Fujita, T.
Osaka J. Math.
27 (1990), 229-245

ON DEL PEZZO FIBRATIONS OVER CURVES

TAKAO FUJITA

(Received May 15, 1989)

Introduction

Let $f: M \to C$ be a proper surjective holomorphic mapping of complex manifolds M, C with $\dim M = n+1 \geqq 3$, $\dim C = 1$ and let L be an f-ample line bundle on M. Such a quadruple (f, M, C, L) will be called a *Del Pezzo fibration* if (M_x, L_x) is a Del Pezzo manifold for any general point x on C, where $M_x = f^{-1}(x)$ and L_x is the restriction of L to M_x. This means $K+(n-1)L=0$ in $\mathrm{Pic}(M_x)$ for the canonical bundle K of M.

Let me explain how such a fibration appears in the classification theory of polarized manifolds. Suppose that L is an ample line bundle on a compact complex manifold M with $\dim M = m$. Then we have the following result (cf. [**F7**]):

Fact 1. *$K+mL$ is nef* (i.e. $(K+mL)Z \geqq 0$ for any curve Z in M) *unless $(M, L) \simeq (\boldsymbol{P}^m, \mathcal{O}(1))$. So $K+tL$ is nef for any $t \geqq m+1$.*

Fact 2. *$K+(m-1)L$ is nef unless $(M, L) \simeq (\boldsymbol{P}^m, \mathcal{O}(1))$, a hyperquadric in $\boldsymbol{P}^{m+1}$ with $L=\mathcal{O}(1)$, $(\boldsymbol{P}^2, \mathcal{O}(2))$ or a scroll over a smooth curve.*

For a vector bundle $\mathcal{E}$ over X, the pair $(\boldsymbol{P}(\mathcal{E}), \mathcal{O}(1))$ is called the *scroll* of $\mathcal{E}$ (or a scroll over X).

Fact 3. *Suppose that $K+(m-1)L$ is nef and $m \geqq 3$. Then $K+(m-2)L$ is nef except the following cases:*
a) *There is an effective divisor E on M such that $(E, L_E) \simeq (\boldsymbol{P}^{m-1}, \mathcal{O}(1))$ and the normal bundle of E is $\mathcal{O}(-1)$.*

b0) *(M, L) is a Del Pezzo manifold, $(\boldsymbol{P}^3, \mathcal{O}(j))$ with $j=2$ or 3, $(\boldsymbol{P}^4, \mathcal{O}(2))$ or a hyperquadric in $\boldsymbol{P}^4$ with $L=\mathcal{O}(2)$.*

b1) *There is a fibration $f: M \to C$ over curve C such that (M_x, L_x) is a hyperquadric in $\boldsymbol{P}^m$ with $L=\mathcal{O}(1)$ or $(\boldsymbol{P}^2, \mathcal{O}(2))$ for any general point x on C.*

Research partially supported by Grant in Aid for Co-operative Research 01302001, The Ministry of Education, Science and Culture

b2) *(M,L) is a scroll over a surface.*

This is a polarized version of the following classical

Fact 3_0. The canonical bundle of a smooth algebraic surface S is nef unless S contains a (-1)-curve, is isomorphic to $\boldsymbol{P}^2$ or is a $\boldsymbol{P}^1$-bundle over a curve.

Now we want to study the behaviour of $A=K+(m-2)L$ in case A is nef. By the base point free theorem (cf. [**KMM**]), there is a fibration $f: M \to W$ onto a normal variety W and an ample line bundle B on W such that $A=f^*B$. Moreover dim $W \leqq 3$ unless dim $W=m$ (cf. [**F7**]; (3.3)]). The latter case can be viewed as "general type". When dim $W=3<m$, f looks like a scroll over an open dense subset of W. When dim $W=2$, (M_x, L_x) is a hyperquadric for any general point x on W. When dim $W=0$, we have $K=(2-m)L$. This is a polarized higher-dimensional version of K3-surfaces. When dim $W=1$, f is a Del Pezzo fibration. This corresponds to elliptic surfaces in the surface theory.

Thus, the study of Del Pezzo fibrations can be viewed as a polarized version of the famous theory of Kodaira. If L is spanned by global sections, we get in fact an elliptic surface by taking general members of $|L|$ successively $(m-2)$ times. Such cases were studied partly by D'Souza [**D**].

This paper is organized as follows. In § 1 we recall the theory of minimal reduction and review basic results. In § 2 we classify reducible fibers. Apparently the result (2.20) is surprizingly simple, but this is no wonder, since the smoothness of M and the f-ampleness of L are very strong conditions. In § 3 we study irreducible non-normal fibers. Using these results and the theory in [**F6**], we study the global structure of f in § 4.

Basically we empoly the same notation as in my preceding papers on polarized varieties. In particular, vector bundles are not distinguished notationally from the locally free sheaves of their sections. Tensor products of line bundles are denoted additively, while we use multiplicative notation for intersection products in Chow rings. The pull-back of a line bundle F on Y by a morphism $f: X \to Y$ is usually denoted by F_X, or often just by F when confusion is impossible or harmless. In case Y is a projective space $\boldsymbol{P}^n_\xi$, we denote $f^*\mathcal{O}_Y(1)$ by H_ξ, using the same Greek index for the sake of identification.

1. Preliminaries

(1.1) Throughout this paper let (f, M, C, L) be a Del Pezzo fibtation over a curve C as in the introduction. Suppose that M_0 is a singular fiber while any nearby fiber M_x is smooth.

(1.2) **Theorem.** *$K+(n-1)L=f^*A$ for some line bundle A on C unless there exists a divisor E contained in a fiber of f such that $(E, L_E) \simeq (\boldsymbol{P}^n, \mathcal{O}(1))$ and $\mathcal{O}[E]_E=\mathcal{O}(-1)$.*

Proof. Suppose that $K+(n-1)L$ is not f-nef, which means, $(K+(n-1)L)Z<0$ for some curve Z such that $f(Z)$ is a point. By virtue of the theory in **[KMM]**, we may assume that Z is an extremal curve and we have the contraction morphism ψ of it. Z must be contained in a singular fiber of f since $K+(n-1)L=0$ on any general fiber. Hence ψ cannot be of fibration type. By the argument in [**F7**; (2.11: a)], we infer that there exists an exceptional divisor E with the required properties.

When $K+(n-1)L$ is nef, by the relative base point free theorem (cf. [**KMM**]), there exist a factorization $M\to W\to C$ of f and a relatively ample line bundle A on W such that $K+(n-1)L=A_M$. Then $W\to C$ is birational since $K+(n-1)L=0$ on any general fiber of f. So $W\simeq C$ and we are done.

(1.3) Thus, if $K+(n-1)L$ does not come from $\mathrm{Pic}(C)$, we find a divisor E as above. E can be blown down to a smooth point on another manifold M' and $L+E=L'_M$ for some ample line bundle L' on M'. Then (f', M', C, L') is a Del Pezzo fibration for the natural map $f'\colon M'\to C$.

Continuing such process if necessary, we finally obtain a model $f^\flat\colon M^\flat\to C$ such that $K^\flat+(n-1)L^\flat$ comes from Pic (C). This is called the *minimal reduction of f*.

From now on, we assume that f is minimal, i.e., $M=M^\flat$, or equivalently, $K+(n-1)L=f^*A$ for some $A\in\mathrm{Pic}(C)$.

REMARK. If L is nef, we can show $\deg(A)\geqq 2g(C)-2$. But we do not use this fact in this paper.

(1.4) **Proposition.** *f has no multiple fiber.*

Proof. (almost the same as that of [**M**; (3.5.2)]). Suppose that $M_o=mD$ in $\mathrm{Div}(M)$ for some $m\geqq 2$. Since $[D]$ is numerically trivial on the subscheme D, we have $\chi(\mathcal{O}_C[tD])=\chi(\mathcal{O}_D)$ for any t. Hence $\chi(\mathcal{O}_{M_o})=\sum_{t=0}^{m-1}\chi(\mathcal{O}_D[-tD])=m\chi(\mathcal{O}_D)$. On the other hand $\chi(\mathcal{O}_{M_o})=\chi(\mathcal{O}_{M_x})=1$ by the flatness. Thus we get a contradiction.

(1.5) In § 2 we study the case in which M_o is reducible. Here we recall some general results on irreducible fibers.

By (1,4), $V=M_o$ is reduced and (V, L_V) is a polarized variety. By the flatness of f we have $\chi(V, tL_V)=\chi(M_x, tL_x)$ for any t and $g(V, L_V)=1$, whre g is the sectional genus. We have $\Delta(V, L_V)\leqq\Delta(M_x, L_x)=1$ for the Δ-genus by the upper semicontinuity theorem. So $\Delta(V, L_V)=1$ since $\Delta=0$ would imply $g=0$. Thus (V, L_V) is a Del Pezzo variety.

