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ON A MARTINGALE METHOD FOR SYMMETRIC DIFFUSION PROCESSES AND ITS APPLICATIONS

Masayoshi TAKEDA

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1. Introduction

Let X be a locally compact separable metric space and m be a Radon measure on X whose support is the whole space X. Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet space on $L^2(X, m)$ and denote by $\boldsymbol{M}=(\Omega, X_t, P_x)$ a symmetric Markov process associated with the Dirichlet space $(\mathcal{E}, \mathcal{F})$. For $u \in \mathcal{F}$ denote by $\tilde{u}$ the quasi-continuous version of u and let $A_t^{[u]}=\tilde{u}(X_t)-\tilde{u}(X_0)$. Then, it is known in Fukushima [6] that the additive functional (abbreviated by AF) $A^{[u]}$ can be written as

$$A_t^{[u]} = M_t^{[u]}+N_t^{[u]}\,, \quad P_x\text{—}a.e. \quad q.e.\ x \tag{1.1}$$

where $M_t^{[u]}$ is a martingale AF of finite energy and $N_t^{[u]}$ is a continuous AF of zero energy (for notions see [6]). This decomposition is regarded as an extension of the notion of semimartingale AF's in the sense that the quadratic variation of $N^{[u]}$ vanishes (see (5.2.10) in [6]).

On the other hand, under the assumption that the Markov process $\boldsymbol{M}$ is conservative, Lyons-Zheng [10] obtained another expression of $A^{[u]}$: for $T>0$

$$A_t^{[u]} = \frac{1}{2} M_t^{[u]} - \frac{1}{2} (M_T^{[u]}(r_T) - M_{T-t}^{[u]}(r_T)),\ 0 \leq t \leq T,\ P_m\text{—}a.e. \tag{1.2}$$

where r_T is a time reverse operator at T, *i.e.*, $X_t(r_T)=X_{T-t}$, and P_m is a σ-finite measure defined by $\int_X P_x[\cdot]\, dm$. Denote by $\mathcal{F}_t$ (resp. $\mathcal{G}_t$) the σ-field generated by $\{X_s;\ 0 \leq s \leq t\}$ (resp. $\{X_s;\ T-t \leq s \leq T\}$). Then we see that $M_t^{[u]}(r_T)$ is a $(P_m, \mathcal{G}_t)$—martingale. Thus, the AF $A^{[u]}$ is the sum of a $(P_m, \mathcal{F}_t)$—martingale and a $(P_m, \mathcal{G}_t)$—martingale. The formula (1.2) is derived from the fact that the symmetry of $\boldsymbol{M}$ implies the time reversibility: for $\mathcal{F}_T$-measurable function F

$$E_m[F(r_T)] = E_m[F]\,. \tag{1.3}$$

One can say that the decomposition (1.2) reflects the symmetry of the Markov process $\boldsymbol{M}$ faithfully. Furthermore (1.2) would enable us to use the martingale theory more effectively than (1.1) in the study of symmetric Markov processes. The purpose of the present paper is to demonstrate this in getting a conservativeness criterion, a tightness criterion and also some sample path properties for symmetric diffusion processes. We shall further consider an extension of the method to non-symmetric situations.

In §2, we shall give a sufficient condition for symmetric diffusion processes to be conservative (Theorem 2.2). In some important cases, our criterion is sharper than Ichihara's test [7] for the conservation of probability.

In §3, we shall give a sufficient condition for a certain class of symmetric diffusion processes on $\boldsymbol{R}^d$ to be tight. Lyons-Zheng [10] have proved the tightness property for a similar class of diffusion processes but in the "pseudo-path topology" which is even weaker than the Skorohod one. We shall prove the tightness in the usual uniform topology (Theorem 3.1). As an application, we can strengthen those results in Albeverio-Høegh-Krohn-Streit [2] and Albeverio-Kusuoka-Streit [3] on the semi-group convergence of energy forms to weak convergence results.

In §4, we shall present two elementary estimates. The first one (Lemma 4.1) was obtained in Kusuoka [9] by an analytic method but the present method is simpler in that we only use the decomposition (1.2) and the representation theorem of continuous local martingales by Brownian motions. The second one (Lemma 4.3) is applicable to showing certain sample path properties of symmetric diffusion processes as we shall see in §5 for the upper estimate of the law of the iterated logarithm.

