valued function r

$$(2.6) \qquad \sum_{\sigma_G^n \nu=\psi} |\exp(S_n r(\nu))| = \sum_{\sigma_B^n \nu=\psi} \exp(S_n \operatorname{Re} r(\nu)) \leqq C_2 e^{nP(\operatorname{Re} r)}.$$

Now we state the main result in this section, which shows the property P of $\mathcal{L}'_{0,s}$.

Proposition 2.2. *Let $s_0 \in \boldsymbol{R}$ be the one in Proposition 2.1. There exists a neighborhood D_0 of s_0 in $\boldsymbol{C}$ such that, for every s in D_0, $\mathcal{L}'_{0,s}$ has the following decomposition*:

$$\mathcal{L}'_{0,s} = \tilde{\lambda}_s E_{0,s} + S_{0,s}$$

(i) $E_{0,s} E_{0,s} = E_{0,s}$, $E_{0,s} u = \chi_s(u\,|\,\Sigma_0^+)\, w_s$, *where $w_s \in \mathcal{F}_\theta(\Sigma_A^+)$ is an eigenfunction of $\mathcal{L}'_{0,s}$, that is,*

$$\mathcal{L}'_{0,s}\, w_s = \tilde{\lambda}_s w_s .$$

(ii) $E_{0,s} S_{0,r} = S_{0,s} E_{0,s} = 0$.

(iii) *the spectral radius of* $\tilde{\lambda}_s^{-1} S_{0,s} \big|_{\mathcal{F}_\theta(\Sigma_0^+)} < 1-\delta$.

(iv) *for* $s \in \boldsymbol{R} \cap D_0$,

$$w_s(\xi) \geq c > 0 \quad \text{for} \quad \xi \in \Sigma(1),$$
$$w_s(\xi) = 0 \quad \text{for} \quad \xi \in \Sigma(2).$$

As preparations for the proof of Proposition 2.2 we shall show several lemmas.

Lemma 2.3. *Let $r \in \mathcal{F}_\theta(\Sigma_A^+)$. If ξ and ψ in $\Sigma(1)$ satisfy*

$$\xi_i = \psi_i \quad \text{for} \quad i \leq m,$$

we have for $n \leq m$

$$(2.7) \qquad e^{-\|r\|_\theta \theta^{m-n}/1-\theta} \leq \left|\frac{\exp(S_n r(\xi))}{\exp(S_n r(\psi))}\right| \leq e^{\|r\|_\theta \theta^{m-n}/1-\theta}$$

and

$$(2.8) \qquad \left|\frac{\exp(S_n r(\xi))}{\exp(S_n r(\psi))} - 1\right| \leq \|r\|_\theta (\theta^{m-n}/1-\theta)\, e^{\|r\|_\theta \theta^{m-n}/1-\theta}.$$

Proof. Since

$$|r(\sigma_A^j \xi) - r(\sigma_A^j \psi)| \leq \|r\|_\theta \theta^{m-j}$$

holds for $j \leq m$, we have

$$(2.9) \qquad |S_n r(\xi) - S_n r(\psi)| \leq \sum_{j=1}^{n} |r(\sigma_A^l \xi) - r(\sigma_A^j \psi)| \leq \|r\|_\theta \theta^{m-n}/1-\theta,$$

which implies the right hand side inequality of (2.7). To show the left hand side inequality it suffices to exchange the roll of ξ and ψ. Note that it holds that

$$|\exp(S_n r(\xi)-S_n r(\psi))-1| \leq |S_n r(\xi)-S_n r(\psi)| \exp(|S_n r(\xi)-S_n r(\psi)|).$$

The subsutitution of (2.9) into the above inequality gives (2.8). Q.E.D.

Lemma 2.4. *For all $\xi\in\Sigma(1)$, it holds that*

$$\sum_{\sigma_B^n \eta=\xi} |\exp S_n r(\eta)| \leq C_3 \exp(n\tilde{P}(\operatorname{Re} r)) \quad (n=1,2,\cdots). \tag{2.10}$$

When $r(\eta)$ is real for all $\eta\in\Sigma_A^+$ satisfying $B(\eta_1,\eta_2)=1$, we have for all $\xi\in\Sigma(1)$

$$\sum_{\sigma_B^n \eta=\xi} \exp S_n r(\eta) \geq C_4 \exp(n\tilde{P}(r)) \quad (n=1,2,\cdots). \tag{2.11}$$

Here the constants C_3 and C_4 are positive and independent of n.

Proof. From the definition of K and (1.2) we see immediately that, for every k such that $1\leq k\leq K$, there exists $1\leq\eta\leq K$ satisfying $B(k,\eta)=1$. Thus, we can choose a sequence $\eta_j^{(k)}$, $j=1,2,\cdots$, satisfying $B(k,\eta_1^{(k)})=1$, and $B(\eta_j^{(k)},\eta_{j+1}^{(k)})=1$ for all $j\geq1$. Then obviously we have $\eta^{(k)}=(\eta_1^{(k)},\eta_2^{(k)},\cdots)\in\Sigma_C^+$. Now we fix arbitrarily K elements $\eta^{(l)}$, $l=1,2,\cdots,K$, in Σ_C^+ satisfying $B(l,\eta_1^{(l)})=1$. For $\nu_1,\nu_2,\cdots,\nu_n\in\{1,2,\cdots,K\}$ and $\psi\in\Sigma_A^+$ we denote by $(\nu_1,\nu_2,\cdots,\nu_n,\psi)$ the element in $\xi=(\xi_i)\in\prod_{i=1}^{\infty}\{1,2,\cdots,L\}$ such that

$$\xi_i=\nu_i \ (i=1,2,\cdots,n), \quad \xi_{n+j}=\psi_j \quad (j=1,2,\cdots).$$

For η such that $B(\eta_i,\eta_{i+1})=1$ $(i=1,2,\cdots,n-1)$ we correspond the element $\Psi_n(\eta)\in\Sigma_C^+$ defined by

$$\Psi_n(\eta)=(\eta_1,\eta_2,\cdots,\eta_n,\eta^{(l)}) \quad \text{when} \quad \eta_n=l.$$

Then for each $\xi\in\Sigma(1)$ and $l\in\{1,2,\cdots,K\}$ Ψ_n gives a one to one correspondance between

$$\{\eta\in\Sigma_A^+;\sigma_B^n\eta=\xi,\ \eta_n=l\} \quad \text{and} \quad \{\psi\in\Sigma_C^+;\sigma_C^n\psi=\eta^{(l)},\ \psi_n=l\}.$$