By [**F8**; (1.2)], (V, L_V) has a ladder. Hence, similarly as [**F2**; Theorem 4.1], we have the following results:

1) (D, L_D) is a Del Pezzo variety for any general member D of $|L_V|$.

2) $\mathrm{Bs}|L_V|=\emptyset$ if $d=L_V^n\geqq 2$.
3) L_V is simply generated and very ample if $d\geqq 3$.
4) The image of V via the embedding given by $|L_V|$ is defined by quadratic equations if $d\geqq 4$.

Furthermore we can apply the theory in **[F6]** if $d\geqq 5$, since V has only hypersurface singularities and hence cannot be isomorphic to a cone over another Del Pezzo variety when $d>3$.

2. Reducible singular fibers

(2.1) In this section we study the case in which M_o is a reducible fiber in a minimal Del Pezzo fibration (f, M, C, L). Let $M_o=\sum \mu_\alpha D_\alpha$ be the prime decomposition as a divisor on M. The restriction of L to each component D_α will be denoted by L_α, or just by L when confusion is impossible.

We set $d_\alpha=d(D_\alpha, L_\alpha)=L_\alpha^n\{D_\alpha\}$. Then $\sum \mu_\alpha d_\alpha=d=L_x^n\{M_x\}$. By the classification theory of Del Pezzo manifolds we have $d\leqq 9$ (resp 8, 6, 5, 5, 4) if $n=2$ (resp. 3, 4, 5, 6, $\geqq 7$).

(2.2) We set $\gamma_{\alpha\beta}=L^{n-1}D_\alpha D_\beta\in \boldsymbol{Z}$ for each α, β. Then $\gamma_{\alpha\beta}\geqq 0$ if $\alpha\neq\beta$, and the equality holds if and only if $D_\alpha\cap D_\beta=\emptyset$, since L is ample on M_o. For each α, there exists $\beta\neq\alpha$ such that $\gamma_{\alpha\beta}>0$ since M_o is connected. This implies $\gamma_{\alpha\alpha}<0$ since $\sum_\beta \mu_\beta\gamma_{\alpha\beta}=0$.

(2 3) Let ω_α be the canonical sheaf of D_α. Then $\omega_\alpha=[K+D_\alpha]_{D_\alpha}=[D_\alpha+(1-n)L]_{D_\alpha}$ by the adjunction formula. So $2g(D_\alpha, L)-2=(\omega_\alpha+(n-1)L_\alpha)L_\alpha^{n-1}=D_\alpha L_\alpha^{n-1}\{D_\alpha\}=\gamma_{\alpha\alpha}<0$. Hence $g(D_\alpha, L)\leqq 0$.

Set $R_\alpha=\sum_{\beta\neq\alpha}\mu_\beta D_\beta$ and $\Gamma_\alpha=R_{\alpha|D_\alpha}\in \mathrm{Div}(D_\alpha)$ from now on.

(2.4) **Lemma.** *For each α, (D_α, L_α) is a polarized variety of one of the types below. In particular $\Delta(D_\alpha, L_\alpha)=g(D_\alpha, L_\alpha)=0$.*
1) $(\boldsymbol{P}^n, \mathcal{O}(1))$.
2) *(possibly singular) hyperquadric with $L=\mathcal{O}(1)$.*
3) *A scroll over $\boldsymbol{P}^1$.*
4) *Veronese surface $(\boldsymbol{P}^2, \mathcal{O}(2))$.*

Proof. Γ_α is an effective divisor on D_α and $\Gamma_\alpha\neq 0$. So there is a curve Y in D_α such that $\Gamma_\alpha Y>0$. Note that $\mu_\alpha D_\alpha Y=-\Gamma_\alpha Y<0$ and $KY=(1-n)LY<0$. By the theory of **[KMM]**, Y is a lineaı combination of extremal curves Z_i in the group of f-relative 1-cycles modulo numerical equivalence. From this we infer that there exists an extremel curve Z such that $D_\alpha Z<0$. Of course such a curve must be contained in D_α. Hence the contraction morphism ϕ of Z cannot be of fibration type. By the argument **[F7**; (2.14)], we infer that ϕ is a birational morphism with an exceptional divisor, which must be

D_α. Moreover, the type of (D_α, L) is classified into four types a1), a2), a3) and a4) in [**F7**; Theorem 4] when $n \geqq 3$.

In case a4), we have $(D_\alpha, L) \simeq (\boldsymbol{P}^n, \mathcal{O}(1))$ and 1) is the case.

In case a3), we have $(D_\alpha, L) \simeq (\boldsymbol{P}^3, \mathcal{O}(2))$. This is ruled out by (2.3).

In case a2), D_α is a hyperquadric and we are in case 2).

In case a1), there is a surjection $\pi: D_\alpha \to X$ onto a possibly singular curve X such that $(F, L_F) \simeq (\boldsymbol{P}^{n-1}, \mathcal{O}(1))$ for any fiber F over a smooth point on X. Let $D'_\alpha \to X'$ be the induced map of normalizations. Then (D'_α, L) is a polarized variety and in fact a scroll over X' by [**F1**; Corollary 5.4]. Note that $g(D_\alpha, L) \geqq g(D'_\alpha, L)$ and the equality holds if and only if codim (singular locus of D_α)>1. Since $g(D'_\alpha, L)$ is equal to the genus of X', we infer $g(D_\alpha, L) = g(D'_\alpha, L) = 0$ by (2.3). Moreover D_α is normal by Serre's criterion. Thus $D'_\alpha = D_\alpha$ is a scroll over $\boldsymbol{P}^1$, and we are in case 3).

When $n=2$, the type of (D_α, L) is classified by Mori [**M**]. By similar arguments as above we complete the proof of (2.4).

(2.5) **Corollary.** *$\Delta(D_\alpha, L_\alpha) = g(D_\alpha, L_\alpha) = 0$ and $\gamma_{\alpha\alpha} = -2$ for every α. Moreover L_α is very ample.*

(2.6) **Corollary.** *If the normal bundle N of an effective divisor Y on D_α is not ample, then (D_α, L_α) is a rational scroll as in (2.4.3) or possibly $(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathcal{O}(1, 1))$, a special case of (2.4.2). If $(Y, N) \simeq (\boldsymbol{P}^{n-1}, \mathcal{O}(-e))$ with $e>0$, then D_α is a Hirzebruch surface Σ_e (so $n=2$) and Y is the minimal section.*

Proof. The first assertion is obvious, since $\mathrm{Pic}(D_\alpha) \simeq \boldsymbol{Z}$ in the other cases. In the second assertion, Y cannot be a fiber of $D_\alpha \to \boldsymbol{P}^1$. So we have a surjection $Y \to \boldsymbol{P}^1$ and hence $n=2$. The rest is easy.

(2.7) Suppose in addition that $\mu_\alpha = 1$. Then, in $\mathrm{Pic}(D_\alpha)$, we have $-\Gamma_\alpha = [D_\alpha] = \omega_\alpha - K = \omega_\alpha + (n-1)L_\alpha$. So $\Gamma_\alpha \in |2L_\alpha|$ in case (2.4.1), $\Gamma_\alpha \in |L_\alpha|$ in case (2.4.2), and Γ_α is a line in $D_\alpha \simeq \boldsymbol{P}^2$ in case (2.4.4). In these cases Γ_α is an ample divisor.

In case (2.4.3), the restriction of Γ_α to any general fiber of $\phi_\alpha: D_\alpha \to X_\alpha \simeq \boldsymbol{P}^1$ is a hyperplane. Therefore it has a unique component mapped onto X_α. Any other component (if exists) is a fiber of ϕ_α.

From these observations we obtain:

0) Γ_α is always connected.

1) (2.4.1) is the case if $\Gamma_\alpha = 2Y$ for some Cartier divisor Y.

2) If $\Gamma_\alpha = 2Y$ for some Weil divisor Y, then (2.4.1) is the case unless D_α is a singular hyperquadric.

(2.8) **Lemma.** *$\mu_\alpha = 1$ for every α. Moreover, changing the indices suitably, we have one of the following conditions.*

(1) $M_0=D_1+\cdots+D_b$, $b\geqq 3$, $\gamma_{12}=\gamma_{23}=\cdots=\gamma_{b-1,b}=\gamma_{b1}=1$, $\gamma_{\alpha\beta}=0$ *for other* $\alpha\neq\beta$.
(2) $M_0=D_1+D_2$, $\gamma_{12}=2$.

Proof. Combining (2.2) and (2.5), we infer that $\gamma_{\alpha\beta}$'s satisfy the same condition as the intersection numbers of components in a singular fiber of an elliptic surface. Therefore we can classify them by Kodaira's method. Moreover we have $\sum \mu_\alpha\leqq d\leqq 9$ by (2.1). Therefore it suffices to rule out the case corresonding to the type I_b^* in Kodaira's notation.

