CODEGREE OF SIMPLE LIE GROUPS

Dedicated to Professor Shôrô Araki on his 60th birthday

HIDEAKI ŌSHIMA

(Received December 13, 1988)

1. Introduction

Let $G$ be a connected Lie group such that $\pi_3(G) \approx \mathbb{Z}$. Connected simple Lie groups satisfy the condition. Let $f: S^3 \to G$ be a map generating $\pi_3(G)$. The purpose of this note is to study the image of $f^*: \pi_3^s(G) \to \pi_3^s(S^3) \approx \mathbb{Z}$ where $\pi_3^s(\cdot)$ is the reduced stable cohomotopy. The order of the cokernel of $f^*$ which is finite by Proposition 2.5 below is called the codegree of $G$ and denoted by $\text{cd}(G)$. Since $G$ and its maximal compact subgroup have the same homotopy type, it suffices to study the case that $G$ is compact. We denote by $\text{cd}_p(G)$ the exponent of the prime number $p$ in the prime decomposition of $\text{cd}(G)$. Our result is

**Theorem.** Let $G$ be a compact simply connected simple Lie group. Then

1. $\text{cd}_p(G) = 0$ if and only if $G$ is $p$-regular in the sense of Serre [22];
2. $\text{cd}_p(G)$ can be determined except for the following cases:
   
   - $E_6, E_7, E_8, F_4, \text{Spin}(n)$ ($n \geq 9$) for $p = 2$;
   - $E_6, E_7, E_8, F_4$ for $p = 3^*$;
   - $E_8$ for $p = 5$.

More precise statement and upper bounds of the excluded cases in (2) can be seen in 3.3, 4.1, and 4.2 below. For non simply connected groups, see 2.7 below.

In §2, we prove general properties of codegrees, though some of them are not used in this note. In §3, for classical groups $G$, we prove that $\text{cd}(G)$ is equal to the codegree of some canonical vector bundle. As a consequence, $\text{cd}(SU(n))$ can be determined from Crabb-Knapp [4] and then $\text{cd}(Sp(n))$ and odd components of $\text{cd}(\text{Spin}(n))$ can also be determined (see Proposition 3.3).

*) A. Kono informed me that we could prove $\text{cd}_3(E_6) = \text{cd}_3(F_4) = 2$ by using Harper’s mod 3 decomposition of $F_4$. 
In §4, we study relations of \( cd(G) \) and \( p \)-regularity [13, 22], quasi \( p \)-regularity [15], and mod \( p \) decomposability [17] of \( G \).

In this note all cohomology theories are reduced.

I am indebted to H. Minami for useful conversation and suggestions on Lemma 4.4; to A. Kono for informing me that any map \( E_{q} \to U(240) \) factorizes through \( U(124) \); and to M. Mimura for information on the topology of \( E_{6} \) and \( F_{4} \).

2. Codegree of a map

Let \( f: S^{*} \to X \) be a based map (or more generally stable map), where \( X \) is a connected finite \( CW \)-complex having a base point. Let \( E \) be a generalized reduced cohomology theory, that is, \( E \) is a spectrum, and let \( m \in \mathbb{Z} \). The \( E^{m} \)-codegree of \( f, cd(f; E^{m}) \), is defined to be 0 or the cardinal number of the cokernel of

\[
E^{m+n}(X) \xrightarrow{f^{*}} E^{m+n}(S^{*}) \to E^{m+n}(S^{*})/\text{Tor} \cong E^{m}(S^{0})/\text{Tor}
\]

according as \( qf^{*} \) is 0 or not, where \( \text{Tor} \) denotes the torsion subgroup and \( q \) is the quotient homomorphism. We set \( cd(f) = cd(f; \pi_{i}^{0}) \) which is most interesting, because, by 2.3(2) below, \( cd(f; E^{m}) \mid cd(f) \) provided there exists a natural transformation \( \pi_{i}^{0} \to E^{*+m} \) inducing an isomorphism \( \pi_{i}^{0}(S^{0}) \cong E^{m}(S^{0})/\text{Tor} \). Here \( a \mid b \) means that an integer \( b \) is an integral multiple of an integer \( a \).

Remark 2.1. Our codegree of a map \( f: S^{*} \to X \) is a multiple of Gottlieb's codegree of \( f_{*}: H_{*}(S^{*}; \mathbb{Z}) \to H_{*}(X; \mathbb{Z}) \) and degree of \( f^{*}: H^{*}(X; \mathbb{Z}) \to H^{*}(S^{*}; \mathbb{Z}) \) [7].

Proposition 2.2 ([21; 2.1]). For any integer \( k \) we have

1. If \( f^{*} = 0: H^{*}(X; \mathbb{Z}) \to H^{*}(S^{*}; \mathbb{Z}) \), then the image of \( f^{*}: E^{*}(X) \to E^{*}(S^{*}) \) is contained in \( \text{Tor}(E^{*}(S^{0})) \), the torsion subgroup of \( E^{*}(S^{*}) \).

2. If \( E^{*}(S^{*}) \otimes \mathbb{Q} \neq 0 \), then the converse of (1) holds.

Proposition 2.3. (1) \( cd(f; E^{m}) = cd(f; E[0, \infty]); \) if \( m \leq n \) and \( X \) is \( (m-1) \)-connected, where \( E[0, \infty) \to E \) is the \( (m-1) \)-connective covering.

2. If \( \theta: E^{*} \to F^{*+k} \) is a natural transformation of cohomology theories of degree \( k \) such that \( \theta_{*}: E^{m}(S^{0})/\text{Tor} \cong F^{m+k}(S^{0})/\text{Tor} \), then \( cd(f; F^{m+k}) \mid cd(f; E^{m}) \).

3. Suppose that there exist stable maps \( g: X \to X' \) and \( f': S^{*} \to X' \) such that \( g \circ f = f' \circ k \), where \( X' \) is a connected finite based \( CW \)-complex, \( \iota_{*}: S^{*} \to S^{*} \) is the identity map, and \( k \in \mathbb{Z} \). Then \( cd(f) \mid k \cdot cd(f') \). If \( k = \pm 1 \), then \( cd(f; E^{m}) \mid cd(f' ; E^{m}) \), in particular, \( cd(f; E^{m}) \mid cd(g \circ f; E^{m}) \).

4. Suppose that \( E^{m}(S^{0}) \otimes \mathbb{Q} \neq 0 \). Then \( cd(f; E^{m}) = 0 \) if and only if \( f^{*} = 0: H^{*}(X; \mathbb{Z}) \to H^{*}(S^{*}; \mathbb{Z}) \).