First suppose that r is real valued. Then, for $\xi\in\Sigma(1)$ we have

$$\sum_{\sigma_B^n\eta=\xi} \exp S_n r(\eta) = \sum_{\sigma_B^n\eta=\xi} \exp S_n r(\Psi_n(\eta)) \frac{\exp(S_n r(\eta))}{\exp(S_n r(\Psi_n(\eta)))}$$

from (2.7)

$$\geq e^{-\|r\|_\theta/1-\theta} \sum_{l=1}^{K} \sum_{\sigma_C^{n-1}\nu=(l,\eta^{(l)})} \exp S_n r(\nu)$$

$$\geq e^{-\|r\|_\theta/1-\theta}\, C_1\, e^{(n-1)\tilde{P}(\mathrm{Re}\, r)}\, .$$

Thus by setting $C_4=C_1\, e^{-\|r\|_\theta/1-\theta}\, e^{-\tilde{P}(\mathrm{Re}\, r)}$ we have (2.11). When r is complex valued, by using the right hand side inequalities of (2.5) and (2.7) we get (2.11) by a similar argument, where we set $C_3=C_2K\, e^{\|r\|_\theta/1-\theta}\, e^{-\tilde{P}(\mathrm{Re}\, r)}$. Q.E.D.

Corollary 2.5. *Suppose that* ξ *and* ψ *in* $\Sigma(1)$ *satisfy*

$$\xi_i = \psi_i \quad \textit{for} \quad i\leq m\, .$$

Then, it holds that

$$\left| \sum_{\sigma_B^n \eta=\xi} \exp S_n\, r(\eta) - \sum_{\sigma_B^n \nu=\psi} \exp S_n\, r(\nu) \right| \leq C_5\, e^{n\tilde{P}(\mathrm{Re}\, r)}\, \theta^m\, . \tag{2.12}$$

Proof. Note that there is a one to one correspondance F between $\{\eta;\, \sigma_B^n\, \eta=\xi\}$ and $\{\nu;\, \sigma_B^n\, \nu=\psi\}$ such that $\Psi_n(F(\eta))=\Psi_n(\eta)$. Note that $F(\eta)=\nu$ implies that $\nu_i=\eta_i$ for $i\leq m+n$. Then we have from (2.8) replaced m by $m+n$ that

$$\text{the left hand sice of (2.12)} \leq C_5'\, \theta^m \sum_{\sigma_B^n \eta=\xi} |\exp S_n\, r(\eta)|\, ,$$

where $C_5'=(\|r\|_\theta/1-\theta)\, e^{\|r\|_\theta/1-\theta}$. The substitution of (2.10) into the right hand side of the above inequality gives (2.12) for $C_5=C_5'\, C_3$. Q.E.D.

Lemma 2.6. *Suppose that*

$$\gamma_1 = |e^{\tilde{P}(\mathrm{Re}\, r)}\, \theta^{1/2}\, \tilde{\lambda}_s^{-1}|<1\, . \tag{2.13}$$

Then we have an estimate

$$\left|\left|\left| \left(\frac{\mathcal{L}'_{0,s}}{\tilde{\lambda}_s}\right)^m u \right|\right|\right|_\theta \leq C_3\, \theta^{m/2}\, |||u|||_\theta + C_{6,m}\, \|u\|_\infty \quad \textit{for all} \quad u\in\mathcal{F}_\theta(\Sigma_A^+)\, . \tag{2.14}$$

where $C_{6,m}$ *is a constant depending on* m.

Proof. Suppose that ξ, $\psi\in\Sigma(1)$ satisfy $\xi_i=\psi_i$ for $i\leq p\, (p\geq 1)$. Then

$$\begin{aligned} &\mathcal{L}'^{\,m}_{0,s}\, u(\xi) - \mathcal{L}'^{\,m}_{0,s}\, u(\psi) \\ &= \sum_{\sigma_B^m \eta=\xi} \{e^{S_m r(\eta)}\, u(\eta) - e^{S_m r(F(\eta))}\, u(F(\eta))\} \\ &= \sum_{\sigma_B^m \eta=\xi} \{e^{S_m r(\eta)} - e^{S_m r(F(\eta))}\}\, u(\eta) + \sum_{\sigma_B^m \eta=\xi} e^{S_m r(F(\eta))}\, \{u(\eta)-u(F(\eta))\} \\ &= I+II\, . \end{aligned}$$

From Corollary 2.5 it follows that

$$|\tilde{\lambda}_s|^{-m}\, |I| \leq C\, \|u\|_\infty\, (e^{\tilde{P}(\mathrm{Re}\, r)}\, |\tilde{\lambda}_s|^{-1})^m\, \theta^p\, .$$

On the other hand, since

$$|u(\eta)-u(F(\eta))| \leq \|u\|_\theta \, \theta^{p+m},$$

with the aid of Lemma 2.4 we have

$$|\tilde{\lambda}_s|^{-m} |II| \leq C_3 \, \theta^p \, \|u\|_\theta (e^{\tilde{P}(\mathrm{Re}\, r)} \, \theta^{1/2} |\tilde{\lambda}_s|^{-1})^m \, \theta^{m/2}.$$

By using the right hand inequality of (2.6) we have for all ξ

$$|\mathcal{L}'^{\,m}_{0,s} \, u(\xi)| \leq C_2 \, e^{m\tilde{P}(\mathrm{Re}\, r)} \, \|u\|_\infty.$$

Therefore under the assumption (2.13) we have (2.14) by setting $C_{6,m}=(C_2+C_5)$ $(e^{\tilde{P}(\mathrm{Re}\, r)} |\tilde{\lambda}_{\tilde{s}}|^{-1})^m$. Q.E.D.

Now we fix $s \in D$ and set $r=-sf_0+h_0$. Let $u \in \mathcal{F}_\theta(\Sigma_A^+)$. Suppose that $\nu=(\nu_1, \nu_2, \cdots, \nu_q, \xi)$ satisfies $\sigma_B^q \, \nu=\xi$, and that $\nu_q=l$. Since $\sigma_B^p \, \psi \in \Sigma_C^+$ implies that $\psi \in \Sigma_C^+$, we have

$$\begin{aligned}(2.15) \qquad \mathcal{L}'^{\,p}_{0,s} \, u(\nu_1, \nu_2, \cdots, \nu_q, \eta^{(l)}) &= \sum_{\sigma_B^p \psi=(\nu_1, \nu_2, \cdots, \nu_q, \eta^{(l)})} \exp(S_p \, r(\psi)) \, u(\psi) \\ &= \tilde{\mathcal{L}}_s^p \, \tilde{u}(\nu_1, \nu_2, \cdots, \nu_q, \eta^{(l)}),\end{aligned}$$