In case I_b^*, we may assume that $\mu_1=\mu_2=1$, $\mu_3=2$, $\gamma_{13}=\gamma_{23}=1$, $\gamma_{12}=0$ and $\gamma_{\alpha\beta}=0$ for $\alpha\leqq 2$, $\beta\geqq 3$, by changing the indices if necessary. So $[2D_3]=\Gamma_\alpha$ in $\mathrm{Pic}(D_\alpha)$ for $\alpha=1, 2$. Hence $(D_\alpha, L)\simeq(\boldsymbol{P}^n, \mathcal{O}(1))$ for $\alpha=1, 2$ by (2.7.1). Moreover $Y_{\alpha 3}=D_\alpha\cap D_3$ is a hyperplane in D_α. Since $[D_\alpha]=\mathcal{O}(-2)$ on D_α, the normal bundle of $Y_{\alpha 3}$ in D_3 is $\mathcal{O}(-2)$. By (2.6), $Y_{\alpha 3}$ is the minimal section of $D_3\simeq\Sigma_2$ for each α. This contradicts $D_1\cap D_2=\emptyset$.

(2.9) From now on, we study the case (2.8.1) until (2.12). We set $Y_{\alpha\beta}=D_\alpha\cap D_\beta$ for $\alpha\neq\beta$. First we claim $b=3$.

Indeed, $Y_{12}+Y_{23}=\Gamma_2$. So $Y_{12}\cap Y_{23}\neq\emptyset$ by (2.7.0). Hence $D_1\cap D_3\neq\emptyset$ and $\gamma_{13}>0$. Thus $b>3$ cannot occur.

Note that $(Y_{\alpha\beta}, L)\simeq(\boldsymbol{P}^{n-1}, \mathcal{O}(1))$ since $L^{n-1}Y_{\alpha\beta}=\gamma_{\alpha\beta}=1$ and L_α is very ample.

(2.10) *Claim.* $Y_{\alpha\beta}$*'s are all different.*

Indeed, if $Y_{12}=Y_{23}$ for example, then $Y_{12}=Y_{23}\subset D_1\cap D_3$, So $Y_{12}=Y_{23}=Y_{31}$. Hence $(D_\alpha, L_\alpha)\simeq(\boldsymbol{P}^n, \mathcal{O}(1))$ by (2.7.1). Then $D_\alpha=\mathcal{O}(-2)$ on $Y=Y_{\alpha\beta}$. This cannot occur since $D_1+D_2+D_3=0$ in $\mathrm{Pic}(Y)$.

(2.11) We have $Y_{12}\cap D_3\neq\emptyset$ by the argument (2.9), while $Y_{12}\not\subset D_3$ by (2.10). So $[D_3]=\mathcal{O}(\delta)$ in $\mathrm{Pic}(Y_{12})$ for some $\delta>0$. Hence $[D_1]$ or $[D_2]$ is negative on Y_{12}. By symmetry we may assume that $[D_1]$ is negative on Y_{12}. This is the normal bundle of Y_{12} in D_2. So, by (2.6), $n=2$, D_2 is a Hirzebruch surface Σ_e with $e>0$, and Y_{12} is the minimal section of it. Y_{23} is the other component of Γ_2, and hence is a fiber of ϕ_2: $D_2\to X_2$. Therefore $D_1\cap D_2\cap D_3=Y_{12}\cap Y_{23}$ is a simple point. So $D_1Y_{23}=1$. Moreover $D_3Y_{23}=0$ since $[D_3]$ is the normal bundle of Y_{23} in D_2. Hence $D_2Y_{23}=-(D_1+D_2)Y_{23}=-1$. Again by (2.6), this implies $D_3\simeq\Sigma_1$, Y_{23} is the minimal section (the unique (-1)-curve in this case), and Y_{31} is a fiber. Proceeding similarly, for each $\alpha\in\boldsymbol{Z}/3\boldsymbol{Z}$ we get: $D_{\alpha+1}Y_{\alpha-1,\alpha}=1$, $D_\alpha Y_{\alpha-1,\alpha}=0$, $D_{\alpha-1}Y_{\alpha-1,\alpha}=-1$, $D_\alpha\simeq\Sigma_1$, $Y_{\alpha-1,\alpha}$ is the minimal section of D_α, and $Y_{\alpha,\alpha+1}$ is a fiber of $D_\alpha\to\boldsymbol{P}^1$.

(2.12) Thus we describe the structure of M_0 completely in the case (2.8.1). Since $LY_{\alpha\beta}=1$, we infer $L_\alpha=Y_{\alpha-1,\alpha}+2Y_{\alpha,\alpha+1}$ in $\mathrm{Pic}(D_\alpha)$. So $d_\alpha=L_\alpha^2=3$ and $d=d_1+d_2+d_3=9$.

Any D_α can be blown down smoothly to $X_\alpha \simeq \boldsymbol{P}^1$. Suppose that we first blow down D_3. Then D_2 is mapped isomorphically onto its image, which we denote by D_2'. D_1 is mapped onto $\boldsymbol{P}^2$ since Y_{31} is mapped to a point. Next we can blow down D_2' to $\boldsymbol{P}^1$. Then M_o is transformed to a smooth fiber $\boldsymbol{P}^2$.

Conversely, a singular fiber of this type can be obtained from a $\boldsymbol{P}^2$-scroll (M'', H) as follows: First blow up along a line X in a fiber $\simeq \boldsymbol{P}^2$. The exceptional divisor E is isomorphic to Σ_1. Next blow up along a fiber of $E \to X$. Then we get a fiber of the type (2.11). The polarization is given by $L = 3H - E_1 - E_2$, where E_i is the total transform of the exceptional divisor of the i-th blow up.

REMARK. Any D_α can be blown down first, and we get a fiber of the type (2.19.4) below by this step. Thus there are several ways to get a $\boldsymbol{P}^2$-bundle by blowing-down. This can be viewed as a 3-dimensional version of elementary transformations of ruled surfaces.

(2.13) From now on, we study the case (2.8.2). First we claim that $Y = D_1 \cap D_2$ is reduced.

Indeed, otherwise, (2.7.2) applies. So the normal bundle of Y in D_α is ample for each α. This contradicts $[D_1 + D_2]_Y = 0$.

(2.14) Suppose that Y is reducible. Since $L^{n-1} Y = \gamma_{12} = 2$, Y has two components Y_1, Y_2 such that $L^{n-1} Y_i = 1$. So $(Y_i, L) \simeq (\boldsymbol{P}^{n-1}, \mathcal{O}(1))$ since L_α is very ample on D_α. We have $D_1 + D_2 = 0$ in $\mathrm{Pic}(Y_i)$. Hence, by symmetry, we may assume that $[D_1]$ is not ample on Y_1. Then Γ_2 is not ample on D_2. Hence D_2 is a rational scroll by (2.6). In particular D_2 is smooth and both Y_i's are Cartier on D_2. Moreover $Y_1 \cap Y_2 \neq \emptyset$. Since $[Y_1 + Y_2]_{Y_1} = [D_1]_{Y_1}$ is not ample, the normal bundle of Y_1 in D_2 is negative. By (2.6) this implies $n=2$, $D_2 \simeq \Sigma_e$ for some $e > 0$ and Y_1 is the minimal section of it. Y_2 must be a fiber. Hence $1 = (Y_1 + Y_2) Y_2 \{D_2\} = D_1 Y_2$.. So $D_2 Y_2 = -1$. Similarly as above, we get $D_1 \simeq \Sigma_2$ and Y_2 is the minimal section of it, since $-1 = D_2 Y_2 = (Y_1 + Y_2) Y_2 \{D_1\}$. This in turn implies $1 = D_2 Y_1 = -D_1 Y_1$ and $D_2 \simeq \Sigma_2$. As for the polarization, we have $L Y_1 = L Y_2 = 1$. So $L_1 = Y_2 + 3Y_1$ on D_1 and $L_2 = Y_1 + 3Y_2$ on D_2. Hence $d_\alpha = 4$ and $d = d_1 + d_2 = 8$. Thus we get a complete description of M_o.

(2.15) In the above case either divisor D_α can be blown down smoothly to $X_\alpha \simeq \boldsymbol{P}^1$. When we blow down D_2, then D_1 is mapped onto a singular quadric since Y_2 is contracted to a point. The result is a hyperquadric fibration and any nearby fiber is $\boldsymbol{P}^1 \times \boldsymbol{P}^1$.

Conversely, we get a singular fiber of the type (2.14) by blowing up a hyperquadric fibration (M', H) with a singular fiber (M_o', H_o) isomorphic to a normal singular quadric in $\boldsymbol{P}^3$. The center should be a line l on M_o' passing the singular point. The polarization is given by $L = 2H - E$, where E is the exceptional divisor over l.