In §6, we shall consider how we can extend the formula (1.2) to the case of special non-symmetric diffusion processes with an invariant measure.

We emphasize that the diffusion processes we are treating include those whose generators are of divergence form with non-smooth coefficients and accordingly they can not be handled by the method of stochastic differential equations based on the Brownian motions. Nevertheless, the present method enables us to reduce their study to elementary properties of the Brownian motion. One may use known powerful estimates of fundamental solutions in the uniformly elliptic cases, but it seems quite difficult to derive such probabilistic results as Lemma 4.3 and Theorem 5.1 by using only the analytical estimates of this kind.

A part of the present results has been announced in [12].

2. A conservativeness test for symmetric diffusion processes

We use the notions and the notations in [6]. Let X and m be as in §1 and $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet space on $L^2(X, m)$ such that $\mathcal{E}(u, v)$ vanishes whenever v is constant on the support of u. If, for any relatively compact open set G, there exists a function $v \in \mathcal{F}$ such that $u=v$, $m-a.e.$ on G, the function u is said to be locally in $\mathcal{F}$ ($u \in \mathcal{F}_{\text{loc}}$ in notation). We see that the formulas (1.1) and (1.2) are extended to $u \in \mathcal{F}_{\text{loc}}$. As is shown in Chapter 5 in [6], for

$u \in \mathcal{F}_{\mathrm{loc}}$ there exists a Radon measure $\mu_{\langle u \rangle}$ corresponding to the quadratic variation $\langle M^{[u]} \rangle$, and the Dirichlet form $\mathcal{E}$ is written as

$$\mathcal{E}(u, v) = \frac{1}{2} \int_X d\mu_{\langle u,v \rangle}, \quad u, v \in \mathcal{F} \tag{2.1}$$

where $d\mu_{\langle u,v \rangle} = \frac{1}{2} (d\mu_{\langle u+v \rangle} - d\mu_{\langle u \rangle} - d\mu_{\langle v \rangle})$.

Let $\mathcal{K}$ denote the class of compact sets K satisfying that $m(K) > 0$, supp $(\chi_K m) = K$ and that the bilinear form

$$\mathcal{E}^K(u, v) = \frac{1}{2} \int_K d\mu_{\langle u,v \rangle}, \quad u, v \in \mathcal{F}$$

(which can be seen to be dependent only on restrictions to K of $u, v \in \mathcal{F}$) is closable on $L^2(X, \chi_K m)$. Here, χ_K is the indicator functions of K. For $K \in \mathcal{K}$, we denote by $\mathcal{F}^K$ the domain of the closure of $\mathcal{E}^K$. Then, the pair $(\mathcal{E}^K, \mathcal{F}^K)$ can be regarded as a regular Dirichlet space on $L^2(K, \chi_K m)$ and the diffusion process on K associated with $(\mathcal{E}^K, \mathcal{F}^K)$ is conservative because $\chi_K \in \mathcal{F}^K$ and $\mathcal{E}^K(\chi_K, \chi_K) = 0$.

We set

$$\mathcal{F}_{\mathrm{loc,ac}} = \left\{ \rho \in \mathcal{F}_{\mathrm{loc}};\ \begin{matrix} \mu_{\langle \rho \rangle} \text{ is absolutely continuous} \\ \text{with respect to } m. \end{matrix} \right\}$$

and denote by $\Gamma(\rho)$ the density of $\mu_{\langle \rho \rangle}$ with respect to the Radon measure m for $\rho \in \mathcal{F}_{\mathrm{loc,ac}}$. Furthermore, we set

$$\mathcal{A} = \left\{ \rho \in \mathcal{F}_{\mathrm{loc,ac}} \cap C(X);\ \begin{matrix} \lim_{x \to \Delta} \rho(x) = \infty \quad \text{and for any} \quad r > 0 \\ \text{the set } \{x \in X;\ \rho(x) \leq r\} \text{ belongs to } \mathcal{K}. \end{matrix} \right\},$$

where $C(X)$ is the family of the continuous functions on X and Δ is the extra point in the one-point compactification of X.

