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A DISCRIMINANT CRITERION FOR THE TWO DIMENSIONAL JACOBIAN PROBLEM

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1. Let $f(X, Y)$ and $g(X, Y)$ be any two polynomials with complex coefficients, i.e., $f, g\in \boldsymbol{C}[X, Y]$. The pair (f, g) is called an *automorphic pair* if there exist $u, v \in \boldsymbol{C}[X, Y]$, such that,

$$X = u(f(X, Y), g(X, Y))$$

and

$$Y = v(f(X, Y), g(X, Y))$$

The pair (f, g) is called a *Jacobian pair* if the determinant of the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial f}{\partial X} & \dfrac{\partial f}{\partial Y} \\ \dfrac{\partial g}{\partial X} & \dfrac{\partial g}{\partial Y} \end{pmatrix}$$

of (f, g) with respect to the variables X and Y, is a nonzero element of $\boldsymbol{C}$. It is easily seen that every automorphic pair is a Jacobian pair. The Jacobian problem is to determine whether every Jacobian pair is an automorphic pair or not.

2. Various equivalent formluations of this problem are known. We shall recall some of these results, relevant to our discussion, from [1].

A polynomial $f\in \boldsymbol{C}[X, Y]$ is said to have r *points at infinity*, if its homogeneous component of maximal degree (i.e., the degree form) is a product of r coprime factors. If $F(X, Y, Z)$ is the homogenization of $f(X, Y)$, and $\mathcal{E}:=\{F(X, Y, Z)=0\}$ is the curve in $\boldsymbol{P}^2$ then, $\mathcal{E}$ intersects the line at infinity, $L:=\{Z=0\}$ in precisely r distinct points. The total number of local branches of $\mathcal{E}$ at all of these r points taken together is called the *number of places of f at infinity*. Note that the number of points at infinity is not an automorphic invariant, whereas, the number of places at infinity of a nonconstant polynomial

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is an automorphic invariant. We recall a portion of the result proved in Theorem 19.4 of [1], in the following proposition.

Proposition: *Let (f, g) be a Jacobian pair. Then the following three statements are equivalent:*
i) (f, g) is an automorphic pair.
ii) f has only one point at infinity.
iii) f has only one place at infinity.

REMARK: As stated and proved, 19.4 of [1] contains only the equivalence of *i*) and *ii*). But the implications *i*)$\Rightarrow$*iii*)$\Rightarrow$*ii*) are obvious.

3. The following notion is introduced purely for technical convenience.

A polynomial $f(X, Y)$ is said to be in *ready form with respect to* X, if the following conditions hold:
a) $f(0, 0)=0$
b) the degree form of f is equal to X^n for some $n\geq 2$ and
c) f has no multiple factors.

Apart from the results from [1], quoted above, the only nontrivial deep result that we use is the theory of equisingularity of plane algebroid curves, from [2]. Given a polynomial f in the ready form with respect to X, we let $f_\lambda=f+\lambda$, $\lambda\in\boldsymbol{C}$; $F_\lambda(X, Y, Z)$ be the homogenization of f_λ and let $\varphi_\lambda(X, Z)=F_\lambda(X, 1, Z)$. Let $\mathcal{C}_\lambda:=\{f_\lambda=0\}$, $\mathcal{D}_\lambda:=\{\varphi_\lambda=0\}$ and $\bar{\mathcal{C}}_\lambda:=\{F_\lambda=0\}$. Then, $\mathcal{D}_\lambda\subset\bar{\mathcal{C}}_\lambda$, and $\bar{\mathcal{C}}_\lambda=\mathcal{C}_\lambda\cup\{P\}$, where $P:=[0, 1, 0]$ in $\boldsymbol{P}^2$. The discriminant, $\Delta_X(f_\lambda)$ with respect to X, of f_λ is a polynomial in Y and λ. From condition a) and b) above, it follows that, $\varphi_\lambda(0, 0)=0$ and φ_λ is regular in X, (i.e, by definition, $\varphi_\lambda(X, 0)=tX^n$, for some nonzero constant t; see, for instance, p. 145 of [3]). Condition c) implies that the X-discriminant of f is not identically zero. Hence, the X-discriminant of φ_0 is not identically zero. This means that the X-discriminant of φ_λ is not identically zero, considered as a polynomial in Z and λ. Thus we are in a situation as described in §6 of [2]. We now state our main result:

Theorem: *Let (f, g) be a Jacobian pair with f in the ready form with respect to X. Then each statement in the Proposition is equivalent to each of the following two statements:*
iv) $\mathcal{D}_\lambda:=\{\varphi_\lambda=0\}$ is an equisingular family of plane algebroid curves at $P:=\{X=0=Z\}$
v) The Y-degree of $\Delta_X(f_\lambda)$ is independent of λ.