(2.16) From now on we suppose in addition that Y is irreducible. Since $L^{n-1}Y=2$, Y is a hyperquadric which is possibly singular. By symmetry we may assume that $[D_1]_Y$ is not ample. Then Γ_2 is not ample on D_2, so D_2 is a rational scroll by (2.6). We have a surjection $Y\to X_2\simeq \boldsymbol{P}^1$. This is possible only when $n=2$, or $n=3$ and $Y\simeq \boldsymbol{P}^1\times\boldsymbol{P}^1$.

(2.17) Here we suppose $Y\simeq \boldsymbol{P}^1\times\boldsymbol{P}^1$. A factor is identified with $X\simeq \boldsymbol{P}^1_\xi$ and the other will be denoted by $\boldsymbol{P}^1_\zeta$. The pull-backs of $\mathcal{O}(1)$'s will be denoted by H_ξ and H_ζ respectively. Then the normal bundle of Y in D_2 is $aH_\xi+H_\zeta$ for some $a\in\boldsymbol{Z}$, since a fiber of $Y\to X_2$ is a line in a fiber of $D_2\to X_2$. So $[D_2]_Y=[-D_1]_Y=-aH_\xi-H_\zeta$. Hence Y is not ample on D_1. By (2.6), D_1 is a scroll over $X_1\simeq\boldsymbol{P}^1$. The restriction of D_2 to a fiber of $Y\to X_1$ is $\mathcal{O}(1)$. Therefore $Y\to X_1$ is identified with the projection $Y\to\boldsymbol{P}^1_\zeta$ and hence $a=-1$.

We have $L_1=Y+zH_\zeta$ for some $z\in\boldsymbol{Z}$ in $\mathrm{Pic}(D_1)$, where H_ζ is the pull-back of $\mathcal{O}(1)$ on $X_1\simeq\boldsymbol{P}^1_\zeta$. Restricting to Y we get $H_\zeta+H_\xi=(H_\xi-H_\zeta)+zH_\zeta$, so $z=2$. Now, using the exact sequence $0\to\mathcal{O}_{D_1}(2H_\zeta)\to\mathcal{O}_{D_1}(L_1)\to\mathcal{O}_Y(L_1)\to 0$, we obtain an exact sequence $0\to\mathcal{O}(2)\to\mathcal{E}_1\to\mathcal{O}(1)\oplus\mathcal{O}(1)\to 0$ of locally free sheaves on X_1, where $\mathcal{E}_1=(\phi_1)_*\mathcal{O}_{D_1}(L_1)$. So (D_1, L_1) is the scroll associated to $\mathcal{E}_1\simeq\mathcal{O}(2)\oplus\mathcal{O}(1)\oplus\mathcal{O}(1)=\mathcal{O}(2, 1, 1)$. Quite similarly (D_2, L_2) is the scroll associated to $\mathcal{E}_2\simeq\mathcal{O}(2, 1, 1)$ over $X_2\simeq\boldsymbol{P}^1$. In particular we have $d_\alpha=4$ and $d=d_1+d_2=8$.

(2.18) In the above case either divisor D_α can be blown down smoothly to X_α. When D_2 is blown down, D_1 is mapped onto $D_1'\simeq\boldsymbol{P}^3$ and the pull-back to D_1 of $\mathcal{O}(1)$ on D_1' is L_1-H_ζ. Thus we get a $\boldsymbol{P}^3$-bundle.

Conversely, we get a singular fiber of the type (2.17) by blowing up a $\boldsymbol{P}^3$-scroll (M', H). The center is a line in a fiber. The polarization is given by $L=2H-E$, where E is the exceptional divisor.

There are two ways of blowing-down, and they give a 4-dimensional version of elementary transformations.

(2.19) In the remaining cases we have $n=2$. So Y is a smooth quadric curve. By symmetry we may assume $D_1Y\leqq 0$. Then Y is not ample on D_2. By (2.6) $D_2\simeq\Sigma_e$ for some $e\geqq 0$. Let $H_2\in\mathrm{Pic}(D_2)$ be the class of a fiber and let Z be the class of a section with $Z^2=e$. When $e>0$, the minimal section is the unique member of $|Z-eH_2|$. In any case we have $L_2=Z+aH_2$ for some $a>0$ and $\omega_2=-2Z+(e-2)H_2$. So $Y=-(\omega_2+L_2)=Z+(2-a-e)H_2$ in $\mathrm{Pic}(D_2)$. Recall that $0\geqq D_1Y=Y^2=4-2a-e$. If $a>2$, then $Y^2<0$ and Y must be the minimal section, but this contradicts $Y^2=4-2a-e<-e$. If $a=1$, then $0\geqq 2-e$ and $e\geqq 2$, so $Y^2\leqq 0$ is possible only when Y is the minimal section. Thus, in any case, we have $D_1Y=Y^2=-e$, $a=2$ and $d_2=L_2^2=e+4$. Note

that D_2 can be blown down smoothly to $X_2 \simeq \boldsymbol{P}^1$. According to the type of (D_1, L_1), we now divide the cases as follows:
1) $(\boldsymbol{P}^2, \mathcal{O}(1))$.
2) (D_1, L_1) is a singular quadric.
3) A scroll over $X_1 \simeq \boldsymbol{P}^1$, including the case $(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathcal{O}(1, 1))$.
4) $(\boldsymbol{P}^2, \mathcal{O}(2))$.

(2.19.1) In this case Y is a conic on D_1. So $4 = Y^2 = D_2 Y$ and hence $-4 = D_1 Y = -e$, $d = d_1 + d_2 = 1 + 8 = 9$. We get a $\boldsymbol{P}^2$-bundle by blowing down D_2 to X_2. The center of the converse blowing-up is a conic. The contraction of D_1 yields a quotient singularity.

(2.19.2) In this case Y is a smooth hyperplane section on D_1. So $2 = Y^2\{D_1\} = D_2 Y$ and hence $2 = -D_1 Y = e$, $d = d_1 + d_2 = 2 + 6 = 8$. We get a singular quadric by blowing down D_2. Any nearby fiber is $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. The contraction of D_1 yields a hypersurface singularity of the type $x^2 + y^2 + z^2 + u^3 = 0$.

(2.19.3) D_1 is a Hirzebruch surface Σ_s with $s \geqq 0$. Let H_1 be the class of a fiber and let S be a section with $S^2 = s$ as before. Then $L_1 = S + bH_1$ for some $b > 0$ and $Y \in |S + (2 - b - s)H_1|$. So $e = -D_1 Y = D_2 Y = 4 - 2b - s$. Hence $(b, s, e) = (2, 0, 0)$, $(1, 0, 2)$, $(1, 1, 1)$ or $(1, 2, 0)$.

(2.19.3a) In case $(b, s, e) = (2, 0, 0)$, we have $d = d_1 + d_2 = 4 + 4 = 8$ and $D_1 Y = D_2 Y = 0$. We get Σ_0 by blowing down etiher divisor D_α.

(2.19.3b) In case $(b, s, e) = (1, 0, 2)$, we have $d = d_1 + d_2 = 2 + 6 = 8$ and $-D_1 Y = D_2 Y = 2$. We get Σ_0 by blowing down D_2. We get Σ_2 by blowing down D_1, but in this case the ampleness of the polarization is not preserved and the result is not a Del Pezzo fibration in our sense.

(2.19.3c) In case $(b, s, e) = (1, 1, 1)$, we have $d = d_1 + d_2 = 3 + 5 = 8$ and $-D_1 Y = D_2 Y = 1$. We get Σ_1 by blowing down D_2. Any nearby fiber is also Σ_1. We get a Σ_1-bundle by blowing down D_1 too, but the situation is not symmetric (the center of the converse blow-up is different).

(2.19.3d) In case $(b, s, e) = (1, 2, 0)$, Y is a member of $|S - H_1|$ on $D_1 \simeq \Sigma_2$. This case is ruled out since Y must be irreducible.

(2.19.4) In this case Y is a line in $D_1 \simeq \boldsymbol{P}^2$. So $1 = Y^2\{D_1\} = D_2 Y$ and hence $1 = -D_1 Y = e$, $d = d_1 + d_2 = 4 + 5 = 9$. We get a $\boldsymbol{P}^2$-bundle by blowing down D_2. The center of the converse blowing-up is a line. The contraction of D_1 yields also a $\boldsymbol{P}^2$-bundle, but the center of the converse blowing-up is a point.

These two processes yield a 3-dimensional version of elementary transformation.