where $\tilde{u}=u|_{\Sigma_C^+}$. Now we have

$$\begin{aligned}(2.16) \qquad & \mathcal{L}'^{\,p}_{0,s} \, u(\nu_1, \nu_2, \cdots, \nu_q, \xi)-\mathcal{L}'^{\,p}_{0,s} \, u(\nu_1, \nu_2, \cdots, \nu_q, \eta^{(l)}) \\ &= \sum_{\sigma_B^p \eta=(\nu_1, \nu_2, \cdots, \nu_q, \xi)} \{e^{S_p \, r(\eta)} \, u(\eta)-e^{S_p \, r(\Psi_{p+q}(\eta))} \, u(\Psi_{p+q}(\eta))\} \\ &= \sum_{\sigma_B^p \eta=(\nu_1, \nu_2, \cdots, \nu_q, \xi)} \{e^{S_p \, r(\eta)}-e^{S_p \, r(\Psi_{p+q}(\eta))}\} \, u(\eta) \\ &\quad + \sum_{\sigma_B^p \eta=(\nu_1, \nu_2, \cdots, \nu_q, \xi)} e^{S_p \, r(\Psi_{p+q}(\eta))} \{u(\eta)-u(\Psi_{p+q}(\eta))\} = I+II.\end{aligned}$$

By using Corollary 2.5 we have

$$|I| \leq C_5 \, e^{p\tilde{P}(\mathrm{Re}\, r)} \, \theta^q \, \|r\|_\theta \, \|u\|_\infty.$$

Concerning II, since

$$|u(\eta)-u(\Psi_{p+q}(\eta))| \leq \|u\|_\theta \, \theta^{p+q},$$

it follows from (2.10) that

$$|II| \leq C_3 \|u\|_\theta \, \theta^{p+q} \, e^{p\tilde{P}(\mathrm{Re}\, r)}.$$

Thus we have for $C_7=C_3+C_5$

$$(2.17) \qquad |\text{the left hand side of (2.16)}| \leq C_7 \, \theta^q \, e^{p\tilde{P}(\mathrm{Re}\, r)} \, |||u|||_\theta.$$

By applying Proposition 2.1 to the right hand side of (2.15) we have

$$\tilde{\lambda}_s^{-p}\,\mathcal{L}_{0,s}'^{\,p}u(\nu_1,\nu_2,\cdots,\nu_q,\eta^{(l)}) \\ =\chi_s(\tilde{u})\,p_s(\nu_1,\nu_2,\cdots,\nu_q,\eta^{(l)})+\tilde{\lambda}_s^{-p}\,\tilde{S}_{0,s}^{p}\,\tilde{u}(\nu_1,\nu_2,\cdots,\nu_q,\eta^{(l)})\,. \tag{2.18}$$

The difference of the representations of (2.18) for p and $p+1$ gives the estimate

$$|(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{p+1}u(\nu_1,\nu_2,\cdots,\nu_q,\eta^{(l)})-(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{p}u(\nu_1,\nu_2,\cdots,\nu_q,\eta^{(l)})| \\ \leq|||(\tilde{\lambda}_s^{-1}\,\tilde{S}_{0,s})^{p+1}\,\tilde{u}\,|||_\theta+|||(\tilde{\lambda}_s^{-1}\,\tilde{S}_{0,s})^{p}\,\tilde{u}\,|||_\theta \\ \leq 2((1-2\delta)\,|\tilde{\lambda}_s|^{-1})^p\,|||u|||_\theta\,. \tag{2.19}$$

From the combination of (2.17) for p and $p+1$ and (2.19) we have

$$|(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{p+1}u(\nu_1,\cdots,\nu_q,\xi)-(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{p}u(\nu_1,\cdots,\nu_q,\xi)| \\ \leq|\tilde{\lambda}_s|^{-p}\{2(1-2\delta)^p+C_7\,\theta^q\,e^{p\tilde{P}(\mathrm{Re}\,r)}\}\,|||u|||_\theta\,. \tag{2.20}$$

Let $m=p+q$. Note that

$$\mathcal{L}_{0,s}'^{\,m}\,u(\xi)=\sum_{\nu_1,\cdots,\nu_q}e^{S_q r(\nu_1,\cdots,\nu_q,\xi)}\,\mathcal{L}_{0,s}'^{\,p}\,u(\nu_1,\cdots,\nu_q,\xi)\,, \tag{2.21}$$

where the summation is taken over all $(\nu_1,\nu_2,\cdots,\nu_q)$ such that $B(\nu_i,\nu_{i+1})=1$ $(i=1,2,\cdots,q-1)$ and $B(\nu_q,\xi_1)=1$. Then, it follows from the representations (2.21) for $m=p+q$ and $m+1=(p+1)+q$, and (2.20) that

$$|(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{m+1}u(\xi)-(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{m}u(\xi)| \\ \leq|\tilde{\lambda}_s^{-q}|\sum_{\nu_1,\cdots,\nu_q}|e^{S_q r(\nu_1,\cdots,\nu_q,\xi)}| \\ \cdot|(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{p+1}u(\nu_1,\cdots,\nu_q,\xi)-(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{p}u(\nu_1,\cdots,\nu_q,\xi)| \\ \leq C_3|\tilde{\lambda}_s^{-q}|\,e^{q\tilde{P}(\mathrm{Re}\,r)}|\tilde{\lambda}_s|^{-p}\{2(1-2\delta)^p+C_7\,\theta^q\,e^{pP(\mathrm{Re}\,r)}\}\,|||u|||_\theta\,. \tag{2.22}$$

Assume that

$$\gamma_2=e^{\tilde{P}(\mathrm{Re}\,r)/2}((1-2\delta)\,|\tilde{\lambda}_s|^{-1})^{1/2}<1\,. \tag{2.23}$$

Then for large m, if we choose p and q as

$$q=\left[\frac{1}{2}m\right],\ p=m-q\,,$$

where $[x]$ denotes the integer part of x, it holds that

$$|\text{the right hand side of (2.22)}|\leq C_8(\gamma_1^m+\gamma_2^m)\,|||u|||_\theta\,,$$

which implies that for all $\xi\in\Sigma_A^+$

$$(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{m}\,u(\xi)\quad\text{converges in } C \text{ as } m \text{ tends to the infinity.} \tag{2.24}$$

By using the formula (2,.21) we have from (2.17) and (2.10)

$$|(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{m}u(\xi)-\tilde{\lambda}_s^{-q}\sum_{l=1}^{K}\sum_{\nu_1,\cdots,\nu_{q-1},l}e^{S_q r(\xi_1,\cdots,l,\xi)} \\ \cdot(\tilde{\lambda}_s^{-1}\,\mathcal{L}_{0,s}')^{p}u(\nu_1,\cdots,l,\eta^{(l)})|\leq C_3\,C_7\,\gamma_1^m\,|||u|||_\theta\,. \tag{2.25}$$