Proof. We have (4) by 2.2. Others can be easily proved. We omit the details.
We call $f: S^n \to X$ orientable if $f^*: H^n(X; \mathbb{Z})/\text{Tor} \cong H^n(S^n; \mathbb{Z})$, and call $f$ quasi-orientable if $f^*: H^n(X; \mathbb{Z})/\text{Tor} \to H^n(S^n; \mathbb{Z})$ is a non-trivial injection. In both cases an element $U(f) \in H^n(X; \mathbb{Z})$ which generates $H^n(X; \mathbb{Z})/\text{Tor}$ is called an orientation class of $f$.

**Example 2.4.** Let $V$ be an $n$-dimensional vector bundle over a connected finite $CW$-complex $Y$, and let $f: S^n = T(V; \{y\}) \to T(V; Y)$ be the inclusion of Thom spaces for some point $y$ of $Y$. Then $f$ is quasi-orientable if and only if $f$ is orientable if and only if $V$ is orientable. Put $cd(V) = cd(f)$. See [20].

**Proposition 2.5.** Let $G$ be a compact connected Lie group such that $\pi_3(G) \approx \mathbb{Z}$ and let $f: S^3 \to G$ be a generator of $\pi_3(G)$. Then $cd(G) \neq 0$. Also $f$ is quasi-orientable if and only if $\dim Z(G) \leq 2$, where $Z(G)$ is the centre of $G$. Moreover $f$ is orientable if $G$ is simply connected.

**Proof.** As is well-known, $G$ has a covering group $q: T^k \times G' \to G$ whose kernel is a central finite subgroup of $T^k \times G'$, where $G'$ is a compact simply connected simple Lie group, and $T^k = (S^1)^k$ is the $k$-dimensional toral group with $k = \dim Z(G)$. Let $i: G' = \{1\} \times G' \to T^k \times G'$ be the inclusion and let $f^*: S^3 \to G'$ be a map such that $q \circ i \circ f = f$. Then $q^*: H^*(G; \mathbb{Q}) \approx H^*(T^k \times G'; \mathbb{Q})$ and $f^*: H^*(G'; \mathbb{Z}) \approx H^*(S^3; \mathbb{Z})$. Hence $f^* \neq 0: H^3(G; \mathbb{Q}) \to H^3(S^3; \mathbb{Q})$, therefore $cd(G) \neq 0$ by 2.3 (4). Also we have

\[
\begin{align*}
 f \text{ is quasi-orientable} & \iff f^*: H^3(G; \mathbb{Q}) \approx H^3(S^3; \mathbb{Q}) \\
 & \iff i^*: H^3(T^k \times G'; \mathbb{Q}) \approx H^3(G'; \mathbb{Q}) \\
 & \iff k \leq 2.
\end{align*}
\]

The last assertion of 2.5 follows from the fact that if $G$ is simply connected then $G$ is 2-connected. This completes the proof.

**Proposition 2.6.** Let $g: X \to Y$ be a map between connected finite $CW$-complexes. Suppose that $H^n(Y; \mathbb{Q}) \approx \mathbb{Q}$ and $f: S^n \to X$ ($n \geq 1$) is quasi-orientable, and that there exists a stable map $\tau: Y_+ \to X_+ (Z_+ is the union of $Z$ and the disjoint base point) such that $\tau^* g^*: H^n(Y_+; \mathbb{Z}) \to H^n(Y_+; \mathbb{Z})$ is the multiplication by an integer $\chi$. Then $cd(g \circ f) | \chi \cdot cd(f)$.

**Proof.** Consider the following commutative diagram in which $h$ are the Hurewicz homomorphisms.

\[
\begin{array}{ccccccc}
\pi^*_n(S^n) & \cong & \pi^*_n(S^n) & \leftarrow & \pi^*_n(X_+) & \cong & \pi^*_n(Y_+) \\
\downarrow h & & \downarrow h & & \downarrow h & & \downarrow h \\
H^n(S^n; \mathbb{Z}) & \cong & H^n(S^n; \mathbb{Z}) & \leftarrow & H^n(X_+; \mathbb{Z}) & \cong & H^n(Y_+; \mathbb{Z})
\end{array}
\]
Let \( a \in H^\ast(X_+; \mathbb{Z}) \), \( b \in H^\ast(Y_+; \mathbb{Z}) \) be generators of respective free parts. There exist integers \( k, l, m \) such that

\[
\begin{align*}
    f^\ast(a) &= k[S^\ast], \\
    g^\ast(b) &\equiv la \mod \text{Tor}(H^\ast(X_+; \mathbb{Z})), \\
    \tau^\ast(a) &\equiv mb \mod \text{Tor}(H^\ast(Y_+; \mathbb{Z}))
\end{align*}
\]

where \([S^\ast]\) is a generator of \( H^\ast(S^\ast; \mathbb{Z}) \). We have \( \chi b = \tau^\ast g^\ast(b) \equiv lmb \mod \text{Tor}(H^\ast(Y_+; \mathbb{Z})) \), hence \( \chi = lm \). By definition, there exists \( \beta \in \pi_\ast(X_+) \) with \( f^\ast h(\beta) = h f^\ast(\beta) = cd(f) [S^\ast] \). Hence \( h(\beta) \equiv (cd(f)/k)a \mod \text{Tor}(H^\ast(X_+; \mathbb{Z})) \), and so \( h \tau^\ast(\beta) = \tau^\ast h(\beta) \equiv (m \cdot cd(f)/k)b \mod \text{Tor}(H^\ast(Y_+; \mathbb{Z})) \), and therefore \( hf^\ast g^\ast \tau^\ast(\beta) = f^\ast g^\ast h(\tau^\ast(\beta)) = f^\ast g^\ast((m \cdot cd(f)/k)b) = lm \cdot cd(f) [S^\ast] = \chi \cdot cd(f) [S^\ast] \). Thus \( cd(g \circ f) \mid \chi \cdot cd(f) \) as desired.

**Corollary 2.7.** Let \( q: G' \to G \) be a covering group such that \( G' \) and \( G \) are compact connected Lie groups, \( \pi_3(G) \approx \mathbb{Z} \), and \( \dim Z(G) \leq 2 \). Then

\[
cd(G') \mid cd(G) \mid m \cdot cd(G')
\]

where \( m \) is the order of the kernel of \( q \).