(2.20) Summing up the preceding observations we get the following classification table of reducible singular fibers in minimal Del Pezzo fibrations.

type	n	(D_α, L_α)'s	d & d_α	can blow down first	to	nearby fiber	$D_\alpha \cap D_\beta$
(2.12)	2	three $\Sigma(2, 1)$'s	$9=3+3+3$	any	$\boldsymbol{P}^2$	$\boldsymbol{P}^2$	3 lines
(2.19.1)	2	$\boldsymbol{P}^2(1)+\Sigma(6, 2)$	$9=1+8$	D_2	$\boldsymbol{P}^2$	$\boldsymbol{P}^2$	conic
(2.19.4)	2	$\boldsymbol{P}^2(2)+\Sigma(3, 2)$	$9=4+5$	either	$\boldsymbol{P}^2$	$\boldsymbol{P}^2$	conic
(2.19.3c)	2	$\Sigma(2, 1)+\Sigma(3, 2)$	$8=3+5$	either	Σ_1	Σ_1	conic
(2.19.3b)	2	$\Sigma(1, 1)+\Sigma(4, 2)$	$8=2+6$	D_2	Σ_0	Σ_0	conic
(2.19.2)	2	$\Sigma'(2, 0)+\Sigma(4, 2)$	$8=2+6$	D_2	Σ'_2	Σ_0	conic
(2.19.3a)	2	$\Sigma(2, 2)+\Sigma(2, 2)$	$8=4+4$	either	Σ_0	Σ_0	conic
(2.14)	2	$\Sigma(3, 1)+\Sigma(3, 1)$	$8=4+4$	either	Σ'_2	Σ_0	$\boldsymbol{P}^1+\boldsymbol{P}^1$
(2.17)	3	$\Sigma(2, 1, 1)+\Sigma(2, 1, 1)$	$8=4+4$	either	$\boldsymbol{P}^3$	$\boldsymbol{P}^3$	$\boldsymbol{P}^1\times\boldsymbol{P}^1$

Here $\boldsymbol{P}^2(\mu)$ denotes $(\boldsymbol{P}^2, \mathcal{O}(\mu))$ and $\Sigma(\delta_1, \cdots, \delta_n)$ denotes the scroll of the vector bundle $\mathcal{O}(\delta_1)\oplus\cdots\oplus\mathcal{O}(\delta_n)$ over $\boldsymbol{P}^1$. In particular the base space of $\Sigma(a, b)$ is $\Sigma_{|a-b|}$. Σ' denotes the projective image via the rational map defined by the tautological line bundle. So $\Sigma'(2, 0)$ is the singular quadric.

(2.21) *Remark.* In any case L is relatively very ample. When we cut a fiber of the type (2.17) by a general member of $|L|$, we get a fiber of the type (2.19.3a). If we cut $(n-1)$-times, then we get a singular fiber of Kodaira's type I_2 except in case (2.12), which yields type I_3.

3. Non-normal fibers

In this section we study irreducible non-normal fibers in minimal Del Pezzo fibrations. The main result here is the following

(3.1) **Theorem.** *If $d\geqq 5$, an irreducible non-normal fiber appears only when $d=6$ and $n\leqq 3$.*

(3.2) Beginning of the proof. As we see in (1.5), L is very ample on $V=M_o$ if $d\geqq 5$, and such a pair (V, L_V) is classified in **[F6]**. Since V is not normal, its singular locus is of dimension $n-1$. So, by **[F6**; (2.7)], $n\leqq 3$ and V is of the type (c) there. Let us recall what this means precisely.

Embed V in $P=\boldsymbol{P}_\alpha^{n+d-2}$ by $|L|$. Take a singular point v of V and let W be the projective join $v*V$. By "type (c)" we mean that W is a generalized cone over a Veronese curve $Z\simeq\boldsymbol{P}_\beta^1$ of degree $d-2$ in $\boldsymbol{P}^{d-2}$.

(3.3) In such a case the structure of V is described in **[F6]** as follows. Both singular loci of V and W coincide with $R=\text{Ridge}(W)$, the set of points x such that $x*W=W$. R is a linear $\boldsymbol{P}^{n-1}$ in P. Let $\tilde{P}$ be the blowing-up of P along R and let $\tilde{W}$, $\tilde{V}$ be the proper transforms of W, V. Then $(\tilde{W}, H_\alpha)$ is the scroll of the vector bundle $(d-2)H_\beta\oplus\mathcal{O}\oplus\cdots\oplus\mathcal{O}$ over $Z\simeq \boldsymbol{P}^1_\beta$, where H_α is the pull-back of $\mathcal{O}_P(1)$. $\tilde{V}$ is a smooth member of $|H_\alpha+2H_\beta|$ on $\tilde{W}$ by [**F6**; (c.1)]. The unique member D of $|H_\alpha-(d-2)H_\beta|$ on $\tilde{W}$ is the exceptional divisor of $\tilde{W}\to W$. The natural mappings $D\to Z$ and $D\to P$ yield $D\simeq Z\times R\simeq \boldsymbol{P}^1_\beta\times\boldsymbol{P}^{n-1}_\alpha$. The intersection $\tilde{V}\cap D$ is defined by $\alpha_0\beta_0^2+\alpha_1\beta_0\beta_1+\alpha_2\beta_1^2=0$ when $n=3$ for suitable homogeneous coordinates $(\alpha_0\colon\alpha_1\colon\alpha_2)$ and $(\beta_0\colon\beta_1)$ of R and Z. When $n=2$, there are two cases as in [**F6**; p. 153, (ci)].

(3.4) Here and in (3.5), we assume $n=3$. In this case the exceptional divisor $\tilde{V}_D=\tilde{V}\cap D$ of the birational map $\tilde{V}\to V$ is a $\boldsymbol{P}^1$-bundle over Z and is a double coevring of R branched along a smooth hyperquadric. Therefore $\tilde{V}_D\simeq \boldsymbol{P}^1_\beta\times\boldsymbol{P}^1_\tau$ with $H_\alpha=H_\beta+H_\tau$ in $\mathrm{Pic}(\tilde{V}_D)$. The normal bundle of $\tilde{V}_D$ in $\tilde{V}$ is the restriction of $[D]=H_\alpha-(d-2)H_\beta=H_\tau+(3-d)H_\beta$.

(3.5) On the other hand, V is a hypersurface in M. Let S be its singular locus and let $\mathcal{E}$ be the conormal bundle of S in M. Clearly $S\simeq R\simeq \boldsymbol{P}^2_\alpha$ and $\det\mathcal{E}=K_{|S}-\omega_S=H_\alpha$. Let $\tilde{M}$ be the blowing-up of M along S, let E be the exceptional divisor over S, let $\tilde{V}_M$ be the proper transform of V and set $\tilde{V}_E=\tilde{V}_M\cap E$. Then $(E,[-E])$ is the scroll of $\mathcal{E}$ over S and $(\tilde{V}_M,\tilde{V}_E)\simeq(\tilde{V},\tilde{V}_D)$. Hence $\tilde{V}_E\to S$ is a double covering, $\tilde{V}_E\simeq\boldsymbol{P}^1_\beta\times\boldsymbol{P}^1_\tau$ and the restriction of $[E]$ to $\tilde{V}_E$ is $H_\tau+(3-d)H_\beta$. Set $H(\mathcal{E})=[-E]\in\mathrm{Pic}(E)$. Then, since $\tilde{V}\in|[V]-2E|$ on $\tilde{M}$, $\tilde{V}_E$ is a member of $|2H(\mathcal{E})|$. So $2H(\mathcal{E})^2H_\alpha\{E\}=H(\mathcal{E})H_\alpha\{\tilde{V}_E\}=(-H_\tau+(d-3)H_\beta)(H_\tau+H_\beta)\{\boldsymbol{P}^1_\beta\times\boldsymbol{P}^1_\tau\}=d-4$. On the other hand $H(\mathcal{E})^2H_\alpha\{E\}=c_1(\mathcal{E})H_\alpha\{S\}=H_\alpha^2\{\boldsymbol{P}^2_\alpha\}=1$. Thus we get $d=6$, as desired.

(3.6) Now we study the case $n=2$. First we consider the case in which $\tilde{V}_D$ is irreducible. Then $\tilde{V}_D$ is a section of $D\to Z$ and $D\to R\simeq\boldsymbol{P}^1_\alpha$ makes $\tilde{V}_D$ a double covering of R. The normal bundle of $\tilde{V}_D$ in $\tilde{V}$ is $H_\alpha-(d-2)H_\beta$, which is of degree $4-d$. On the other hand, let S be the singular locus of V, $\mathcal{E}$ the conormal bundle of S, $\tilde{M}$ the blowing-up of M along S, E the exceptional divisor over S, $\tilde{V}_M$ the proper transform of V, and $H(\mathcal{E})=[-E]_E\in\mathrm{Pic}(E)$. Then, similarly as in (3.5), we have $(\tilde{V}_M,\tilde{V}_E)\simeq(\tilde{V},\tilde{V}_D)$ for $\tilde{V}_E=\tilde{V}_M\cap E$, $S\simeq R\simeq\boldsymbol{P}^1_\alpha$, $\det\mathcal{E}=H_\alpha$ and $\tilde{V}_E\in|2H(\mathcal{E})|$ on E. So $4-d=[E]\{\tilde{V}_E\}=-H(\mathcal{E})\{\tilde{V}_E\}=-2H(\mathcal{E})^2\{E\}=-2c_1(\mathcal{E})=-2$, hence $d=6$.