On the other hand, we have from (2.15) and Proposition 2.1

$$(2.26)\qquad |\tilde{\lambda}_s^{-q} \sum_{\nu_1,\cdots,\nu_{q-1},l} e^{S_q r(\nu_1,\cdots,l,\xi)} \{(\tilde{\lambda}_s^{-1} \mathcal{L}'_{0,s})^p u(\nu_1, \cdots, l, \eta^{(l)}) \\ -\chi_s(\tilde{u})\, p_s(\nu_1, \cdots, l, \eta^{(l)})\} | \leq C_3\, \gamma_2^m\, |||u|||_\theta\,.$$

It follows from (2.25) and (2.26) that for $C_9 = C_3(1+C_7)$

$$(2.27)\qquad |\left(\frac{\mathcal{L}'_{0,s}}{\tilde{\lambda}_s}\right)^m u(\xi) - \tilde{\lambda}_s^{-q} \sum_{\nu_1,\cdots,\nu_{q-1},l} e^{S_q r(\nu_1,\cdots,l,\xi)}\, \chi_s(\tilde{u}) \\ \cdot p_s(\nu_1, \cdots, l, \eta^{(l)})| \leq C_9(\gamma_1^m + \gamma_2^m)\, |||u|||_\theta\,.$$

Now we have

Lemma 2.7. *Suppose that* (2.13), (2.23) *are satisfied. Then we have for all* $\xi \in \Sigma_A^+$

$$(2.28)\qquad \lim_{m\to\infty} (\tilde{\lambda}_s^{-1} \mathcal{L}'_{0,s})^m u(\xi) = \chi_s(\tilde{u})\, w_s(\xi)\,,$$

where $w_s(\xi)$ *is a function defined by*

$$(2.29)\qquad w_s(\xi) = \lim_{q\to\infty} \tilde{\lambda}_s^{-q} \sum_{l=1}^{K} \sum_{\nu_1,\cdots,\nu_{q-1}} e^{S_q r(\nu_1,\cdots,\nu_{q-1},l,\xi)}\,. \\ \cdot p_s(\nu_1, \cdots, \nu_{q-1}, l, \eta^{(l)}) \quad \text{for } \xi \in \Sigma(1)\,, \\ w_s(\xi) = 0 \quad \text{for} \quad \xi \in \Sigma(2)$$

satisfying

$$(2.30)\qquad w_s(\xi) \in \mathcal{F}_\theta(\Sigma_A^+)\,,$$

$$(2.31)\qquad \mathcal{L}'_{0,s}\, w_s(\xi) = \tilde{\lambda}_s\, w_s(\xi)$$

and

$$(2.32)\qquad \text{if} \quad s \in \boldsymbol{R} \cap D,\ w_s(\xi) \geq C_{10} \quad \text{for all} \quad \xi \in \Sigma(1)$$

where C_{10} *is a positive constant.*

Proof. The convergence of the right hand side of (2.29) follows from (2.24) and (2.27), and at the same time (2.28) also follows. By applying (2.12) to the right hand side of (2.29) we have (2.30). We have immediately (2.31) from (2.28). If $s \in \boldsymbol{R}$, we have $p_s(\psi) > c$ for all $\psi \in \Sigma_C^+$ from (iv) of Proposition 2.1. Therefore (2.11) and the fact that $\tilde{\lambda}_s = e^{\tilde{P}(r)}$ for s real imply (2.32) for $C_{10} = cC_4$.

Lemma 2.8. *Assume* (2.13) *and* (2.23) *are satisfied, and let* $s \in \boldsymbol{R} \cap D$. *Then, there is no eigenvalue* λ *of* $\mathcal{L}'_{0,s}$ *such that*

$$|\lambda| \geq |\tilde{\lambda}_s| \quad \text{and} \quad \lambda \neq \tilde{\lambda}_s\,.$$

Proof. Let λ be an eigenvalue of $\mathcal{L}'_{0,s}$, and let $v\in\mathcal{F}_\theta(\Sigma_A^+)$ be an eigenfunction associated with λ, that is, $v\neq 0$ and

$$\lambda^{-1}\mathcal{L}'_{0,s}\,v(\xi)=v(\xi)\,.$$

Then, we have

$$(\lambda/\tilde{\lambda}_s)^m\,v(\xi)=(\tilde{\lambda}_s^{-1}\mathcal{L}'_{0,s})^m\,v(\xi)\to\chi_s(\tilde{v})\,w_s(\xi)\,.$$

which shows that $(\lambda/\tilde{\lambda}_s)^m\,v(\xi)$ converges for all $\xi\in\Sigma_A^+$. Since there is a ξ such that $v(\xi)\neq 0$, this implies that $(\lambda/\tilde{\lambda}_s)^m$ converges as m tends to the infinity. Thus we have $\lambda=\tilde{\lambda}_s$ or $|\lambda|<|\tilde{\lambda}_s|$. Q.E.D.

With the preparation of the above lemmas, we proceed to show Proposition 2.2. First remark that the pair of spaces $C(\Sigma_A^+)$ and $\mathcal{F}_\theta(\Sigma_A^+)$ satisfies the required conditions on the pair of E and B in Ionescu, Turcea and Marinescu [6]. Since $\tilde{\lambda}_{s_0}=e^{\tilde{P}(r_0(\cdot,s_0))}=1$, we have for all m

$$|(\tilde{\lambda}_{s_0}^{-1}\mathcal{L}'_{0,s_0})^m\,u(\xi)|\leq\|u\|_\infty|\tilde{\lambda}_{s_0}^{-1}|^m\sum_{\sigma_B^m\eta=\xi}\exp S_m\,r_0(\eta,s_0) \tag{2.35}$$

by using (2.10)

$$\leq\|u\|_\infty|\tilde{\lambda}_{s_0}^{-1}|^m\,C_2\,e^{m\tilde{P}(r_0(\cdot,s_0))}=C_2\,\|u\|_\infty\,.$$

Then (2.35) and Lemma 2.6 for m satisfying $C_3\,\theta^{m/2}<1$ assure that the result of [6, Section 9] is applicable to $\tilde{\lambda}_{s_0}^{-1}\mathcal{L}'_{0,s_0}$, and it follows that

$$\tilde{\lambda}_{s_0}^{-1}\mathcal{L}'_{0,s_0}=\sum_{i=1}^{p}c_i\,E_i+S\,,$$

where

$$|c_i|=1,\ E_i\,E_i=E_i,\ E_i\,E_j=0\,(i\neq j),\ SE_i=E_iS=0$$

and

$$\text{spectral radius of } S<1\,.$$

Now Lemmas 2.7 and 2.8 imply that

$$p=1\quad\text{and}\quad c_1=\tilde{\lambda}_{s_0}=1\,,$$

and Lemma 2.7 shows that the dimension of the space of eigenfunctions for c_1 is 1, namely

$$\text{dimension of range of } E_1=1\,.$$

Thus we have

$$\tilde{\lambda}_{s_0}^{-1}\mathcal{L}'_{0,s_0}=E_1+S\,.$$

Note that $\tilde{\lambda}_s^{-1}\mathcal{L}'_{0,s}$ is continuous in s as a bounded operator valued function.