**Proof.** Note that \( m \) is finite and \( \dim Z(G') = \dim Z(G) \). By [2], there exists a transfer map \( \tau: G_+ \to G'_+ \) which is a stable map such that \( \tau^\ast q^\ast: H^\ast(G_+; \mathbb{Z}) \to H^\ast(G'_+; \mathbb{Z}) \) is the multiplication by \( m \). Then the result follows from 2.5 and 2.6.

We recall \( K \)-theory codegree from Crabb-Knapp [3]. Let \( KF \) denote the complex \( K \)-theory \( K(F=C) \) and the real \( K \)-theory \( KO(F=R) \). If \( f: S^n \to X \) is quasi-orientable, then we define

\[
\text{cd}^{KF}(f) = \text{Min} \{ m > 0; m \cdot U(f) \in \text{Im}(ch_F: KF^n(X) \to H^\ast(X; \mathbb{Q})) \}
\]

where \( ch_G = ch \), the Chern character, \( ch_R \) is the composition of the complexification and \( ch \), and \( \text{Min} \) and \( \text{Im} \) denote the minimum and the image, respectively. If \( f \) is non quasi-orientable, then we define \( cd^{KF}(f) = 0 \).

**Proposition 2.8** ([20; §2]). If \( f: S^n \to X \) is quasi-orientable, then \( \text{cd}^{KF}(f) \) is well-defined and satisfies

\[
\text{cd}(f; KF^0) \mid \text{cd}(f; H^0) \mid \text{cd}^{KF}(f) \quad \text{and} \quad \text{cd}^{KF}(f) \mid \text{cd}(f).
\]

Next we consider \( J \)-theory codegrees

\[
\begin{align*}
    cd^{JF}(f) &= \Pi_p cd(f; jF^p), \\
    cd^{JF}(f) &= \Pi_p cd(f; jF^p).
\end{align*}
\]

See [3], [4] or [20] for their definitions. We denote the exponents of \( p \) in the
prime decompositions of $cd(f)$, $cd^R(f)$, $cd^I(f)$, ..., by $cd_p(G)$, $cd^R_p(f)$, $cd^I_p(f)$, ..., respectively. We use also the notations $cd(G; E^n)$, $cd^R(G)$, $cd^I(G)$, ..., in canonical sense. Notice that the $p$-components of $cd^I_p(f)$ and $cd^R_p(f)$ are $cd(f; jF^p)$ and $cd(f; jF^p)$ respectively.

The following theorem can be proved by modifying the proof of Theorem 3.3 of [20]. So we omit the details.

**Theorem 2.9.** Suppose that $f: S^n \to X$ is quasi-orientable.

1. $cd^R(f) | cd^R_G(f) | 2cd^R(f)$.
2. $cd^I(f) | cd^I_G(f) | cd(f)$.
3. If $f$ is orientable, then $cd^R(f) | cd^I(f)$.
4. If $H^{n-1}(X; \mathbb{Z}_p) = 0$, then $cd^I_G(f) = cd^I_G(f)$.
5. $cd^I^G(f) | cd^I(f)$.
6. $p > 2$.
7. $p = 2$.

**Lemma 2.10.**

1. $A_\Gamma A_\Sigma A_\Xi A_\Omega A_6 \cong A_{10}$.
2. If $H^{n-1}(X; \mathbb{Z}) = 0$, then $A_3 = A_5$.
3. If $X$ is $(n-1)$-connected, then $A_3 \cong A_5$.
4. If $H^n(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and $Sq^2$ is non trivial on $H^n(X; \mathbb{Z}_2)$, then $A_5 = A_6$.
5. If $f^*; H^n(X; \mathbb{Z}_2) \cong H^n(S^n; \mathbb{Z}_2)$ and $f^*; KO[1, \infty)^n(X) \to KO[1, \infty)^n(S^n)$ is surjective, then $f^*_*; KO[2, \infty)^n(X) \to KO[1, \infty)^n(X)$ is injective, where $h; KO[2, \infty)^n(X) \to KO[1, \infty)^n(X)$ is the connective fibering. If moreover $X$ is $(n-1)$-connected, then $A_1 = A_{10} \cong A_{11}$, so $A_3 = A_5 = A_6$.

**Proposition 2.11.** If $G$ is a compact simply connected simple Lie group, then $\gamma_*; KO[4, \infty)^n(G) \to KO[0, \infty)^n(G)$ is injective, hence $cd^I^G(G) = cd^I^R(G)$. 

Given $f: S^n \to X$ with $n \geq 1$, we define abelian groups $A_1, \ldots, A_{11}$ by using $(f, X)$ instead of $(i, T)$ in [20; Lemma 4.7]. For example $A_i = \text{Ker}(\gamma_*; KO[4, \infty)^n(X) \to KO[0, \infty)^n(X)$, where $\gamma; KO[4, \infty) \to KO[0, \infty)$ is the connective fibering. Then the following lemma can be proved by almost the same proof as [20; Lemma 4.7]. We omit the details.
Proof. We have $A_1 \simeq \cdots \simeq A_5 \simeq A_6 \simeq \cdots \simeq A_{10}$ by 2.10, since $G$ is 2-connected. As is well-known, $H^2(G; \mathbb{Z}_2) = 0$, hence $A_{10} = 0$, and if $G = Sp(n)$ for every $n$ then $S^3$ is non trivial on $H^3(G; \mathbb{Z}_2)$, hence $A_5 = A_6$ by 2.10 (4), so $A_1 \simeq A_{10} = 0$ and $\gamma_*$ is injective on $KO[4, \infty)^3(G)$. On the other hand, we have the following commutative diagram, where $f: S^3 = Sp(1) \subset Sp(n)$.

$$
\begin{array}{ccc}
KO[1, \infty)^3(Sp(n)) = KO^3(Sp(n)) = [Sp(n), Sp/U] & \xrightarrow{q_*} & [Sp(n), Sp] \\
\downarrow f^* & \downarrow f^* & \downarrow f^* \\
KO[1, \infty)^3(S^3) = KO^3(S^3) & = [Sp(1), Sp/U] & \xleftarrow{f_*} [Sp(1), Sp]
\end{array}
$$

Since $\pi_2(U) = 0$, the above lower $q_*$ is surjective. Let $i: Sp(n) \subset Sp$ be the inclusion. Then $f^*(i)$ is a generator of $\pi_3(Sp) \simeq \mathbb{Z}$. Hence the fourth $f^*$ is surjective, and so is the first $f^*$. Thus we have $A_1 \simeq A_{10}$ by 2.10 (5), so $A_1 = 0$ and $\gamma_*$ is injective on $KO[4, \infty)^3(Sp(n))$. This completes the proof.