Next we consider the case in which $\tilde{V}_D$ is not irreducible. As we saw in [**F6**], $\tilde{V}_D$ does not contain any fiber of $D\to R$ and hence $\tilde{V}_D$ is of the form Y_1+Y_2 on $D\simeq Z\times R$, where Y_1 is a fiber of $D\to Z$ and Y_2 is a smooth member of $|H_\beta+H_\alpha|$. Thus $Y_1^2=0$, $Y_1Y_2=1$, $Y_2^2=2$ in D. The normal bundle of $\tilde{V}_D$ in $\tilde{V}$ is the pull-back of $[D]=H_\alpha-(d-2)H_\beta$, so $[D]Y_1=1$ and $[D]Y_2=3-d$. On the other hand let S, $\mathcal{E}$, $\tilde{M}$, E, $\tilde{V}_M$, $H(\mathcal{E})$ and $\tilde{V}_E$ be as above. Then $S\simeq R\simeq\boldsymbol{P}^1_\alpha$, $\det\mathcal{E}=H_\alpha$, $\tilde{V}_E\in|2H(\mathcal{E})|$ and $\tilde{V}_E$ is of the form Z_1+Z_2 with $[E]Z_1=1$ and $[E]Z_2=3-d$. Both Z_i's are sections of $E\to S$. Thus $4-d=[E]\{\tilde{V}_E\}=-2$ again.

Thus we complete the proof of (3.1).

(3.7) REMARK. It is uncertain whether such non-normal fibers of degree six appear really in Del Pezzo fibrations or not.

4. Global structures

Let (f, M, C, L) be a minimal Del Pezzo fibration as in § 1.

(4.1) First we study the case $d=L^n=1$. In [**F4**-III; § 14] we studied the structure of Del Pezzo manifold of degree one, which can be generalized in the present relative situation as follows.

Every fiber $V=M_x$ is irreducible and reduced since $d=1$ and L_V is ample. Moreover every member of $|L_V|$ is irreducible and reduced. Hence (V, L_V) has a ladder. $\mathrm{Bs}|L_V|$ is the intersection of n general members of $|L_V|$ and is a simple point since $d=1$. Set $\mathcal{L}=\mathcal{O}_M(L)$, $\mathcal{E}=f_*\mathcal{L}$ and let $\mathcal{C}$ be the cokernel of the natural homomorphism $f^*\mathcal{E}\to\mathcal{L}$. The above observation implies that the scheme-theoretical support Z of $\mathcal{C}$ is a section of f. Let $M^\natural$ be the blowing-up of M along Z, let E be the exceptional divisor over Z and set $\mathcal{L}^\natural=\mathcal{O}(L-E)$. Then $\mathcal{E}=f_*\mathcal{L}^\natural$ and $f^*f_*\mathcal{L}^\natural\to\mathcal{L}^\natural$ is surjective. Hence we have a morphism $\phi\colon M^\natural\to \boldsymbol{P}$ of C-schemes such that $\phi^*H_\beta=L-E$, where $(\boldsymbol{P}, H_\beta)$ is the scroll of $\mathcal{E}$. Similarly as in [**F4**-III, § 13], $\mathrm{rank}(\mathcal{E})=n$ and the restriction $\phi_E\colon E\to \boldsymbol{P}$ is an isomorphism. Every fiber of ϕ is an irreducible reduced curve of arithmetic genus one.

Set $\mathcal{D}=\mathcal{O}_{M^\natural}(2L)$ and $\mathcal{F}=\phi_*\mathcal{D}$. Similarly as in [**F4**; (14.2)], $\mathcal{F}$ is a locally free sheaf of rank two on $\boldsymbol{P}$, and we have a finite morphism $\rho\colon M^\natural\to W$ of degree two such that $\rho^*H_\alpha=\mathcal{D}$, where (W, H_α) is the scroll of $\mathcal{F}$ over $\boldsymbol{P}$. The branch locus of ρ is of the form $S+B_2$, where $S=\rho(E)$ and B_2 is a smooth divisor such that $S\cap B_2=\emptyset$. S is a section of the $\boldsymbol{P}^1$-bundle $W\to \boldsymbol{P}$ and $E\simeq S\simeq \boldsymbol{P}$.

We have an exact sequence $0\to\mathcal{O}_{M^\natural}(2L-2E)\to\mathcal{O}_{M^\natural}(2L-E)\to\mathcal{O}_E(2L-E)\to 0$. The map $\phi_*\mathcal{O}_{M^\natural}(2L-E)\to\phi_*\mathcal{O}_E(2L-E)$ vanishes everywhere on $\boldsymbol{P}$. So $\phi_*\mathcal{O}_{M^\natural}(2L-E)\simeq\mathcal{O}_P(2H_\beta)$. Using the exact sequence $0\to\mathcal{O}_{M^\natural}(2L-E)\to\mathcal{O}_{M^\natural}(2L)\to\mathcal{O}_{E^\natural}(2L)\to 0$, we get an exact sequence (*) $0\to\mathcal{O}_P(2H_\beta)\to\mathcal{F}\to\mathcal{O}_P(2A_P)\to 0$, where A is the line bundle on C corresponding to L_Z via $Z\simeq C$. Pull back (*) to B_2 by the branched covering $B_2\to \boldsymbol{P}$. Then $\boldsymbol{P}_{B_2}(\mathcal{F})$ is the fiber product of W and B_2 over $\boldsymbol{P}$, and hence has two disjoint sections. Therefore (*) splits on B_2. So it splits on $\boldsymbol{P}$ itself. Thus $\mathcal{F}\simeq\mathcal{O}_P(2H_\beta)\oplus\mathcal{O}_P(2A_P)$. Clearly S is the unique member of $|H_\alpha-2H_\beta|$ on $W=\boldsymbol{P}_P(\mathcal{F})$ and $B_2\in|3(H_\alpha-2A_W)|$.

Conversely, (f, M, C, L) is obtained from the data C, $\mathcal{E}$, $A\in\mathrm{Pic}(C)$ and B_2 above as follows. Let $(\boldsymbol{P}, H_\beta)$ be the scroll of the vector bundle $\mathcal{E}$ over C. Set $\mathcal{F}=\mathcal{O}_P(2H_\beta)\oplus\mathcal{O}_P(2A_P)$ and let (W, H_α) be the scroll of $\mathcal{F}$ over $\boldsymbol{P}$. Let S be the unique member of $|H_\alpha-2H_\beta|$. Suppose that there is a smooth member B_2

of $|3(H_\alpha-2A_W)|$ such that $S\cap B_2=\emptyset$. Let $M^\natural$ be the finite double covering of W branched along $S\cup B_2$. The ramification locus E over S can be blown down smoothly along the direction $E\simeq P\to C$. Let M be the manifold obtained by this blowing-down. The line bundle $L=H_\beta+E$ comes from $\mathrm{Pic}(M)$. Clearly we have $f\colon M\to C$ and (f, M, L, C) is a Del Pezzo fibration of degree one.

Various invariants of (f, M, L, C) can be calculated by the above description. For example, we have $K(P/C)=-nH_\beta+\det\mathcal{E}$, $K(W/P)=-2H_\alpha+(2H_\beta+2A_W)$, $K(M^\natural/W)=2H_\alpha-H_\beta-3A$, $K(M^\natural/M)=(n-1)E$ and hence $K(M/C)+(n-1)L=f^*(\det\mathcal{E}-A)$, where $K(X/Y)$ denotes the relative canonical bundle of $X\to Y$.

The singular fibers of f appear exactly where the mapping $B_2\to C$ is not smooth. Moreover, every fiber has only isolated singularities (hence normal). Indeed, otherwise, there is a curve Y on B_2 contained in a fiber F of $W\to C$ such that B_2 and F are tangent along Y. The tangent spaces of B_2 and F at each point y on Y are the same viewed as subspaces of the tangent space of W. Hence their quotient spaces are the same. So $[B_2]_Y\simeq[F]_Y$. But $[B_2]_Y$ is an ample line bundle while $[F]_Y$ is trivial. This contradiction proves the assertion.

Finally we remark that every fiber of f is a weighted hypersurface of degree 6 in the weighted projective space $\boldsymbol{P}(3, 2, 1, \cdots, 1)$.