Let $B_{r,\rho} = \{x \in X;\ \rho(x) \leq r\}$ and $M_\rho(r) = \operatorname{ess.sup}_{x \in B_{r,\rho}} \Gamma(\rho)(x)$ for $\rho \in \mathcal{F}_{\mathrm{loc,ac}}$. Then, we have

Lemma 2.1. For $\rho \in \mathcal{A}$

$$P_{\chi_{B_{R,\rho}} m} [\sup_{0 \leq t \leq T} (\rho(X_t) - \rho(X_0)) \geq r] \leq 6m(B_{R+r,\rho})\, l\left(\frac{2r}{3\sqrt{M_\rho(R+r)\, T}}\right), \tag{2.2}$$

where $l(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2}\, dx$.

Proof. Put $(\mathcal{E}^r, \mathcal{F}^r) = (\mathcal{E}^{B_{r,\rho}}, \mathcal{F}^{B_{r,\rho}})$ and $m_r = \chi_{B_{r,\rho}} m$. Let $\boldsymbol{M} = (P_x, X_t)$ and $\boldsymbol{M}^r = (P_x^r, X_t)$ be the diffusion processes corresponding to $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^r, \mathcal{F}^r)$ respectively. Then, we have, for $R, r > 0$

$$\begin{aligned}(2.3)\qquad & P_{m_R}[\sup_{0\leq t\leq T}(\rho(X_t)-\rho(X_0))\geq r] \\ &= P^{R+r}_{m_R}[\sup_{0\leq t\leq T}(\rho(X_t)-\rho(X_0))\geq r] \\ &\leq P^{R+r}_{m_{R+r}}[\sup_{0\leq t\leq T}(\rho(X_t)-\rho(X_0))\geq r]\,.\end{aligned}$$

Since the diffusion process $\boldsymbol{M}^{R+r}$ is conservative, it follows from the formula (1.2) that

$$\rho(X_t)-\rho(X_0)=\frac{1}{2}M^{[\rho]}_t-\frac{1}{2}(M^{[\rho]}_T(r_T)-M^{[\rho]}_{T-t}(r_T)),\ P^{R+r}_{m_{R+r}}\text{—}a.e.\,.$$

Thus, we see that the right hand side of (2.3) is not greater than

$$\begin{aligned}(2.4)\qquad & P^{R+r}_{m_{R+r}}[\sup_{0\leq t\leq T}M^{[\rho]}_t\geq\frac{2}{3}r]+P^{R+r}_{m_{R+r}}[\sup_{0\leq t\leq T}M^{[\rho]}_t(r_T)\geq\frac{2}{3}r] \\ &+P^{R+r}_{m_{R+r}}[-M^{[\rho]}_T(r_T)\geq\frac{2}{3}r] \\ &\leq 2\,P^{R+r}_{m_{R+r}}[\sup_{0\leq t\leq T}M^{[\rho]}_t\geq\frac{2}{3}r]+P^{R+r}_{m_{R+r}}[\sup_{0\leq t\leq T}(-M^{[\rho]}_t)\geq\frac{2}{3}r]\end{aligned}$$

by the relation (1.3). Using a one-dimensional Brownian motion $\boldsymbol{B}(t)$ with respect to P^{R+r}_x for *q.e. x*, we see that the right hand side of (2.4) is dominated by

$$\begin{aligned}& 2\,P^{R+r}_{m_{R+r}}[\sup_{0\leq t\leq T}\boldsymbol{B}(\int_0^t\Gamma(\rho)\,(X_u)\,du)\geq\frac{2}{3}r]+P^{R+r}_{m_{R+r}}[\sup_{0\leq t\leq T}-\boldsymbol{B}(\int_0^t\Gamma(\rho)\,(X_u)\,du)\geq\frac{2}{3}r] \\ &\leq 3P^{R+r}_{m_{R+r}}[\sup_{0\leq t\leq M_\rho(R+r)T}\boldsymbol{B}(t)\geq\frac{2}{3}r] \\ &= 6\,m(B_{R+r,\rho})\,l\Big(\frac{2r}{3\sqrt{M_\rho(R+r)\,T}}\Big)\,.\end{aligned}$$

q.e.d.

We shall prove the following general criterion for the conservation of probability.

Theorem 2.2. If there exist $\rho\in\mathcal{A}$ and $T>0$ such that for any $R>0$

$$(2.5)\qquad \lim_{r\to\infty}m(B_{R+r,\rho})\,l\Big(\frac{r}{\sqrt{M_\rho(R+r)\,T}}\Big)=0\,,$$

then the diffusion process corresponding to $(\mathcal{E},\mathcal{F})$ is conservative.