Recall from Serre [22] that a connected finite CW-complex $Y$ is $p$-regular if there exists a map $g: \Pi S^{n(i)} \rightarrow Y$ such that $g_*: \pi_*(\Pi S^{*n(i)}) \rightarrow \pi_*(Y)$ has the kernel and the cokernel which are finite and have orders prime to $p$ for every $k$, that is, $g$ is a $p$-equivalence. If moreover $Y$ is a simply connected $p$-regular $H$-space, then there exists a $p$-equivalence $n: Y \rightarrow \Pi S^{n(i)}$ [16; Theorem 1.7].

Proposition 2.12. If a compact simply connected simple Lie group $G$ is $p$-regular, then $cd_p(G) = 0$.

Proof. Suppose that a compact simply connected simple Lie group $G$ is $p$-regular. Then there exists a $p$-equivalence $g: G \rightarrow \Pi S^{n(i)}$. Hence there is only one $n(i)$ with $n(i) = 3$. Let $q: \Pi S^{n(i)} \rightarrow S^3$ be the projection and $f: S^3 \rightarrow G$ be a generator of $\pi_3(G)$. Then the degree of $q \circ g \circ f$ is prime to $p$. Hence $cd_p(G) = 0$.

3. $cd(G)$ for classical groups

Proposition 3.1. (1) $cd(U(n)) = cd(SU(n)) = cd(\xi_{n-1})$ ($n \geq 2$), where $\xi_{n-1}$ is the Hopf bundle over the complex projective $(n - 2)$-space $P(C^{n-1})$.

(2) $cd(Sp(n)) = cd(\xi_n)$ ($n \geq 1$), where $\xi_n$ is the vector bundle over the quaternionic projective $(n - 1)$-space $P(H^n)$ associated with the canonical principal $S^3$-bundle $S^{n-1} \rightarrow P(H^n)$ and the adjoint representation of $S^3$ on its Lie algebra.

(3) $cd(SO(n))$ ($n \geq 5$) is the order of the cokernel of $i^*: \pi_3(SO(n)) \rightarrow \pi_3(SO(3))$, where $i: SO(3) \subset SO(n)$.

(4) $cd(SO(n)) = cd(ad_2 \oplus can_n)$ ($n \geq 5$), where $ad_2$ (resp., $can_n$) is 1 (resp., 2) dimensional vector bundle associated with the principal $O(2)$-bundle $O(n-1)/O(n-3) \rightarrow O(n-1)/(O(2) \times O(n-3))$ and the non-trivial representation $O(2) \rightarrow O(1)$ (resp., the canonical representation $O(2)$ on $\text{Hom}_R(R^2, R)$).
Proof. We have $cd(U(n)) = cd(SU(n))$, since $U(n)$ is homeomorphic to $S^1 \times SU(n)$. By James [12; (7.11)], the inclusions $\Sigma P(C^r) = \Sigma T(\xi_{n-1}; P(C^{r-1})) \subset SU(n)$ and $Q_n = T(\xi_n; P(H^\infty)) \subset Sp(n)$ are stable retracts with which the compositions of $S^3 = \Sigma P(C^2) \subset \Sigma P(C^r)$ and $S^3 = Q_1 \subset Q_n$ generate $\pi_3(SU(n))$ and $\pi_3(Sp(n))$ respectively. We then obtain (1) and (2).

To prove (3), let $n \geq 5$ and $i^\prime: S^3 = Spin(3) \to Spin(n)$ be the inclusion. Then $Im(i^\prime_\ast: \pi_3(S^3) \to \pi_3(Spin(n))) = 2\pi_3(Spin(n))$, since $\pi_3(Spin(n)/Spin(3)) \simeq \mathbb{Z}_2$. Hence there exists $f^\prime: S^3 \to Spin(n)$ which generates $\pi_3(Spin(n))$ and $i^\prime = f^\prime \circ 2\iota_3$. Let $q_n: Spin(m) \to SO(m)$ be the double covering and set $f = q_n \circ f^\prime: S^3 \to SO(n)$. Then $f$ generates $\pi_3(SO(n))$, and $f \circ 2\iota_3 = q_n \circ f^\prime \circ 2\iota_3 = q_n \circ i^\prime = i \circ q_3$, hence $2 \cdot cd(SO(n)) = cd(i \circ q_3)$. On the other hand, we have $\pi_3(SO(3)) \simeq \mathbb{Z}$ and $Im(q_3^\ast: \pi_3(SO(3)) \to \pi_3(S^3)) = 2\pi_3(S^3)$. Hence $cd(SO(n))$ is the order of the cokernel of $i^\ast: \pi_3(SO(n)) \to \pi_3(SO(3))$ and this proves (3).

To prove (4), we prepare some notations. Let $ad_k$ denote the adjoint representation of $O(k)$ on its Lie algebra, $can_k$ denote the canonical representation of $O(k)$ on the form $\Lambda_k \Lambda$, and $V_{n,k} = O(n)/O(n-k) \to G_{n,k} = O(n)/(O(k) \times O(n-k))$ denote the canonical $O(k)$-principal bundle. We may form the associated vector bundles over $G_{n,k}$ for $ad_k$ and $can_k$; we denote them by $ad_k$ and $can_k$ for saving of notations. Note that dim $ad_k = k(k-1)/2$ and dim $can_k = k$.

By Miller [14], there exists a stable homotopy equivalence which are natural with respect to $n$:

$$SO(n) \simeq \bigvee_{i=1}^{n-1} T_k$$

where $T_k = T(ad_k \oplus can_k; G_{n-1,k})$. As is easily seen, we have $T_2 = P(R^k)$, the real projective $(n-1)$-space, the 3-skeleton of $T_2$ is $S^3 = T(ad_2 \oplus can_2; G_{2,2})$, and $T_k (k \geq 3)$ is 5-connected. The inclusion $i: SO(3) \to SO(n)$ corresponds under the above equivalence to

$$P(R^3) \vee S^3 \simeq P(R^k) \vee T_2 \subset \bigvee_{i=1}^{n-1} T_k.$$ 

Hence $Im(i^\ast: \pi_3^i(SO(n)) \to \pi_3^i(SO(3))) \simeq Im(i^\ast: \pi_3^i(T_2) \to \pi_3^i(S^3))$, where $i$ is the inclusion, thus $cd(ad_2 \oplus can_2)$ is equal to the order of the cokernel of $i^\ast: \pi_3^i(SO(n)) \to \pi_3^i(SO(3))$. Therefore we obtain (4) from (3). This completes the proof.