Then the perturbation theory (see for example, Kato [7, Theorem 3.16, Chapter IV]) assures the following fact:

There exist a neighborhood D_0 of s_0 in $\boldsymbol{C}$ and a positive constant δ such that we have for all $s \in D_0$

$$\tilde{\lambda}_s^{-1} \mathcal{L}'_{0,s} = \mu_s E_s + S_s$$

where μ_s is holomorphic in s and $\mu_{s_0}=1$,

$$E_s E_s = E_s,\ E_s S_s = S_s E_s = 0\,, \quad \text{dimension of range } E_s = 1$$

and

$$\text{spectral radius of } S_s < 1-2\delta\,.$$

On the other hand, Lemmas 2.7 and 2.8 show that $\mu_s \equiv 1$ and the eigenspace is spanned by w_s. Thus Proposition 2.2 is proved.

3. Proof of Teorem 1

As remarked in Section 1, we compare the spectrum of $\mathcal{L}_{\varepsilon,s}$ and $\mathcal{L}'_{0,s}$ in order to get a decomposition of $\mathcal{L}_{\varepsilon,s}$. Note that $\mathcal{L}_{\varepsilon,s}$ is represented as

$$\begin{aligned}\mathcal{L}_{\varepsilon,s}\, u(\xi) &= \sum_{\sigma_B \eta=\xi} e^{-sf_\varepsilon(\eta)+h_\varepsilon(\eta)}\, u(\eta) + \sum_{\sigma_A \eta=\xi, B(\eta_1,\eta_2)=0} \varepsilon^{k(\eta)}\, e^{-sf_\varepsilon(\eta)+h_\varepsilon(\eta)}\, u(\eta) \\ &= \mathcal{L}'_{\varepsilon,s}\, u(\xi) + \mathcal{L}''_{\varepsilon,s}\, u(\xi)\,.\end{aligned}$$

We have from (1.6) by a direct calculus

$$|||\mathcal{L}'_{0,s} - \mathcal{L}'_{\varepsilon,s}|||_\theta \to 0 \quad \text{as} \quad \varepsilon \to 0\,. \tag{3.1}$$

Suppose that $\eta_i = \nu_i$ for $i \leq n+1$. Then it follows that

$$|\varepsilon^{k(\eta)} - \varepsilon^{k(\nu)}| \leq \theta^n\, ||k||_\theta\, \varepsilon^{c_0} |\log \varepsilon| \exp(\theta^n\, ||k||_\theta)\,,$$

where $c_0 = \inf_{B(\xi_1,\xi_2)=0} k(\xi) > 0$, namely

$$||\varepsilon^{k(\cdot)}||_\theta \leq ||k||_\theta\, \varepsilon^{c_0} |\log \varepsilon| \exp(\theta^n\, ||k||_\theta)\,.$$

Therefore we have

$$|||\mathcal{L}''_{\varepsilon,s}|||_\theta \leq C(|||f_\varepsilon|||_\theta + |||h_\varepsilon|||_\theta + |||k|||_\theta)\, \varepsilon^{c_0} |\log \varepsilon|\,. \tag{3.2}$$

Thus we have from (3.1) and (3.2)

Lemma 3.1.

$$|||\mathcal{L}_{\varepsilon,s} - \mathcal{L}'_{0,s}|||_\theta \to 0 \quad \text{as} \quad \varepsilon \to 0\,. \tag{3.3}$$

Now we shall show the following properties of $\mathcal{L}_{\varepsilon,s}$.

Proposition 3.2. *There exist $\varepsilon_2>0$, and $\delta_2>0$ and a neighborhood D_2 of s_0 in $\boldsymbol{C}$ such that, for all $0<\varepsilon\leq\varepsilon_2$ and $s\in D_2$, $\mathcal{L}_{\varepsilon,s}$ is decomposed as follows*:

(i) $$\mathcal{L}_{\varepsilon,s}=\lambda_{\varepsilon,s}E_{\varepsilon,s}+S_{\varepsilon,s},$$

(ii) $$E_{\varepsilon,s}E_{\varepsilon,s}=E_{\varepsilon,s},\quad \mathcal{L}_{\varepsilon,s}E_{\varepsilon,s}=\lambda_{\varepsilon,s}E_{\varepsilon,s}.$$

(iii) *dimension of the range of $E_{\varepsilon,s}=1$, and there exists $h_{\varepsilon,s}\in\mathcal{F}_\theta(\Sigma_A^+)$ satisfying $\mathcal{L}_{\varepsilon,s}h_{\varepsilon,s}=\lambda_{\varepsilon,s}h_{\varepsilon,s}$ and*

(3.4) $$|h_{\varepsilon,s}(\xi)|\geq c\,\varepsilon^{c_0}\ (c>0),$$

(iv) $$|\lambda_{\varepsilon,s}-1|\leq\delta_2\quad \textit{for all}\quad s\in D_2,$$

(v) $$E_{\varepsilon,s}S_{\varepsilon,s}=S_{\varepsilon,s}E_{\varepsilon,s}=0,$$

$$\textit{the spectral rasdius of}\quad S_{\varepsilon,s}\big|_{\mathcal{F}_\theta(\Sigma_A^+)}\leq 1-2\delta_2,$$

(vi) $$|\lambda_{\varepsilon,s}-\tilde{\lambda}_s|\to 0\quad \textit{uniformly in } s\in D_2 \textit{ as } \varepsilon \textit{ tends to zero.}$$

Proof. As a direct result of the perturbation theory, the above proposition except (3.4) follows from Lemma 3.1 and Proposition 2.2. Then it suffices to show (3.4). By using the fact that $\{\xi\in\Sigma_A^+;\ B(\xi_1,\xi_2)=0\}$ is compact, it follows from (1.9) and (1.11) that, for an $a>0$,