Proof. Let $\boldsymbol{M}=(P_x, X_t)$ be the diffusion process corresponding to $(\mathcal{E},\mathcal{F})$. By Lemma 2.1 and assumption (2.5), we have for $T'=\frac{4}{9}T$

$$\begin{aligned}& P_{m_R}[\sup_{0\leq t\leq T'}(\rho(X_t)-\rho(X_0))=\infty]=\lim_{r\to\infty}P_{m_R}[\sup_{0\leq t\leq T'}(\rho(X_t)-\rho(X_0))\geq r] \\ &\leq\lim_{r\to\infty}6\,m(B_{R+r,\rho})\,l\Big(\frac{r}{\sqrt{M_\rho(R+r)\,T}}\Big)\end{aligned}$$

$$= 0\,,$$

and so

$$\begin{aligned} P_{T'}\,1(x) &= P_x[T'<\zeta] \\ &= P_x[\sup_{0\le t\le T'}(\rho(X_t)-\rho(X_0))<\infty] \\ &= 1,\ m\text{—}a.e.\,, \end{aligned}$$

where ζ is the life time of the diffusion process $\boldsymbol{M}$. By virtue of the semi-group property we can conclude that for any $t>0$, $P_t\,1=1$, m—$a.e.$. q.e.d.

To give examples, we deal with a more concrete Dirichlet space for $X=\boldsymbol{R}^d$. Let $\mathcal{E}$ be a symmetric bilinear form on $L^2(\boldsymbol{R}^d, m)$ defined by

$$\mathcal{E}(u, v) = \frac{1}{2}\sum_{i,j=1}^{d}\int_{\boldsymbol{R}^d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dm,\ u, v\in C_0^\infty(\boldsymbol{R}^d)\,, \tag{2.6}$$

where $C_0^\infty(\boldsymbol{R}^d)$ is the space of infinitely differentiable functions with compact support. Let the coefficients a_{ij} be locally integrable Borel measurable functions satisfying

$$\text{(2.7)}\qquad \begin{aligned} &\text{i)}\quad a_{ij} = a_{ji} \\ &\text{ii)}\quad \sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\,\xi_j \ge 0 \quad \text{for any} \quad x, \xi\in\boldsymbol{R}^d\,. \end{aligned}$$

Together with (2.6), we consider for each closed ball $B_r=\{x\in\boldsymbol{R}^d;\ |x|\le r\}$ a symmetric form

$$\mathcal{E}^{B_r}(u, v) = \frac{1}{2}\sum_{i,j=1}^{d}\int_{B_r} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dm,\ u, v\in C^\infty(B_r)\,, \tag{2.8}$$

where $C^\infty(B_r)$ is the restrictions to B_r of functions in $C_0^\infty(\boldsymbol{R}^d)$.

We assume the closability of the form (2.6) on $L^2(\boldsymbol{R}^d, m)$ and also that of (2.8) on $L^2(B_r, m)$ for each r. This closability requirement is satisfied if m is the Lebesgue measure and if a_{ij} are either locally uniformly elliptic or smooth. See [11] for the closability for more general m and a_{ij}.

EXAMPLE 1 Consider the case that there exists a constant λ such that $\sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\,\xi_j\le\lambda|\xi|^2$, for any $x, \xi\in\boldsymbol{R}^d$. Since the function $|x|$ belongs to $\mathcal{A}$, we see according to Theorem 2.2 with $\rho(x)=|x|$ that, if there exists $T>0$ such that

$$\lim_{r\to\infty} m(B_{R+r})\,l\Big(\frac{r}{\sqrt{\lambda T}}\Big) = 0 \quad \text{for any} \quad R>0\,, \tag{2.9}$$

then the corresponding diffusion is conservative. Noting that $l(a)\le\frac{1}{a}\,e^{-a^2/2}$, for $a>0$, we see that if $m(B_r)\leq c_1 e^{c_2 r^2}$ with some constants c_1 and c_2, (2.9) is fulfilled by choosing $T<\frac{1}{2\lambda c_2}$. This improves a result of Ichihara [7: Example 3.2] not only in the growth order but also in that we require neither the absolutely continuity of m nor the non-degeneracy of the density except for the closability requirement. The diffusion process corresponding to $a_{ij}=\delta_{ij}$ and $m(dx)=e^{|x|^{2+\varepsilon}}dx$ is known to be explosive for any $\varepsilon>0$.