Remark 3.2. Of course we can prove (1) and (2) of 3.1 by using Miller’s stable splitting [14] of $SU(n)$ and $Sp(n)$.

Proposition 3.3. (1) For $n \geq 3$, we have

$$cd_p(SU(n)) = \begin{cases} \max \{i \mid p^i < n\}, & p > 2 \\ \max \{i \mid 2^i \leq n\}, & p = 2. \end{cases}$$

(2) For $p > 2$, we have
\[ cd_p(SO(2n+1)) = cd_p(Spin(2n+1)) = cd_p(Sp(n)) \quad (n \geq 1), \]
\[ cd_p(SO(2n)) = cd_p(Spin(2n)) = cd_p(Spin(2n-1)) \quad (n \geq 3). \]

(3) \[ cd_2(SU([n/2]]) \leq cd_2(SO(n)) \leq cd_2(SU([n/2])) + 2 \quad (n \geq 6), \] where \([n/2]\] is the largest integer smaller than or equal to \(n/2\).

(4) \[ cd_p(Sp(n)) = cd(SU(2n)). \]

Proof. By 3.1(1), \(cd(U(n)) = cd(SU(n)) = cd(ξ_{n-1})\) for \(n \geq 2\). Crabb-Knapp [4; 5.16] has determined \(cd(ξ_{n-1})\) as in (1). Moreover they proved that \(cd(ξ_n) = cd^{1R}(ξ_n)\). Notice from Theorem 1.1 of [20] and 2.9 that \(cd^{1R}(ξ_n) = cd^{\mathbb{C}O}(ξ_n)\).

To prove (2), let \(p\) be an odd prime. Then \(cd_p(SO(n)) = cd_p(Spin(n))\) by 2.7, and

\[
Spin(2n+1) \simeq Sp(n),
\]
\[
Spin(2n) \simeq Spin(2n-1) \times S^{2n-1}
\]
by Harris [8]. Hence (2) follows by the technique which will be used in the proof of Theorem 4.1 below.

Concerning the natural inclusions we have

\[
\text{Coker}(\pi_3(SO(n)) \to \pi_3(U(n))) \simeq \pi_3(U(O) \simeq \mathbb{Z}_2 \quad (n \geq 5),
\]
\[
\text{Coker}(\pi_3(U(n)) \to \pi_3(SO(2n))) \simeq \pi_3(O/U) = 0 \quad (n \geq 3).
\]

It then follows from 2.3(3) that

\[
cd(SO(n)) | 2 \cdot cd(U(n)) \quad (n \geq 5),
\]
\[
cd(U(n)) | cd(SO(2n)) \quad (n \geq 3)
\]
and so we obtain (3).

The rest of this section is devoted to the proof of (4). We use Thom classes which are defined by Atiyah, Bott and Shapiro [1]. Let \(g: P(C^{2n}) \to P(H^n)\) be the usual fibration, \(h: T(ξ_{2n}) \to T(ξ_{2n})\) be the map induced by the fibre preserving map \(ξ_{2n} \to ξ_{2n} = ξ_{2n} \otimes s ξ_{2n}\) which sends \(z\) into \(z^2\). Then \(g^* ξ_n \simeq ξ_{2n} \oplus 1\) [26] and the degree of \(h\) on fibres is 2. Let \(γ\) be the canonical quaternionic line bundle over \(P(H^n)\), \(β \in K(S^2)\) be a generator, and \(r: KO \to KO\) be the real restriction. Set \(θ_c = γ - 2 \in K(P(H^n))\), \(μ = ξ_{2n+1} - 1 \in K(P(C^{2n+1}))\), \(μ_3 = r(β^3 μ) \in KO^{-4}(P(C^{2n+1}))\), \(μ_0 = r(μ) \in KO(P(C^{2n+1}))\), and \(t = e_i(ξ_{2n+1}) \in H^n(P(C^{2n+1}); \mathbb{Z})\). Recall from [24] that \(K(P(H^n)) = Z[θ_c]/(θ_c^2)\) and the complexification \(c: KO(P(H^n)) \to K(P(H^n))\) is injective and has the image generated by \(e_i θ_c(0 \leq i \leq n-1)\), where \(e_i\) is equal to 1 if \(i\) is even and equal to 2 if \(i\) is odd. Let \(θ_R ∈ KO(P(H^n))\) be the element with \(c(θ_R^2) = e_i θ_c\). Notice that \(θ_R^2\) does not mean a power of some element \((i \geq 2)\). Recall from [6] that \(KO^{-6}(P(C^{2n+1}))\) is free module with basis \(μ_3, μ_2, μ_0, \ldots, μ_3 μ_0^{-1}\). Note that \(cd^{\mathbb{C}O}(ξ_{2n})\) is equal to

\[
\text{Min}\{m > 0 | mt ∈ \text{Im}(ch_R: KO^{-6}(P(C^{2n+1})) \to H^n(P(C^{2n+1}); \mathbb{Q}))\}.
\]
and that $\text{ch}_F(\mu_3 \mu_j) = (e^t - e^{-t})(e^t + e^{-t} - 2)^i$.

Consider the commutative diagram

\[
\begin{array}{ccc}
KF(P(C^{2a}_+)) & \xrightarrow{g^*} & KF(P(H^*)_+), \\
\downarrow \phi_{KF} & = & \downarrow \phi_{KF}, \\
KF^{-6}(P(C^{2a+1})) & \xrightarrow{\sigma} & KF(T(\xi_{2a} \oplus 6)) \\
\downarrow ch_F & \cong & \downarrow ch_F, \\
H^* - 6(P(C^{2a+1}); Q) & \xrightarrow{\sigma} & H^*(T(\xi_{2a} \oplus 6); Q) \\
\cong & \phi_H & \phi_H, \\
H^* - 6(P(C^{2a+1}); Q) & \leftarrow H^*(T(\xi_{2a} \oplus 6); Q) & \leftarrow H^*(T(\xi_5 \oplus 5); Q)
\end{array}
\]

where vertical isomorphisms are Thom isomorphisms and $\oplus k$ means the Whitney sum with the real $k$-dimensional trivial bundle. Computations show that