(3.5) $$a\leq \operatorname{Im}(-sf_\varepsilon(\xi)+h_\varepsilon(\xi))\leq\pi-a$$

holds for all ξ satisfying $B(\xi_1,\xi_2)=0$, by exchanging ε_0, D_0 and a for smaller ones if necessary. As a result by the perturbation theory it holds that

(3.6) $$|||y_{\varepsilon,s}-w_s|||_\theta\to 0\quad\text{as}\quad \varepsilon\to 0.$$

Therefore we may assume also that

(3.7) $$|\operatorname{Arg} y_{\varepsilon,s}(\xi)|\leq\frac{a}{3}\quad\text{for all}\quad \xi\in\Sigma(1)$$

by exchanging ε_0 once more for a smaller one if necessary, because

$$\operatorname{Re} y_{\varepsilon,s}(\xi)\geq\frac{1}{2}C_4\quad\text{for all}\quad 0<\varepsilon\leq\varepsilon_0,$$

and

$$\operatorname{Im} y_{\varepsilon,s}(\xi)\to 0\quad\text{as}\quad \varepsilon\to 0.$$

By taking account of (1.9), (3.5) and (iv) of Proposition 2.2, we may assume that for all $\eta\in\Sigma(1)$ such taht $B(\eta_1,\eta_2)=0$

(3.8) $$\frac{a}{2}\leq\operatorname{Arg}(\lambda_{\varepsilon,s}^{-1}\exp(-sf_\varepsilon(\eta)+h_\varepsilon(\eta))\,y_{\varepsilon,s}(\eta))\leq\pi-\frac{a}{2}.$$

Since $w_s(\xi)=0$ for $\xi\in\Sigma(2)$, we have from (3.6) that

$$\beta_{\varepsilon,s}=\sup_{\xi\in\Sigma(2)}|y_{\varepsilon,s}(\xi)|\to 0 \quad\text{as}\quad \varepsilon\to 0\,. \tag{3.9}$$

Since $y_{\varepsilon,s}(\xi)$ is an eigenfunction of $\mathcal{L}_{\varepsilon,s}$ it is represented as

$$\begin{aligned} y_{\varepsilon,s}(\xi) &= \lambda_{\varepsilon,s}^{-1}\mathcal{L}_{\varepsilon,s}y_{\varepsilon,s}(\xi) \\ &= \sum_{\sigma_A\eta=\xi,\eta\in\Sigma(1)} \lambda_{\varepsilon,s}^{-1}\varepsilon^{k(\eta)}\exp(-sf_\varepsilon(\eta)+h_\varepsilon(\eta))\,y_{\varepsilon,s}(\eta) \\ &\quad+\sum_{\sigma_A\eta=\xi,\eta\in\Sigma(2)} \lambda_{\varepsilon,s}^{-1}\varepsilon^{k(\eta)}\exp(-sf_\varepsilon(\eta)+h_\varepsilon(\eta))\,y_{\varepsilon,s}(\eta) \\ &= I(\xi)+II(\xi)\,. \end{aligned}$$

Let $\xi\in\Sigma(2)$. Note that $B(\eta_1,\eta_2)=0$ for all η satisfying $\sigma_A\eta=\xi$ and that (1.2) implies $\{\eta\in\Sigma(1);\ \sigma_A\eta=\xi\}\neq\phi$. Then, by using (1.5) we have from (3.8)

$$\operatorname{Im} I(\xi)\geq\varepsilon^{c_0}\frac{C_4}{2}\sin\frac{a}{2}\,.$$

Since we have from (1.5)

$$|II(\xi)|\leq LC\varepsilon^{c_0}\beta_{\varepsilon,s}\,,$$

it follows from (3.9) that for small $\varepsilon>0$

$$\operatorname{Im} y_{\varepsilon,s}(\xi)\geq\varepsilon^{c_0}\frac{C_4}{4}\sin\frac{a}{2}\quad\text{for all}\quad \xi\in\Sigma(2)\,,$$

which shows that (3.4) holds.

The properties of $\mathcal{L}_{\varepsilon,s}$ shown in Proposition 3.2, more precisely (i)~(v) of Proposition 3.2 guarantee that the procedure of Parry [9] can be carried out for $\mathcal{L}_{\varepsilon,s}$ without any modification, and Theorem 1 of [9] implies that $Z_\varepsilon(s)=\zeta(-sf_\varepsilon+h_\varepsilon+\log\varepsilon\ k)$ is well defined for all $0<\varepsilon\leq\varepsilon_0$ and $s\in D_2$ and that $Z_\varepsilon(s)$ is of the form

$$Z_\varepsilon(s)=(1-e^{P_\varepsilon(s)})^{-1}\,\varphi_\varepsilon(s) \tag{3.10}$$

where

$$P_\varepsilon(s)=\log\lambda_{\varepsilon,s}$$

and φ_ε is a holomorphic function in D_2 without zero. Note that $P_\varepsilon(s)$ is well defined and analytic in $s\in D_2$ because of (iv).

Here, we would like to make a remark on the dependency of φ_ε on ε. Fix an $s\in D_2$ and denote $-sf_\varepsilon+h_\varepsilon$ by r_ε. If we cnostruct $\bar{r}_{\varepsilon,m}$ by following the process of [9], which is a function depending on $(\xi_1,\xi_2,\cdots,\xi_m)$ only such that the largest eigenvalue of the Ruelle operator for $\bar{r}_{\varepsilon,m}$ equals $\lambda_{\varepsilon,s}$, it is possible to have an estimate

$$||r_\varepsilon-\tilde{r}_{\varepsilon,m}||_\infty \leq C\,\varepsilon^{-c_0}|\log \varepsilon|\,\theta^m .$$

This estimate is sufficient for the convergence of φ_ε in D_2, but the dependency on ε seems to be singular when ε tensd to zero.

Since $\varphi_\varepsilon(s)\neq 0$ for all $s\in D_2$, in order to show the existence of pole in D_2 it suffices to show the existence of zero of $1-e^{P_\varepsilon(s)}$. The relation (vi) of Proposition 3.2 implies that $|P_\varepsilon(s)-\tilde{P}(s)|\to 0$ uniformly in D_2 as $\varepsilon\to 0$. On the other hand, (i) of Proposition 2.1 shows that $\tilde{s}=s_0$ is a simple zero of $\tilde{P}(s)$. Then according to Rouché's theorem, we see that $P_\varepsilon(s)$ has a zero near s_0 for ε small. Thus $Z_\varepsilon(s)$ has a pole in D_2 for small ε.