EXAMPLE 2 Consider the case that the measure m is the Lebesgue measure on $\boldsymbol{R}^d$ and $\sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\,\xi_j\leq k(2+|x|)^2\log(2+|x|)|\xi|^2$ with some constant k. Employing the function $\rho(x)=\log(2+|x|)\in\mathcal{A}$, we have

$$
\begin{aligned}
& m(B_{R+r,\rho})\,l\Big(\frac{r}{\sqrt{M_\rho(R+r)T}}\Big) \\
&= m(\{x;\log(2+|x|)\leq R+r\})\,l\Big(\frac{r}{\sqrt{k(R+r)T}}\Big) \\
&\leq \frac{\pi^{d/2}}{\Gamma\big(\frac{d}{2}+1\big)}\,e^{d(R+r)}\,\frac{\sqrt{kT(R+r)}}{r}\,e^{-\frac{r^2}{2kT(R+r)}} \\
&\rightarrow 0\,, \quad \text{as} \quad r\rightarrow\infty\,,
\end{aligned}
$$

if $T<\frac{1}{2kd}$. Hence, Theorem 2.2 shows that the corresponding diffusion is conservative, improving again a result of Ichihara [7: Example 3.1]. It was known in Davies [5] that if $a_{ij}(x)=(1+|x|)^2(\log(1+|x|))^\beta\,\delta_{ij}$, $\beta>1$, the corresponding diffusion is not conservative.

3. A tightness criterion for symmetric diffusion processes

Let $\mathcal{E}^n(u,v)$ be a sequence of symmetric bilinear forms on $L^2(\boldsymbol{R}^d, m_n)$ defined by

$$
\text{(3.1)}\qquad \mathcal{E}^n(u,v)=\frac{1}{2}\sum_{i,j=1}^{d}\int_{\boldsymbol{R}^d} a_{ij}^n\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dm_n\,,\quad \text{for}\quad u,v\in C_0^\infty(\boldsymbol{R}^d)
$$

where m_n is everywhere dense positive Radon measure on $\boldsymbol{R}^d$. Let the coefficients a_{ij}^n be Borel measurable functions satisfying

i) $a_{ij}^n=a_{ji}^n$

ii) for each ball B_r there exists a constant $\lambda(r)$ independent of n such that

$$
0\leq\sum_{i,j=1}^{d}a_{ij}^n(x)\,\xi_i\,\xi_j\leq\lambda(r)|\xi|^2\,,\quad \text{for any}\quad (x,\xi)\in B_r\times\boldsymbol{R}^d\,.
$$

For each ball B_r, let $\mathcal{E}^{n,B_r}$ be the symmetric form defined by (2.8) with a_{ij} and m replaced by a_{ij}^n and m_n. We assume the closability of $(\mathcal{E}^n, C_0^\infty(\boldsymbol{R}^d))$ and $(\mathcal{E}^{n,B_r}$,

$C^\infty(B_r)$) on $L^2(\boldsymbol{R}^d, m_n)$ and $L^2(B_r, m_n)$ respectively. The corresponding closure will be denoted by $(\mathcal{E}^n, \mathcal{F}^n)$ and $(\mathcal{E}^{n,r}, \mathcal{F}^{n,r})$. Furthermore, we assume

Condition I There exists a constant $T>0$ such that for any $R>0$

$$\sup_n \{m_n(B_{R+r})\, l\Big(\frac{r}{\sqrt{\lambda(R+r)T}}\Big)\} \to 0\,, \quad \text{as} \quad r \to \infty\,.$$

Note that, by virtue of Theorem 2.2, Condition I implies that the diffusion processes $\boldsymbol{M}^n=(P^n_x, X_t)$ corresponding to $(\mathcal{E}^n, \mathcal{F}^n)$ are conservative.

For probability measures μ_n on $\boldsymbol{R}^d$ we define the probability measures $P^n_{\mu_n}$ on $C([0, \infty)\to\boldsymbol{R}^d)$ by

$$P^n_{\mu_n}[\cdot] = \int_{\boldsymbol{R}^d} P^n_x[\cdot]\, d\mu_n$$

where $C([0, \infty)\to\boldsymbol{R}^d)$ is the space of all continuous functions from $[0, \infty)$ into $\boldsymbol{R}^d$. Now, we give the sufficient conditions for the sequence of probability measures $P^n_{\mu_n}$ to be tight.