\[
\sigma h^* g^* \phi_{KF}(\sum_{i=1}^{n-1} a_i \theta_i^e) = \beta(1 + (1 + \mu)^{-1}) \sum_{i=1}^{n-1} a_i (1 + \mu)^{-1} - 2)^i;
\]

where we consider $H^*(P(C^{2a}); Q)$ as a submodule of $H^*(P(C^{2a+1}); Q)$ naturally. Hence we have

\[
\sigma h^* g^* ch_F \phi_{KF}(\sum_{i=0}^{n-1} a_i \theta_i^e) = (e^t - e^{-t}) \sum_{i=0}^{n-1} a_i e_i^e (e^t + e^{-t} - 2)^i,
\]

where $e_i^e = 1$ and $e_i^e = e_i$. By definition there exist integers $a_0^e, \ldots, a_{n-1}^e$ such that $cd_{K^F}(\xi_{2a}) \cdot 1 = \phi_H \cdot 1 = \phi_H = \phi_{KF}(\sum_{i=0}^{n-1} a_i^e \theta_i^e)$. In this case we have

\[
2 \cdot cd_{K^F}(\xi_{2a}) t = (e^t - e^{-t}) \sum_{i=0}^{n-1} a_i^e e_i^e (e^t + e^{-t} - 2)^i.
\]

Hence $a_0^e = cd_{K^F}(\xi_{2a})$, and also

\[
2 \cdot cd_{K^E}(\xi_{2a}) = cd_{K^O}(\xi_{2a}).
\]

By [19] we have

\[
i ! a_0^e = (-1)^i 2^i \cdot 3 \cdot \cdots \cdot (2i + 1) \cdot a_0^e \quad \text{for} \quad 1 \leq i \leq n - 1
\]

hence

\[
\nu_2(a_0^e) = \alpha(i) + \nu_2(a_0^e)
\]

since $\nu_2(i!) = i - \alpha(i)$, where $\nu_2(m)$ is the exponent of 2 in the prime decomposi-
tion of $m$, and $\alpha(i)$ is the number of 1's in the dyadic expansion of $i$. It follows that
\[
\text{cd}^F(\zeta_n) = \text{Max} \{\alpha(i) \mid 1 \leq i \leq n-1\}
\]
and from the similar reason that
\[
\text{cd}^K(\zeta_n) = \text{Max} \{\alpha(i) + \nu_s(e_i) \mid 1 \leq i \leq n-1\}
\]
\[= 1 + \text{cd}^F(\zeta_n)
\]
so that
\[
\text{cd}^K(\zeta_n) = 2 \cdot \text{cd}^F(\zeta_n)
\]
by 2.9(1). We then have
\[
\text{cd}^K(\zeta_n) = \text{cd}(\zeta_n) = \text{cd}(U(2n)) \ (n \geq 2)
\]
from the equations
\[
\text{cd}^K(\zeta_n) = \text{cd}(U(2n)) = \text{cd}(U(2n+1)) = \text{cd}^K(\xi_n) = \text{cd}^K(\zeta_n).
\]
This proves (4) and completes the proof of Proposition 3.3.

4. Mod $p$ decomposition of Lie groups

The result of this section is

**Theorem 4.1.** Let $G$ be a compact simply connected simple Lie group. Then the following statements hold.

1. $\text{cd}_p(G) = 0$ if and only if $G$ is $p$-regular.
2. $\text{cd}_p(G) = 1$ if $G$ is not $p$-regular but quasi $p$-regular.
3. $\text{cd}_p(E_6) = \text{cd}_p(E_7) = \text{cd}_p(E_8) = 1$.
4. $\text{cd}(G_2) = \text{cd}(\text{Spin}(7)) = \text{cd}(\text{Spin}(8)) = \text{cd}(\text{SO}(7)) = 2 \cdot \text{cd}(SU(7)) = 2^3 \cdot 3 \cdot 5$.
5. $2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 | \text{cd}(G)$ for $G = E_6, E_7, E_8, F_4$.
6. $\text{cd}_p(E_6) \leq 8; \text{cd}_p(E_7) \leq 7; \text{cd}_p(E_8) \leq 5; \text{cd}_p(F_4) \leq 5; \text{cd}_p(E_6) \leq 5; \text{cd}_p(E_7) \leq 4; \text{cd}_p(E_8) = \text{cd}_p(F_4) \leq 3; \text{cd}_p(E_6) \leq 3$.

From 4.1 and Table (4.3) below, we have

**Corollary 4.2.** For simply connected exceptional groups, $\text{cd}_p(G) = 1$ if (and only if provided $G \neq E_6$) $G$ is not $p$-regular and has no $p$-torsion in $H^*(G; \mathbb{Z})$.

Precise values of $p$ in 4.1 and 4.2 can be read from the following Table (4.3).
Table (4.3)

<table>
<thead>
<tr>
<th>$G$</th>
<th>$p$-torsion</th>
<th>quasi $p$-regular</th>
<th>$p$-regular</th>
<th>$cd_p(G)=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(n)$</td>
<td>NO</td>
<td>$n/2 &lt; p$</td>
<td>$n \leq p$</td>
<td>$p &lt; n \leq p^2$ ($p &gt; 2$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n=3$</td>
</tr>
<tr>
<td>$Sp(n)$</td>
<td>NO</td>
<td>$n &lt; p$</td>
<td>$2n \leq p$</td>
<td>$p &lt; 2n &lt; p^2$ ($p &gt; 2$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>NO ($p=2$)</td>
</tr>
<tr>
<td>$Spin(n)$</td>
<td>2</td>
<td>$(n-1)/2 &lt; p$</td>
<td>$n-1 \leq p$</td>
<td>$p &lt; n-1 \leq p^2$ ($p &lt; 2$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>NO ($p=2$)</td>
</tr>
<tr>
<td>$G_2$</td>
<td>2</td>
<td>$5 \leq p$</td>
<td>$7 \leq p$</td>
<td>$2 &lt; p &lt; 7$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>2, 3</td>
<td>$5 \leq p$</td>
<td>$13 \leq p$</td>
<td>$3 &lt; p &lt; 13$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>2, 3</td>
<td>$5 \leq p$</td>
<td>$13 \leq p$</td>
<td>$3 &lt; p &lt; 13$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>2, 3</td>
<td>$11 \leq p$</td>
<td>$19 \leq p$</td>
<td>$3 &lt; p &lt; 19$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>2, 3, 5</td>
<td>$11 \leq p$</td>
<td>$31 \leq p$</td>
<td></td>
</tr>
</tbody>
</table>

Lemma 4.4. We have $cd^K(Sp(2))=2\cdot 3$; $cd^K(Spin(7))=2^3\cdot 3\cdot 5$; $cd^K(G_2)=2^3\cdot 3\cdot 5$; $cd^K(F_4)=2^3\cdot 3^3\cdot 5\cdot 7\cdot 11$; $cd^{KO}(G_2)/cd^K(G_2)=cd^{KO}(F_4)/cd^K(F_4)=2$.  