The following two sections contain the proof of this theorem. In §4, we shall use a geometric argument to show the equivalence of iii) and iv). In §5, we shall use the results from [2] to show the equivalence of iv) and v). We shall use the notations introduced in this section, in the rest of the paper.

4. Proof of *iii)* $\Leftrightarrow$ *iv)*

iii) $\Rightarrow$ *iv)*:

Let (f, g) be a Jacobian pair with f in the ready form with respect to X. By iii), f_λ has only one place at infinity, i.e., the curve $\mathscr{D}_\lambda$ has only one branch at P. Let $m_1 \geq m_2 \geq \cdots \geq m_k$, $(m_k \geq 2)$, be the multiplicity sequence of the generic member of this family of curves and let $m_1' \geq m_2' \geq \cdots \geq m_l'$, $(m_l' \geq 2)$, be the multiplicity sequence of some specialization. By the semicontinuity property, it follows that, $m_i' \geq m_i$ for each i, and hence in particular, $l \geq k$. By the equivalence of i) and iii), it follows that, each C_λ is isomorphic to $\boldsymbol{C}$.

Note that, each $\bar{C}_\lambda$ represents the same element in the divisor class group of $\boldsymbol{P}^2$. Hence, they all have the same arithmetic genus: $p_a(\bar{C}_\lambda) = a$, say. Since, the only possible singular point of $\bar{C}_\lambda$, is the point at infinity P, it follows that,

$$2p_a = \sum_{i=1}^{k} m_i(m_i - 1) = \sum_{i=1}^{l} m_i'(m_i' - 1)\,.$$

Since, the summands in each sum are positive, it follows that, $m_i = m_i'$ for each i and $k = l$. This implies that the family $\mathscr{D}_\lambda$ is equisingular.

iv) $\Rightarrow$ *iii)*;

Assuming iv), given a Jacobian pair (f, g), we want to show that, f has only one place at infinity. So, we are at liberty to change this pair by pre-composing with an automorphic pair. Note that if f is of degree one, then there is nothing to prove. Hence, we assume that, $\deg f \geq 2$. By replacing Y with $Y + tX^2$, for a suitable $t \in \boldsymbol{C}$, if necessary, we can assume that, f satisfies b) of §3. By substracting $f(0, 0)$ from f, we can also assume that, $f(0, 0) = 0$. Finally, condition c) of §3 is satisfied by any f such that (f, g) is a Jacobian pair. Thus f is now in the ready form with respect to X. Statement iv) is now applicable, i.e., the family of curves $\{\mathscr{D}_\lambda\}$ is equisingular.

We shall first show that, each fibre C_λ of the map $f: \boldsymbol{C}^2 \to \boldsymbol{C}$ is irreducible. Note that, $\bar{C}_\lambda = C_\lambda \cup \{P\}$. The Jacobian condition implies that, C_λ is smooth. Hence, by successively blowing-up at P and its pre-images, all the $\bar{C}_\lambda$'s get resolved, simultaneously, because of the equisingularity. Indeed, there exists a morphism $\pi: S \to \boldsymbol{P}^2$, on a smooth surface S, such that,

1) π is a composite of contraction of (-1)-curves,
2) if $\mathscr{F}_\lambda$ is the proper trasnform of $\bar{C}_\lambda$ in S, then each $\mathscr{F}_\lambda$ is smooth and
3) there exists a finite set $R \subset S$ (possibly empty), such that, each $\mathscr{F}_\lambda$ passes through each point of R.