We consider the following conditions:

Condition II i) $\sup\limits_n m_n(K)<\infty$ for any cmpact set K.

ii) μ_n is absolutely continuous with respect to m_n, say $\mu_n=\varphi_n\, m_n$, and a sequence $\{\varphi_n\}$ satisfies that $\sup\limits_n \|\varphi_n\|_{\infty,K}(=\sup\limits_n \operatorname{ess\cdot sup}\limits_{x\in K} |\varphi_n(x)|)<\infty$ for any compact set K.

iii) $\{\mu_n\}$ is tight.

Theorem 3.1. *Under Condition I and II, the sequence of probability measures $P^n_{\mu_n}$ is tight on the space $C([0, \infty)\to\boldsymbol{R}^d)$ equipped with the local uniform topology.*

Proof. For $\delta>0$, put $q^n_{h,L}(x)=P^n_x[\sup\limits_{\substack{0\le s,t\le L\\ |t-s|\le h}} |X^i_t-X^i_s|>\delta]$. Here X^i_t is the i-th component of the diffusion process X_t. Note that $(q^n_{h,L}, \varphi_n)_{m_n}\le\|\varphi_n\|_{\infty,B_R}(q^n_{h,L}, \chi_{B_R})_{m_n}+\mu_n(B^C_R)$. Thus, by Condition II ii) and iii), if we can show that

$$\lim_{h\to 0}\sup_n (q^n_{h,L}, \chi_{B_R})_{m_n} = 0\,, \quad \text{for any} \quad L, R>0\,, \tag{3.2}$$

then we have $\lim\limits_{h\to 0}\sup\limits_n (q^n_{h,L}, \varphi_n)_{m_n}=0$, for any $L>0$, and arrive at this theorem.

Let $T'=\dfrac{4}{9}T$. Then

$$\begin{aligned}(q^n_{h,T'}, \chi_{B_R})_{m_n} = {} & P^{n,R+r}_{\chi_{B_R}m_n}[\sup_{\substack{0\le s,t\le T'\\ |t-s|\le h}} |X^i_t-X^i_s|>\delta;\ \Lambda_r]+ \\ & P^n_{\chi_{B_R}m_n}[\sup_{\substack{0\le s,t\le T'\\ |t-s|\le h}} |X^i_t-X^i_s|>\delta;\ \Lambda^C_r]\,,\end{aligned} \tag{3.3}$$

where $\Lambda_r=\{\omega;\ \sup_{0\leq t\leq T'}(|X_t|-|X_0|)<r\}$ and $\boldsymbol{M}^{n,r}=(P_x^{n,r}, X_t)$ is the diffusion process corresponding to $(\mathcal{E}^{n,r}, \mathcal{F}^{n,r})$. Since it follows from the formula (1.2) that

$$X_t^i-X_s^i=\frac{1}{2}(M_t^{[x_i]}-M_s^{[x_i]})+\frac{1}{2}(M_{T'-t}^{[x_i]}(r_{T'})-M_{T'-s}^{[x_i]}(r_{T'})),\ P_{\chi_{B_{R+r}}m_n}^{n,R+r}\text{—a.e.}\,,$$

the first term of the right hand side of (3.3) is dominated by

$$\begin{aligned}(3.4)\qquad & P_{\chi_{B_{R+r}}m_n}^{n,R+r}[\sup_{\substack{0\leq s,t\leq T'\\ |t-s|\leq h}}|X_t^i-X_s^i|>\delta] \\ &\leq P_{\chi_{B_{R+r}}m_n}^{n,R+r}[\sup_{\substack{0\leq s,t\leq T'\\ |t-s|\leq h}}|M_t^{[x_i]}-M_s^{[x_i]}|>\delta]+ \\ &\quad P_{\chi_{B_{R+r}}m_n}^{n,R+r}[\sup_{\substack{0\leq s,t\leq T'\\ |t-s|\leq h}}|M_t^{[x_i]}(r_{T'})-M_s^{[x_i]}(r_{T'})|>\delta] \\ &= 2\,P_{\chi_{B_{R+r}}m_n}^{n,R+r}[\sup_{\substack{0\leq s,t\leq T'\\ |t-s|\leq h}}|M_t^{[x_i]}-M_s^{[x_i]}|>\delta]\,.\end{aligned}$$