(4.2) Here we study the case $d=2$. Every fiber of f is irreducible and reduced by the results in § 2. Set $\mathcal{L}=\mathcal{O}_M(L)$, $\mathcal{E}=f_*\mathcal{L}$. Then $f^*\mathcal{E}\to\mathcal{L}$ is surjective by (1.5.2). So we have a morphism $\rho\colon M\to\boldsymbol{P}=\boldsymbol{P}(\mathcal{E})$ such that $\rho^*H=L$, where $H=\mathcal{O}_P(1)$. $\mathcal{E}$ is locally free of rank $n+1$ and $\boldsymbol{P}$ is a $\boldsymbol{P}^n$-bundle over C. ρ is a finite double covering and hence its branch locus B is smooth. B is a member of $|4H-2A_P|$ for some $A\in\mathrm{Pic}(C)$ and $K(M/C)+(n-1)L=f^*(\det\mathcal{E}-A)$. Singular fibers of f appear exactly where the mapping $B\to C$ is not smooth and every fiber has only isolated singularities. It is a weighted hypersurface of degree 4 in $\boldsymbol{P}(2, 1, \cdots, 1)$.

The proofs are easier than in (4.1).

(4.3) The case $d=3$. Every fiber of f is irreducible and reduced by the result in § 2. Moreover L is relatively very ample by (1.5). Hence we have an embedding $\rho\colon M\to\boldsymbol{P}=\boldsymbol{P}(\mathcal{E})$ for $\mathcal{E}=f_*\mathcal{O}_M(L)$ such that $\rho^*H=L$ for $H=\mathcal{O}_P(1)$. M is a smooth member of $|3H-A_P|$ for some $A\in\mathrm{Pic}(C)$ and $K(M/C)+(n-1)L=f^*(\det\mathcal{E}-A)$. Every fiber of f has only isolated singularities.

(4.4) The case $d=4$. Every fiber $M_x=f^{-1}(x)$ over $x\in C$ is irreducible and reduced. We have an embedding $\rho\colon M\to\boldsymbol{P}$ such that $\rho^*H=L$, where $(\boldsymbol{P}, H)$ is the scroll of $\mathcal{E}=f_*\mathcal{O}_M(L)$ over C. M_x is a complete intersection of two hyperquadrics in $\boldsymbol{P}_x\simeq\boldsymbol{P}^{n+2}$. Let $\mathcal{J}$ be the ideal defining M in P. The exact sequence $0 \to \mathcal{J}[2H] \to \mathcal{O}_P[2H] \to \mathcal{O}_M[2L] \to 0$ yields an exact sequence $0 \to \mathcal{R} \to S^2\mathcal{E} \to f_*\mathcal{O}_M[2L] \to 0$ on C, where $\mathcal{R}$ is a locally free sheaf of rank two. $\mathcal{C} = \mathcal{J}/\mathcal{J}^2$ is the conormal sheaf of M in P and $\mathcal{R} = f_*(\mathcal{C}[2L])$. Hence $\mathcal{C} \simeq f^*\mathcal{R}[-2L]$ and $K(M/P) = 4L - f^*A$ for $A = \det \mathcal{R}$. So $K(M/C) + (n-1)L = f^*(\det \mathcal{E} - A)$.

We claim that every fiber M_x has only isolated singularities. To see this, take a small neighborhood U of x in C such that $\mathcal{R}_U$ is free. A free basis of $\mathcal{R}_U$ gives two divisors D_1, D_2 on P_U such that $M_U = D_1 \cap D_2$. D_j's are members of $|2H_U|$. Since M is smooth we may assume that D_j's are smooth, by shrinking U and by choosing a suitable basis if necessary. We will derive a contradiction assuming that $\mathrm{Sing}(M_x)$ contains a curve Y. At every point y on Y, the tangent space of M is contained in the tangent space of the fiber P_x of $P \to C$. So we have a surjection $\mathcal{N}(M, P)_Y \to \mathcal{N}(P_x, P)_Y$ where $\mathcal{N}(X, P)$ is the normal bundle of X in P. $\mathcal{N}(M, P)_Y \simeq [D_1]_Y \oplus [D_2]_Y \simeq 2H_Y \oplus 2H_Y$ is ample, while $\mathcal{N}(P_x, P)_Y \simeq \mathcal{O}_Y$. This yields a contradiction, as desired.

(4.5) The case $d=5$. Every fiber is irreducible and reduced, and furthermore normal by (3.1). L is relatively very ample. Any smooth fiber is a linear section of the Grassmann variety $\mathrm{Gr}(5, 2)$ parametrizing $\boldsymbol{C}^2$'s in $\boldsymbol{C}^5$ embedded by Plücker coordinates (cf. [**F4**-II]). In particular $n \leqq 6$. Any singular fiber V is of one of the types classified in [**F6**]. Moreover we claim that it has only isolated singularities. In view of [**F6**; (2.9)], it suffices to rule out the cases (si21i), (si21u), (si111i), (si111u).

In case (si111u), we have $n=5$ and V is embedded in $P \simeq \boldsymbol{P}^8_\alpha$ by $|L_V|$. Let $v \in \mathrm{Sing}(V)$ and $W = v^*V$ as in (3.2). W is a generalized cone over the scroll $\Sigma(1, 1, 1)$ with $R = \mathrm{Ridge}(W) \simeq \boldsymbol{P}^2_\alpha$ in case (si111u). Let $\tilde{W}$ and $\tilde{V}$ be the proper transforms of W and V in the blowing-up of P along R. Since the scroll $\Sigma(1, 1, 1)$ is isomorphic to $(\boldsymbol{P}^2_\tau \times \boldsymbol{P}^1_\eta, H_\tau + H_\eta)$, $(\tilde{W}, H_\alpha)$ is isomorphic to the scroll of the vector bundle $[H_\tau + H_\eta] \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ over $B = \boldsymbol{P}^2_\tau \times \boldsymbol{P}^1_\eta$, where H_α is the pull-back of $\mathcal{O}_W(1)$. The exceptional divisor of $\tilde{W} \to W$ is the unique member D of $|H_\alpha - H_\tau - H_\eta|$ and $D \simeq B \times R$. $\tilde{V}$ is a member of $|H_\alpha + H_\tau|$ on $\tilde{W}$ and $\tilde{V}_D = \tilde{V} \cap D$ is defined by $\alpha_0\tau_0 + \alpha_1\tau_1 + \alpha_2\tau_2 = 0$ in D, where $(\alpha_0 : \alpha_1 : \alpha_2)$ and $(\tau_0 : \tau_1 : \tau_2)$ are homogeneous coordinates of R and $\boldsymbol{P}^2_\tau$. Thus V has double points along R.

Suppose that such a variety V is a fiber of f. Let S be the submanifold of M corresponding to R. Then $\det \mathcal{E} = -H_\alpha$ for the conormal bundle $\mathcal{E}$ of S in M, since $K_S = (1-n)L_S = -4H_\alpha$ and $S \simeq \boldsymbol{P}^2_\alpha$. Let $\tilde{M}$ be the blowing-up of M along S, let E be the exceptional divisor over S and let $\tilde{V}_M$ be the proper transform of V on $\tilde{M}$. Then $(E, [-E])$ is isomorphic to the scroll of $\mathcal{E}$ over S and $\tilde{V}_M$ is a member of $|V_{\tilde{M}} - 2E|$. So $\tilde{V}_E = \tilde{V}_M \cap E$ is a member of $|2H(\mathcal{E})|$ on $E \simeq \boldsymbol{P}(\mathcal{E})$. We have an exact sequence $0 \to \mathcal{O}_E[-H(\mathcal{E})] \to \mathcal{O}_E[H(\mathcal{E})] \to \mathcal{O}_{\tilde{V}_E}[H(\mathcal{E})] \to 0$. This yields $\mathcal{E} \simeq \pi_*\mathcal{O}_E[H(\mathcal{E})] \simeq \pi_*\mathcal{O}_{\tilde{V}_E}[H(\mathcal{E})]$, where π is the map $E \to S$.

Now we use the natural isomorphisms $(\tilde{V}_D \subset \tilde{V}) \simeq (\tilde{V}_E \subset \tilde{V}_M)$ and $(\tilde{V}_D \to R) \simeq (\tilde{V}_E \to S)$. Since $[H(\mathcal{E})] = [-E]$ in $\mathrm{Pic}(\tilde{V}_E)$ is the conormal bundle of $\tilde{V}_E$ in $\tilde{V}_M$, we infer $\mathcal{E} \simeq \phi_* \mathcal{O}_{\tilde{V}_D}[-D] = \phi_* \mathcal{O}_{\tilde{V}_D}[H_\tau + H_\eta - H_\alpha]$, where ϕ is the map $D \to R$. Using the exact sequence $0 \to \mathcal{O}_D[-H_\tau - H_\alpha] \to \mathcal{O}_D \to \mathcal{O}_{\tilde{V}_D} \to 0$, we get an exact sequence $0 \to H^0(B, H_\eta) \otimes \mathcal{O}(-2) \to H^0(B, H_\tau + H_\eta) \otimes \mathcal{O}(-1) \to \mathcal{E} \to 0$ on $R \simeq S$. Since $h^0(B, H_\eta) = 2$ and $h^0(B, H_\tau + H_\eta) = 6$, this implies $c_1(\mathcal{E}) = -2H_\alpha$, contradicting the preceding observation. The case (si111u) is thus ruled out.