By equsingularity, it follows that, each $\mathscr{F}_\lambda$ represents the same element of the divisor class group of S. Now, by further blowing-up at points of R, and their pre-images, we obtain a smooth surface T and a map, $\psi: T \to \boldsymbol{P}^1$, which extends, the map $f\colon \boldsymbol{C}^2 \to \boldsymbol{C}$. The fibres $\mathcal{E}_\lambda$ of ψ are connected, and generically smooth. Let $\mathcal{C}'_\lambda$ denote the proper transform of $\mathcal{C}_\lambda$ on T. Then, $\mathcal{C}'_\lambda \subseteq \mathcal{E}_\lambda$ and generically the equality holds. Again, each $\mathcal{C}'_\lambda$ represents the same element of the divisor class group of T. So do each $\mathcal{E}_\lambda$. Hence, from the generic equality, it follows that, $\mathcal{C}'_\lambda = \mathcal{E}_\lambda$ for each λ. In particular, each $\mathcal{C}'_\lambda$ is connected. Being smooth it is irreducible. Hence, each $\mathcal{C}_\lambda$ is irreducible.

Now the arithmetic genus, $p_a(\bar{\mathcal{C}}_\lambda) = a$, say. By equisingularity, it follows that, the geometric genus, $g(\bar{\mathcal{C}}_\lambda) = \gamma$, is the same for all λ and the number of places at infinity, $\sigma(\bar{\mathcal{C}}_\lambda) = \sigma$ say, is also the same for all λ. So, it follows that the topological Euler characteristic

$$\chi(\mathcal{C}_\lambda) = 2 - 2\gamma - \sigma$$

is the same for all λ. Hence, we have the product formula:

$$\chi(\boldsymbol{C}^2) = \chi(\mathcal{C}_\lambda)\chi(\boldsymbol{C})$$

which implies that, $\chi(\mathcal{C}_\lambda) = 1$. This in turn yields that, $g(\mathcal{C}_\lambda) = 0$ and $\sigma(\mathcal{C}_\lambda) = 1$. Thus, we have shown that, f has one place at infinity, as claimed.

5. Proof of iv)$\Leftrightarrow$v)

v)$\Rightarrow$iv)

As indicated before, we appeal to theorem 7a) of [2] to conclude that $\{\varphi_\lambda = 0\}$ defines an equisingular family of plane algebroid curves at P, if the X-discriminant $\Delta_X(\varphi_\lambda)$ is of the form:

$$\Delta_X(\varphi_\lambda) = \varepsilon(Z, \lambda) Z^N$$

for some unit, $\varepsilon(Z, \lambda)$ in the power series ring $\boldsymbol{C}[[Z, \lambda]]$. Writing

$$\varepsilon(Z, \lambda) = \theta_0 + \theta_1 Z + \cdots + \theta_k Z^k$$

with $\theta_i \in \boldsymbol{C}[\lambda]$, and $\theta_0 \neq 0$, the above condition is the same as saying that θ_0 is a nonzero constant.

Now the discriminant $\Delta_X(F_\lambda)$ of the homogeneous polynomial F_λ is a homogeneous polynomial in Y and Z and we have,

$$\Delta_X(F_\lambda)(1, Z) := \Delta_X(F_\lambda(X, 1, Z)) := \Delta_X(\varphi_\lambda),$$

and

$$\Delta_X(F_\lambda)(Y, 1) := \Delta_X(F_\lambda(X, Y, 1) := \Delta_X(f_\lambda)$$

Hence, it follows that, $\Delta_X(f_\lambda)$ has the form:

$$\Delta_X(f_\lambda) = Y^M(\theta_0 Y^k + \theta_1 Y^{k-1} + \cdots + \theta_k).$$

Thus, the statement v) implies that, θ_0 is a nonzero constant, which in turn, as observed above, implies the statement iv).
iv)$\Rightarrow$*v*):
Here, we need to appeal to the proof of theorem 7 b) in [2], rather than the theorem itself. For in our context, the condition that the line $Z=0$ is not tangential to the curves $\mathcal{D}_\lambda$, is not satisfied. However, the proof of theorem 7 b) of [2] uses only Lemma 6 of [2] and a corollary to it. In this lemma, the required condition is that the corresponding local branches of $\mathcal{D}_\lambda$ under the equivalence, should have the same intersection number with the line L given by $Z=0$. Since, we have already shown that, iv) is equivalent to iii), it follows that, each $\mathcal{D}_\lambda$ has a unique branch at P. Thus, we need to verify that each $\mathcal{D}_\lambda$ has the same intersection number with $Z=0$. But $\varphi_\lambda(X, 0)=X^n$, and hence, the intersection number, $(\mathcal{D}_\lambda, L)=n$ for each λ and hence, we are done.
This completes the proof of the theorem.

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