It is clear that

$$\begin{aligned}&\{\omega;\ \sup_{\substack{0\leq s,t\leq T'\\ |t-s|\leq h}}|M_t^{[x_i]}-M_s^{[x_i]}|>\delta\} \\ &=\{\omega;\ \sup_{\substack{0\leq s,t\leq T'\\ |t-s|\leq h}}|\boldsymbol{B}(\int_0^t a_{ii}(X_u)\,du)-\boldsymbol{B}(\int_0^s a_{ii}(X_u)\,du)|>\delta\} \\ &\subset\{\omega;\ \sup_{\substack{0\leq s,t\leq\lambda(T+r)T'\\ |t-s|\leq\lambda(R+r)h}}|\boldsymbol{B}(t)-\boldsymbol{B}(s)|>\delta\},\ P_x^{n,R+r}\text{-a.e., q.e. } x\,,\end{aligned}$$

where $\boldsymbol{B}(t)$ is the one-dimensional Brownian motion with respect to $P_x^{n,R+r}$. Therefore, denoting by $(W=C([0,\infty)\rightarrow\boldsymbol{R}^d), P^w)$ the standard Wiener space and setting $\gamma(h,r)=P^w[w\in W;\ \sup_{\substack{0\leq s,t\leq\lambda(r)T'\\ |t-s|\leq\lambda(r)h}}|w(t)-w(s)|>\delta]$, the last term of (3.4) is not greater than $2\,m_n(B_{R+r})\,\gamma(h, R+r)$.

On the other hand, according to Lemma 2.1 we have $P_{\chi_{B_R}m_n}^n[\Lambda_r^C]\leq 6\,m_n(B_{R+r})\,l\Big(\frac{r}{\sqrt{\lambda(R+r)\,T}}\Big)$. Hence, we see that the right hand side of (3.3) is dominated by $2\,m_n(B_{R+r})\,\gamma(h, R+r)+6\,m_n(B_{R+r})\,l\Big(\frac{r}{\sqrt{\lambda(R+r)\,T}}\Big)$, and consequently

$$(3.5)\qquad \lim_{h\to 0}\sup_n\,(q_{h,T'}^n, \chi_{B_R})_{m_n}=0\,,\quad \text{for any}\quad R>0$$

by virtue of Condition I and Condition II i).

Note that by the Markov property

$$(3.6)\qquad P_{\chi_{B_R}m_n}^n[\sup_{\substack{\beta\leq s,t\leq T'+\beta\\ |t-s|\leq h}}|X_t^i-X_s^i|>\delta]=(P_\beta^n(q_{h,T'}^n), \chi_{B_R})_{m_n}$$

$$= (q^n_{h,T'}, \chi_{B_{R'}} P^n_\beta(\chi_{B_R}))_{m_n} + (q^n_{h,T'}, \chi_{B^c_{R'}} P^n_\beta(\chi_{B_R}))_{m_n}$$
$$\leq (q^n_{h,T'}, \chi_{B_{R'}})_{m_n} + (\chi_{B_R}, P^n_\beta(\chi_{B^c_{R'}}))_{m_n} .$$

Thus, it follow from (3.5) and Lemma 2.1 that for $0 \leq \beta \leq T'$

$$\overline{\lim_{h\to 0}} \sup_n P^n_{\chi_{B_R} m_n}[\sup_{\substack{\beta \leq s,t \leq T'+\beta \\ |t-s| \leq h}} |X^i_t - X^i_s| > \delta]$$
$$\leq \overline{\lim_{R'\to\infty}} \overline{\lim_{h\to 0}} \sup_n \{(q^n_{h,T'}, \chi_{B_{R'}})_{m_n} + (\chi_{B_R}, P^n_\beta(\chi_{B^c_{R'}}))_{m_n}\} = 0 ,$$

and, consequently, $\lim_{h\to 0} \sup_n (q^n_{h,T'+\beta}, \chi_{B_R})_{m_n} = 0$ for any $R > 0$. By repeating this argument, (3.2) is established. q.e.d.