4. Application to an obstacle consisting of several convex bodies

First we shall give a brief explanation on the definition and fundamental properties of $F_D(s)$. Let $\mathcal{O}_j$, $j=1, 2, \cdots, L$, be bounded open sets with smooth boundary Γ_j satisfying

(H.1) every $\mathcal{O}_j$ is strictly convex,

(H.2) for every $\{j_1, j_2, j_3\}\in\{1, 2, \cdots, L\}^3$ such that $j_l\neq j_{l'}$ if $l\neq l'$,

$$\text{(convex hull of } \bar{\mathcal{O}}_{j_1} \text{ and } \bar{\mathcal{O}}_{j_2})\cap\bar{\mathcal{O}}_{j_3}=\phi .$$

We set

$$\mathcal{O}=\cup_{j=1}^L\mathcal{O}_j\,,\quad \Omega=\boldsymbol{R}^3-\bar{\mathcal{O}}\quad\text{and}\quad\Gamma=\partial\Omega .$$

Denote by γ a periodic ray in Ω, and we shall use the following notations:

d_γ: the length of γ,
T_γ: the primitive period of γ,
i_γ: the number of the reflecting points of γ,
P_γ: the Poincare map of γ.

We define $F_D(s)$ by

$$F_D(s)=\sum_\gamma(-1)^{i_\gamma}T_\gamma|I-P_\gamma|^{-1/2}e^{-sd_\gamma}$$

where the summation is taken over all the periodic rays in Ω and $|I-P_\gamma|$ denotes the determinant of $I-P_\gamma$.

Let $X(s)$ ($s\in\boldsymbol{R}$) be a representation of a broken ray in Ω by the arc length such that $X(0)\in\Gamma$. When $\{|X(s)|\,;\,s\in\boldsymbol{R}\}$ is bounded, $X(s)$ repeats reflections on the boundary Γ infinitely many times as s tends to $\pm\infty$. Let the j-th reflection point X_j be on Γ_{l_j}. Then a bounded broken ray defines an infinite sequence $\xi=\{\cdots, l_{-1}, l_0, l_1, \cdots\}$. Obviousely we have $l_j\neq l_{j+1}$. We call this infinite sequence as the reflection order of $X(s)$.

We deonte by A the $L\times L$ matrix defined by

$$A=(A(i,j))_{i,j=1,\cdots,L}\,,\quad A(i,j)=\begin{cases}1, & \text{if}\quad i\neq j\\ 0, & \text{if}\quad i=j\,,\end{cases}$$

and set

$$\Sigma_A=\{\xi=(\cdots,\xi_{-1},\xi_0,\xi_1,\cdots);\ A(\xi_j,\xi_{j+1})=1\quad\text{for all } j\}\ .$$

Then as was shown in [3], for every $\xi\in\Sigma_A$ there exists uniquely a broken ray whose reflection order is just equal to ξ. We set

$$f(\xi)=|X_0X_1|$$

where X_j denotes the j-th reflection point of the broken ray corresponding to ξ. For each periodic ray it corresponds a periodic element $\xi\in\Sigma_A$, that is, $\sigma_A^n\xi=\xi$ for some n. Denote by $\lambda_1(\xi)$ and $\lambda_2(\xi)$ the eigenvalues of P_γ greater than 1, and by $\kappa_l(\xi)$, $l=1, 2$, the principal curvatures at X_0 of the wave front of the phase function $\varphi_{i,0}^{\infty}$ defined in [3, Section 5], where $\boldsymbol{i}=(\xi_0,\cdots,\xi_{n-1})$. Then we have

$$\lambda_1(\xi)\,\lambda_2(\xi)=\prod_{j=1}^{n}(1+f(\sigma^j\xi)\,\kappa_1(\sigma^j\xi))\,(1+f(\sigma^j\xi)\,\kappa_2(\sigma^j\xi))\,. \tag{4.1}$$

It is easy to check that

$$\lambda_1(\xi)\,\lambda_2(\xi)\geq e^{cn}\ (c>0) \tag{4.2}$$

and

$$\#\{\gamma;\ \text{peridic ray in } \Omega \text{ suin that } d_\gamma\leq r\}\leq e^{a_0r}\,. \tag{4.3}$$

Since the other eigenvalues of P_γ are λ_1^{-1} and λ_2^{-1}, it holds that

$$|\lambda_1\lambda_2-|I-P_\gamma||\leq C(\lambda_1\lambda_2)^{1/2}\quad\text{for all }\gamma\,. \tag{4.4}$$

Define $g(\xi)$ for an periodic element ξ by

$$g(\xi)=-\frac{1}{2}\log(1+f(\xi)\,\kappa_1(\xi))\,(1+f(\xi)\,\kappa_2(\xi))\,.$$

Remark that we have from (4.1)

$$(\lambda_1(\xi)\,\lambda_2(\xi))^{-1/2}=\exp S_n\,g(\xi)\,,$$

and that $g(\xi)$ can be extended to a function in $\mathcal{F}_\theta(\Sigma_A)$. Define $\zeta(s)$ by

$$\zeta(s)=\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\sigma_A^n\xi=\xi}\exp S_n(-sf(\xi)+g(\xi)+\pi i)\Big)\,.$$

The estimates (4.2)$\sim$(4.4) imply that both $F_D(s)$ and $\zeta(s)$ converge absolutely for $\operatorname{Re} s$ large. Denote by ν_0 the abscissa of convergence of $\zeta(s)$, that is,

$$\nu_0 = \inf\{\nu;\ \zeta(s) \text{ converbes absolutely for } \operatorname{Re} s > \nu\}\,.$$

Then it holds that for $\operatorname{Re} s > \nu_0$

$$\begin{aligned}-\frac{d}{ds}\log Z(s) &= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{\sigma_A^n\xi=\xi} S_n f(\xi)\exp\left(S_n(-sf(\xi)+g(\xi)+\pi i)\right)\\ &= \sum_{n=1}^{\infty}\sum_{\sigma_A^n\xi=\xi}\frac{S_n f(\xi)}{n}(-1)^n(\lambda_1(\xi)\,\lambda_2(\xi))^{-1/2}\exp\left(-s\,S_n f(\xi)\right).\end{aligned}$$

Obviousely we have

$$S_n f(\xi) = d_\gamma,\ n = i_\gamma\,.$$

Taking account of the number of elements $\xi\in\Sigma_A$ corresponding to γ, we have

$$\sum_{\xi\in(\gamma)}\frac{S_n f(\xi)}{n} = T_\gamma$$

where the summation is taken over all ξ corresponding to γ. Thus we have

$$(4.5)\quad F_D(s)-\left(-\frac{d}{ds}\log Z(s)\right) = \sum_\gamma T_\gamma(-1)^n\{|I-P_\gamma|^{-1/2}-(\lambda_1\,\lambda_1)^{-1/2}\}\exp(-sd_\gamma)\,.$$

Since $|\,|I-P_\gamma|^{-1/2}-(\lambda_1\,\lambda_2)^{-1/2}|\le C(\lambda_1\,\lambda_2)^{-1}$ the left hand side of (4.5) is absolutely convergent in $\operatorname{Re} s \ge \nu_0-\alpha\,(\alpha>0)$.