In case (si111i), the situation is similar as above and we get a contradiction by the same method. Here we have $n=4$, $V \subset \boldsymbol{P}^7_\alpha$, $R \simeq \boldsymbol{P}^1_\alpha$ and $(\tilde{W}, H_\alpha)$ is the scroll of $[H_\tau + H_\eta] \oplus \mathcal{O} \oplus \mathcal{O}$ over B. Moreover $\det \mathcal{E} = -H_\alpha$ for the conormal bundle $\mathcal{E}$ of $S \simeq \boldsymbol{P}^1_\alpha$ in M since $K_S = (1-n)L_S = -3H_\alpha$. The rest of the argument is completely the same as above.

In case (si21u), $n=4$ and $B \simeq \Sigma_1 = \{(\tau_0 : \tau_1 : \tau_2), (\eta_0 : \eta_1) \in \boldsymbol{P}^2_\tau \times \boldsymbol{P}^1_\eta \mid \tau_0 \eta_1 = \tau_1 \eta_0\}$. $(\tilde{W}, H_\alpha)$ is the scroll of $[H_\tau + H_\eta] \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ over B. $R = \mathrm{Ridge}(W) \simeq \boldsymbol{P}^2_\alpha$ and $D \simeq B \times R$. $\tilde{V}$ is a member of $|H_\alpha + H_\tau|$ on $\tilde{W}$ and $\tilde{V}_D = \tilde{V} \cap D$ is defined by $\alpha_0 \tau_0 + \alpha_1 \tau_1 + \alpha_2 \tau_2 = 0$ in D. Let S be the submanifold of M corresponding to R. Then $\det \mathcal{E} = 0$ for the conormal bundle $\mathcal{E}$ of S in M, since $K_S = -3L_S = -3H_\alpha$. On the other hand, we get an exact sequence $0 \to H^0(B, H_\eta) \otimes \mathcal{O}(-2) \to H^0(B, H_\tau + H_\eta) \otimes \mathcal{O}(-1) \to \mathcal{E} \to 0$ by the same argument as before. This yields a contradiction since $h^0(B, H_\eta) = 2$ and $h^0(B, H_\tau + H_\eta) = 5$.

The case (si21i) is ruled out similarly. This time we have $n=3$, $R \simeq S \simeq \boldsymbol{P}^1_\alpha$ and $\det \mathcal{E} = 0$, but the rest are the same as in the case (si21u).

Thus we complete the proof of the claim. In view of [**F6**; (2.9)], we obtain the following

(4.5.1) **Corollary.** *When $d=5$ and $n \geq 4$, every fiber of f is smooth.*

(4.5.2) REMARK. When $d=5$ and $n \leqq 3$, singular fibers can really appear. An example is obtained by taking a Lefschetz pencil of a Del Pezzo manifold of degree five.

(4.6) The case $d=6$. Every fiber of f is irreducible and reduced, but possibly non-normal (cf. § 3). L is relatively very ample and $n \leqq 4$.

When $n=4$, any singular fiber is normal and of the type (vu) in [**F6**; (2.9)]. It is uncertain whether such a singular fiber exists or not.

(4.7) The case $d=7$. Every fiber of f is irreducible reduced and normal. Smooth fibers are classified in [**F4**] and we have $n \leqq 3$.

When $n=3$, every fiber is smooth by [**F6**; (2.9)] and is isomorphic to the blowing-up of $\boldsymbol{P}^2$ at a point. Hence M is obtained from a $\boldsymbol{P}^3$-bundle over C by blowing-up along a section.

(4.8) The case $d=8$. We have $n \leqq 3$ and reducible fibers are classified in the table (2.20).

When $n=3$, every irreducible fiber is smooth by (3.1) and [**F6**; (2.9)], hence isomorphic to $\boldsymbol{P}^3$.

When $n=2$, every irreducible singular fiber is a singular quadric. In particular, every irreducible fiber is smooth if a general fiber is isomorphic to Σ_1. In this case M is the blowing-up of a $\boldsymbol{P}^2$-bundle along a section, off the fibers of the type (2.19.3c).

(4.9) The case $d\geqq 9$. We have $d=9$ and $n=2$. Reducible fibers are classified in § 2 (cf. table (2.20)). Every irreducible fiber is smooth as before, hence isomorphic to $\boldsymbol{P}^2$. Thus, M is obtained by blowing up a $\boldsymbol{P}^2$-bundle along curves in fibers as in (2.12), (2.19.1), (2.19.4).

(4.10) By the preceding observations altogether, we see that every irreducible fiber has only isolated singularities unless $d=6$. It would be nice if we have a conceptual proof of this fact which does not need so precise a classification as in [**F6**].

(4.11) Now we study non-minimal Del Pezzo fibrations. As we see in § 1, they are obtained from the minimal reductions by blowing up points. In such a case the minimal reduction $(M^\flat, L^\flat)$ has some additional properties.

First of all $d>1$ since L is f-ample and there exists a reducible fiber. Moreover, for any point p at which the reduction $M^\flat$ is blown up, there is no curve Y in $M^\flat$ such that $Y\ni p$, $L^\flat Y=1$ and $f^\flat(Y)$ is a point.

To see this, let M' be the blowing-up of $M^\flat$ at p and let E be the exceptional divisor over p. Then $L^\flat-E$ is relatively ample on M', since (M', L') is an intermediate step of the reduction. Let Y' be the proper transform of Y on M'. Then $0<L'Y'=(L^\flat-E)Y'<L^\flat Y=1$, since $Y\ni p$. Thus we get a contradiction.

This is a very strong condition when $n\geqq 3$. In fact, any general fiber (V, L_V) must be isomorphic to $(\boldsymbol{P}^3, \mathcal{O}(2))$. Indeed, otherwise, we find a line Y in V passing p for any point p on V, by virtue of the classification theory of Del Pezzo manifolds in [**F4**]. Using relative Hilbert schemes we infer that there is a curve Y as above even if p is in a singular fiber. This is ruled out by the preceding observation.

REMARK. When $n=2$, there are many examples where $M\neq M^\flat$.

(4.12) REMARK. A quadruple (f, M, C, L) is called a *hyperquadric fibration* if a general fiber (M_x, L_x) is not a Del Pezzo manifold but a hyperquadric in $\boldsymbol{P}^{n+1}$ instead. In this case M is embedded in the scroll (P, H) of $\mathcal{E}=f_*\mathcal{O}_M(L)$ over C as a member of $|2H-A_P|$ for some $A\in\mathrm{Pic}(C)$ and $L=H_M$. Moreover every fiber has only isolated singularities. In particular it is irreducible if $n\geqq 2$.

These facts should be easily proved by those who have read through this paper.

References

[D] H. D'Souza: *Threefolds whose hyperplane sections are elliptic surfaces*, Pacific J. Math. **134** (1988), 57–78.

[F1] T. Fujita: *On the structure of polarized varieties with Δ-genera zero*, J. Fac, Sci. Univ. Tokyo **22** (1975), 103–115.

[F2] T. Fujita: *Defining equations for certain types of polarized varieties*, in Complex Analysis and Algebraic Geometry, Iwanami, 1977, 165–173.

[F3] T. Fujita: *On the hyperplane section principle of Lefschetz*, J. Math. Soc. Japan **32** (1980), 153–169.

[F4] T. Fujita: *On the structure of polarized manifolds with total deficiency one, part I, II, III*, J. Math. Soc. Japan **32** (1980), 709–725, & **33** (1981), 415–434 & **36** (1984), 75–89.

[F5] T. Fujita: *On polarized varieties of small Δ-genera*, Tôhoku Math. J. **34** (1982), 319–341.

[F6] T. Fujita: *Projective varieties of Δ-genus one*, in Algebraic and Topological Theories— to the memory of Dr. Takehiko MIYATA, Kinokuniya, 1985, 149–175.

[F7] T. Fujita: *On polarized manifolds whose adjoint bundles are not semipositive*, in Algebraic Geometry Sendai 1985, Advanced Studies in Pure Math. 10, Kinokuniya, 1987, 167–178.

[F8] T. Fujita: *On singular Del Pezzo varieties*, in Algebraic Geometry, Proceedings L'Aquila 1988, Lecture Notes in Math. 1417, Springer, 1990, 117–128.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki: *Introduction to the minimal model problem*, in Algebraic Geometry Sendai 1985, Advanced Studies in Pure Math. 10, Kinokuniya, 1987, 283–360.

[Ko] K. Kodaira: *On compact analytic surfaces II*, Ann. Math. **77** (1963), 563–626.

[M] S. Mori: *Threefolds whose canonical bundles are not numerically effective*, Ann. Math. **116** (1982), 133–176.

Department of Mathematics
College of Arts & Science
University of Tokyo
Komaba, Meguro, Tokyo
153 Japan

Present address:
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro, Tokyo
152 Japan