Proof. The computation of $cd^K(G)$ can be easily done by using Watanabe [27]. We omit the details.

Suppose that $G=G_2$ or $F_4$. Set $r=\text{rank } G$. Consider the commutative diagram:

$$
\begin{array}{ccc}
RO(G) & \xrightarrow{\beta} & KO^{-1}(G_+) \\
\downarrow c & & \downarrow c \\
R(G) & \rightarrow & K^{-1}(G_+) \\
\end{array}
$$

where $RO(G)$, $R(G)$ are real, complex representation rings of $G$, and $\beta$ is the natural homomorphism [11, 23], and $c$ is the complexification. By Yokota [28, 29], $c: RO(G) \cong R(G)$. Let $\rho^c_1, \cdots, \rho^c_r \in R(G)$ be basic representations [5, 11] and $\rho^R_1, \cdots, \rho^R_r \in RO(G)$ be representations such that $c(\rho^R_i) = \rho^c_i$. It follows from Seymour [23; 5.6] that $KO^*(G_+)$ is a free $KO^*(S^0)$-module and is generated multiplicatively by $\beta(\rho^R_1), \cdots, \beta(\rho^R_r)$. Let $A^c$ denote the $KF^*(S^0)$-submodule of $KF^*(G_+)$ generated by decomposable elements with respect to $\{\beta(\rho^c_1), \cdots, \beta(\rho^c_r)\}$. Then

$$KO^4(G) = KO^4(G_+) \equiv KO^4(S^0) \{\beta(\rho^R_1), \cdots, \beta(\rho^R_r)\} \mod A^R$$

and so

$$c(KO^4(G_+)) \equiv 2K^4(S^0) \{\beta(\rho^c_1), \cdots, \beta(\rho^c_r)\} \mod A^c.$$


Therefore \( \text{cd}^{\kappa_0}(G) = 2 \cdot \text{cd}(G) \) and this completes the proof.

**Remark 4.5.** \( \text{cd}(Sp(2)) = \text{cd}(\xi_2) = \text{cd}^{\kappa_0}(\xi_2) = 12. \)

**Proof of Theorem 4.1 (1), (2), (5).** If \( G \) is \( p \)-regular then \( \text{cd}_p(G) = 0 \) by 2.12. By 3.3 or [19], \( \text{cd}_p(SU(n)) = 0 \) if and only if \( n \leq p \). Hence (1) holds for \( SU(n) \).

By 3.3, 4.3, and 2.3(3), we have \( \text{cd}_p(Sp(n)) = 0 \) for \( 2 \leq n \) and \( p \leq 2n \). Hence (1) holds for \( Sp(n) \).

By 4.4 and 2.3(3), we have \( 3 \leq \text{cd}_p(Spin(n)) \) for \( n \geq 7 \). It follows from 3.3 that if \( \text{cd}_p(Spin(n)) = 0 \) then \( n - 1 \leq p \) and so \( Spin(n) \) is \( p \)-regular. Hence (1) holds for \( Spin(n) (n \geq 7) \).

It follows from 4.4 that if \( \text{cd}_p(G_2) = 0 \) then \( p \geq 7 \) and so \( G_2 \) is \( p \)-regular, and that if \( \text{cd}_p(F_4) = 0 \) then \( p \geq 13 \) and so \( F_4 \) is \( p \)-regular. Hence (1) holds for \( G_2 \) and \( F_4 \).

Since there exist embeddings \( F_4 \subset E_6 \subset E_7 \subset E_8 \) which induce isomorphisms of the third homotopy groups, we have \( \text{cd}^{\kappa_0}(F_4) = \text{cd}(\xi_2) = \text{cd}(G) \) for \( G = E_6 \), \( E_7 \), \( E_8 \) by 2.3(3). It follows from 4.4 that (5) holds and that if \( \text{cd}_p(E_6) = 0 \) then \( p \geq 13 \) and so (1) holds for \( E_6 \).

By [15], if \( G \) is quasi \( p \)-regular, then there exists a \( p \)-equivalence \( u: B_1(p) \times Y \to G \) where \( B_1(p) = S^3 \cup \varnothing^{p+1} \cup \varnothing^{p+1} \) is a \( S^3 \)-bundle over \( S^{2p+1} \) such that \( H^*(B_1(p); \mathbb{Z}) = \Lambda(x_3, \varnothing^{2p+1} x_3) \), and \( Y \) is a 4-connected finite \( CW \)-complex. There exists a \( p \)-equivalence \( v: G \to B_1(p) \times Y \), since \( G \) is \( p \)-universal [16]. Let \( f: S^3 \to G \) represent a generator of \( \pi_3(G) \), and \( f': S^3 \subset B_1(p) \) be the inclusion. Then \( f' \) generates \( \pi_3(B_1(p)) \simeq \mathbb{Z} \). Since \( f'^*: H^p(B_1(p); \mathbb{Z}) \to H^p(S^3; \mathbb{Z}) \), it follows that \( \text{cd}(f') \neq 0 \) by 2.3(4). Let \( g: B_1(p) \times Y \to B_1(p) \) be the projection, and \( \iota: B_1(p) = B_1(p) \times \{*\} \subset B_1(p) \times Y \) be the inclusion. Since \( g, \iota \) induce isomorphisms on \( \pi_3(\), and since \( u, v \) are \( p \)-equivalences, there exist integers \( m, n \) such that \( m, n \) are prime to \( p \) and \( \pi_3(\), and since \( u, v \) are \( p \)-equivalences, there exist integers \( m, n \) such that \( m, n \) are prime to \( p \), \( q \) and \( f' \) are \( \mathbb{Z} \)-equivariant, and \( w \) is \( f \) modulo \( m \) and \( n \). It follows from 2.3(3) that \( \text{cd}(f) = m \cdot \text{cd}(f') + n \cdot \text{cd}(f) \), so that \( \text{cd}(f') = \text{cd}(f) = \text{cd}(G) \).