Now we consider the case that $\mathcal{O}_j$ are small balls. Let P_j, $j=1, 2, \cdots, L$, be points in $\boldsymbol{R}^3$. Suppose that

(A.1) any triad of P_j's does not lie on a straight line.

Set

$$d_{\max} = \max_{i\neq j}|P_i\,P_j|$$

and

$$B(i,j) = \begin{cases}1 & \text{if}\quad |P_i\,P_j| = d_{\max}\,,\\ 0 & \text{if}\quad |P_i\,P_j| < d_{\max}\,.\end{cases}$$

By changing the numbering of the points if necessary, we may suppose that there exists $2\le K\le L$ such that

$$B(i,j) = 0 \quad \text{for all } i \quad \text{if} \quad j\ge K+1$$

and

$$B(i,j)=1 \quad \text{for some } i \quad \text{if} \quad j\leq K\,.$$

We ssume that

(A.2) $$C^N>0 \quad \text{for a some positive integer } N$$

and

(A.3) $$\min_{\substack{1\leq i\leq K, 1\leq j\leq L\\ i\neq j}} |P_i P_j| \geq \max_{i,j\geq K+1} |P_i P_j|\,.$$

We denote by $\mathcal{O}_{j,\varepsilon}$ the open ball with center P_j and radius ε, and set

$$\mathcal{O}_\varepsilon = \cup_{j=1}^{L} \mathcal{O}_{j,\varepsilon}\,.$$

Note that under the assumption (A.1) $\mathcal{O}_\varepsilon$ satisfies the condition (H.2) for ε small. We denote $f(\xi)$, $g(\xi)$ and $\zeta(s)$ attached to $\mathcal{O}_\varepsilon$ by $f_\varepsilon(\xi)$, $g_\varepsilon(\xi)$ and $\zeta_\varepsilon(s)$ respectively. It is easy to see that, by setting $f_0(\xi)=|P_{\xi_0} P_{\xi_1}|$,

(4.6) $$|\log \varepsilon|\; |||f_\varepsilon - f_0|||_\theta \to 0 \quad \text{as} \quad \varepsilon\to 0\,.$$

From the relationship between the curvatures of the wave fronts of incident and reflected waves we have

$$\kappa_1(\xi)=\frac{2}{\varepsilon}\left(\cos\frac{\Theta(\xi)}{2}\right)^{-1}+O(1)\,, \quad \kappa_2(\xi)=\frac{2}{\varepsilon}+O(1)$$

where $\Theta(\xi)$ is the angle $P_{\xi_{-1}} P_{\xi_0} P_{\xi_1}$. Thus we have immediately

$$|||g_\varepsilon(\xi)-\left(\log\varepsilon+\frac{1}{2}\log\left(\frac{1}{4}\cos\frac{\Theta(\xi)}{2}\right)\right)|||_\theta \to 0 \quad \text{as} \quad \varepsilon\to 0\,.$$

Then, by setting $\tilde{g}_\varepsilon(\xi)=g_\varepsilon(\xi)-\log\varepsilon$ and $\tilde{g}_0(\xi)=\frac{1}{2}\log\left(\frac{1}{4}\cos\frac{\Theta(\xi)}{2}\right)$ we have

(4.7) $$|||\tilde{g}_\varepsilon-\tilde{g}_0|||_\theta \to 0 \quad \text{as} \quad \varepsilon\to 0\,.$$

We see that k defined by

$$k(\xi)=1-f_0(\xi)/d_{\max}$$

satisfies (1.3) and (1.4). Indeed, (1.3) is trivial, and (1.4) follows from (A.3). By putting $s'=s-(\log\varepsilon+\sqrt{-1}\,\pi)/d_{\max}$ we have

$$-sf_\varepsilon+g_\varepsilon+\sqrt{-1}\,\pi = -s'f_\varepsilon+h_\varepsilon+k\log\varepsilon\,,$$

where

$$h_\varepsilon=\tilde{g}_\varepsilon+\sqrt{-1}\,\pi\,k+(\log\varepsilon+\sqrt{-1}\,\pi)\frac{(f_0-f_\varepsilon)}{d_{\max}}\,.$$

Evidently it follows from (4.6) that

$$h_0=\tilde{g}_0+\sqrt{-1}\,\pi\,k\,,$$

hence we have

$$h_0(\xi) = \tilde{g}_0(\xi) \quad \text{for } \xi \text{ satisfying} \quad B(\xi_1, \xi_2) = 1 .$$

Since $\operatorname{Im} h_0(\xi) = \pi\, k(\xi)$,

$$a \leq \operatorname{Im} h_0(\xi) \leq \pi - a \; (a > 0) \quad \text{for all } \xi \text{ satsfying} \quad B(\xi_1, \xi_2) = 0$$

follows from (1.3). Thus $h_\varepsilon, h_\varepsilon, k$ satisfy the conditions required in Theorem 1. Let $Z_\varepsilon(s)$ be the zeta function defined by (1.6) and (1.7) for these $h_\varepsilon, h_\varepsilon, k$. Then, Theorem 1 says that there exist $\varepsilon_0 > 0$, $s_0 \in \boldsymbol{R}$ and D_0 such that $Z_\varepsilon(s)$ has a pole in D_0. Then, it follows that

$$\zeta_\varepsilon(s) = Z_\varepsilon(s - (\log \varepsilon + \sqrt{-1}\, \pi)/d_{\max})$$

Thus we have the following

Theorem 2. *Suppose that the configuration of L points* $\{P_j \in \boldsymbol{R}^3; j = 1, 2, \cdots, L\}$ *satisfies* (A.1), (A.2) *and* (A.3). *Then there exist* $\varepsilon_0 > 0$, $s_0 \in \boldsymbol{R}$ *and a neighborhood* D_0 *of* s_0 *in* $\boldsymbol{C}$ *such that, for every* $0 < \varepsilon < \varepsilon_0$, $\zeta_\varepsilon(s)$ *is meromorphic in* $D_\varepsilon = \{s = z + (\log \varepsilon + \sqrt{-1}\, \pi)/d_{\max};\, z \in D_0\}$ *and has a pole near* $s_0 + (\log \varepsilon + \sqrt{-1}\, \pi)/d_{\max}$.

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