On the other hand, since the attaching map \( a: S^{2p} \to S^3 \) of \( (2p+1) \)-cell of \( B_1(p) \) is a generator of \( \pi_3(S^3; \mathbb{Z}) = \pi_3(S^3) \subset S^3 \) (see [15, 25]), it follows that \( \text{cd}(f') \geq 1 \) and that the map \( p_{S^3} \) extends stably to \( a': S^3 \cup \varnothing^{p+1} \cup \varnothing^{p+1} \to S^3 \). Since \( \pi_3(B_1(p); \mathbb{Z}) = 0 \) (see [25]), \( a' \) extends stably to \( a'' : B_1(p) \to S^3 \) such that \( a'' \circ f' = p_{S^3} \), hence \( \text{cd}(f') \leq 1 \), so \( \text{cd}(G) = \text{cd}(f') = 1 \), and this proves (2). In particular we have \( \text{cd}_p(E_7) = 1 \) for \( 11 \leq p \leq 17 \) and \( \text{cd}_p(E_8) = 1 \) for \( 11 \leq p \leq 29 \). This completes the proof of (1).

To prove Theorem 4.1(3), we need the following theorem. See [17] for notations.

**Theorem 4.6 ([17; 8.1]).** (1) \( E_7 \simeq B_1^1(5) \times B_1^1(5) \).
(2) \( E_8 \simeq B_1^1(7) \times B_1^1(7) \times S^{29} \).
(3) \( E_9 \simeq B_1^1(7) \times B_1^1(7) \).

**Proof of Theorem 4.1.(3).** We will prove only that \( \text{cd}_p(E_8) = 1 \), because
other cases can be proved by the same method. Note that the 16-skeleton of \( B_7^{(1)} \) is \( S^3 \), and that \( B_7^{(1)}(7) \) is 22-connected. Let \( i: S^3 \to B_7^{(1)}(7) \) be the inclusion.

By the same method as the proof of (1), we have 7-equivalences \( u: B_7^{(1)}(7) \times B_7^{(1)}(7) \to E_6 \), \( v: E_6 \to B_7^{(1)}(7) \times B_7^{(1)}(7) \), and \( \text{cd}(E_6) = \text{cd}(i) \). By definition (see [17]), there exists a map \( h: B_7^{(1)}(7) \to SU(20) \) such that \( h^*: H^*(SU(20); \mathbb{Z}_7) \to H^*(B_7^{(1)}(7); \mathbb{Z}_7) \) is a surjection. In particular the index of the image of \( h^*: \pi_3(SU(20)) \to \pi_3(B_7^{(1)}(7)) \) is \( \mathbb{Z} \) is prime to 7. That is, \( h \circ i \cong f \circ m \) for some integer \( m \) which is prime to 7, where \( f: S^3 \to SU(20) \) represents a generator of \( \pi_3(SU(20)) \). By 2.3(3), we have \( \text{cd}(i) | m \cdot \text{cd}(SU(20)) \), so \( \text{cd}(i) \leq \text{cd}(SU(20)) = 1 \) by 3.3, hence \( \text{cd}(E_6) = \text{cd}(i) = 1 \) by (5).

**Proof of Theorem 4.1(4).** There exist homomorphisms \( G_2 \subset Spin(7) \to SO(7) \) which induce isomorphisms on \( \pi_3(\ ) \). Hence

\[
\text{cd}(G_2) \mid \text{cd}(Spin(7)) \mid \text{cd}(SO(7)) \mid 2 \cdot \text{cd}(SU(20))
\]

where the last divisibility was proved in the proof of 3.3(3). Since \( \text{cd}^{\kappa_0}(G_2) = 2 \cdot \text{cd}(SU(7)) = 120 \) by 3.3(1) and 4.4, the above four codegrees are the same. We have also \( \text{cd}(Spin(7)) = \text{cd}(Spin(8)) \), since \( Spin(8) \) is homeomorphic to \( Spin(7) \times S^7 \).

**Remark 4.7.** We can prove \( \text{cd}(G_2) = 120 \) by constructing a stable map \( g: G_2 \to S^3 \) having the degree 120 on \( S^3 \).

**Proof of Theorem 4.1(6).** Let \( \{w_1, \ldots, w_r\} \) be a system of fundamental weights with respect to a system of simple roots of \( G \), \( \rho(w_i) \) be the irreducible representation with highest weight \( w_i \) (\( 1 \leq i \leq r \)), \( \beta \) be a root of \( G \) of maximal length, and set \( \delta = \sum_i w_i \). Then, from an observation of Harris [10] (cf., Naylor [18]), we know that the cokernel of \( \rho(w_i): \pi_3(G) \to \pi_3(U(\dim(\rho(w_i)))) \) is a cyclic group of order

\[
n_i = \frac{2(w_i, w_i + 2\beta)}{(\beta, \beta)} \cdot \frac{\dim(\rho(w_i))}{\dim G}
\]

where \( (\ , \ ) \) is the Killing form. The number \( n_i \) was called by Dynkin [5] the index of the representation \( \rho(w_i) \). It follows from 2.3(3) that

\[
\text{cd}(G) \mid n_i \cdot \text{cd}(U(\dim(\rho(w_i))))
\]

From Tables 5 and 41 of Dynkin [5] (cf., Harris [10]), we know that there exist irreducible representations of minimal dimension

\[
\begin{align*}
G_2 & \to U(7) , \quad n = 2 , \\
F_4 & \to U(26) , \quad n = 6 , \\
E_6 & \to U(27) , \quad n = 6 , \\
E_7 & \to U(56) , \quad n = 12 , \\
E_8 & \to U(240) , \quad n = 60 .
\end{align*}
\]
Hence we obtain the desired upper bounds except $cd_p(E_8)$ for $p=2,5$ by (4.8) and 3.3(1). Since the complex Stiefel manifold $U(240)/U(124)$ is 248-connected and the dimension of $E_8$ is 248, the representation $E_8 \to U(240)$ factorizes up to homotopy $E_8 \sim U(124) \subset U(240)$ as an unstable map, it follows that $cd(E_8) | 60 \cdot cd(U(124))$ so that $cd_6(E_8) \leq 8$ and $cd_6(E_8) \leq 3$.

By [9, Proposition 1], we have $E_6 \sim_3 F_4 \times E_6/F_4$ so that $cd_3(E_6) = cd_3(F_4)$, since $E_6/F_4$ is 8-connected. This proves (6) and completes the proof of Theorem 4.1.

References


[29] I. Yokota: Exceptional Lie group \( F_4 \) and its representation rings, ibid. 3 (1968), 35–60.

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558, Japan