EXAMPLE 3 Let ψ be a positive Borel function such that $\psi \in L^2_{\text{loc}}(\boldsymbol{R}^d, dx)$ and $\{\psi_n\}$ be an increasing sequence of positive Borel functions bounded by ψ, i.e.,

$$0 < \psi_1 \leq \psi_2 \leq \cdots \leq \psi .$$

Putting $m_n = \psi_n^2\, dx$ and $m = \psi^2\, dx$, we define Dirichlet spaces $(\mathcal{E}^n, \mathcal{F}^n)$ and $(\mathcal{E}, \mathcal{F})$ by

$$(3.8) \qquad \begin{cases} \mathcal{E}^n(u,v) = \dfrac{1}{2} \displaystyle\sum_{i=1}^d \int_{\boldsymbol{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dm_n \\ \mathcal{F}^n\text{: the closure of } C_0^\infty(\boldsymbol{R}^d) \text{ in } L^2(\boldsymbol{R}^d, m_n) \text{ with respect to} \\ \qquad \mathcal{E}^n + (\ ,\ )_{m_n} \end{cases}$$

and

$$(3.9) \qquad \begin{cases} \mathcal{E}(u,v) = \dfrac{1}{2} \displaystyle\sum_{i=1}^d \int_{\boldsymbol{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dm \\ \mathcal{F}\text{: the closure of } C_0^\infty(\boldsymbol{R}^d) \text{ in } L^2(\boldsymbol{R}^d, m) \text{ with respect to} \\ \qquad \mathcal{E} + (\ ,\ )_m \end{cases}$$

Let $\boldsymbol{M}^n = (P^n_x, X_t)$ and $\boldsymbol{M} = (P_x, X_t)$ be the diffusion processes associated with Dirichlet spaces (3.8) and (3.9) respectively. Then, if $m(B_r) \leq c_1 e^{c_2 r^2}$, Condition I is satisfied for $T < \dfrac{1}{2c_2}$. Hence, it follows from Theorem 3.1 that for $f \in L^1(m) \cap L^\infty(m)$ with $f \geq 0$, m-a.e. the sequence of probability measures $\{P^n_{k_n f m_n}, k_n = \dfrac{1}{\int f m_n}\}$ is tight on $C[([0,\infty) \to \boldsymbol{R}^d)$.

Suppose further that, for some (possibly empty) closed set K,

$$(3.10) \qquad \begin{array}{ll} \text{i)} & \mathrm{Cap}(K) = 0 \\ \text{ii)} & \dfrac{\psi_n}{\psi} \to 1 \quad \text{uniformly on any compact set} \quad K' \subset \boldsymbol{R}^d - K . \end{array}$$

Then we can conclude that $P^n_{k_n f m_n}$ converges to P_{kfm} weakly. Here Cap is the capacity associated with the Dirichlet form (3.9) and $k=\frac{1}{\int f dm}$. In fact, let $O_{r,\delta}=\{x\in\boldsymbol{R}^d;\ \inf\{|x-y|;\ y\in B_r\cap K\}>\delta\}$ and $\tau_{r,\delta}=\inf\{t;\ X_t\notin\mathring{B}_t\cap O_{r,\delta}\}$. In the similar manner as we have in the derivation of (3.2) from (3.5), we can show that for any $R, L>0$ $\lim_{r\to\infty}\sup_n P^n_{\chi_{B_R}m_n}[\sup_{0\le t\le L}(|X_t|-|X_0|)\ge r]=0$. Therefore, in view of Lemma 5.11 in [6], we see that for any $\varepsilon>0$ and any $L>0$ there exist r and δ such that

$$\sup_n P^n_{k_n f m_n}[L\ge\tau_{r,\delta}]+P_{kfm}[L\ge\tau_{r,\delta}]<\varepsilon\,.$$

Note that for $\Lambda=\{\omega;\ X_{t_1}\in A_1,\cdots,X_{t_p}\in A_p\}$ $(0<t_1<\cdots<t_p,\ A_i\in\mathcal{B}(\boldsymbol{R}^d))$

$$\begin{aligned}(3.11)\quad &|P^n_{k_n f m_n}[\Lambda]-P_{kfm}[\Lambda]|\\ &\le|P^n_{k_n f m_n}[\Lambda]-P^n_{k_n f m_n}[\Lambda\cap\{t_p<\tau_{r,\delta}\}]|+|P^n_{k_n f m_n}[\Lambda\cap\{t_p<\tau_{r,\delta}\}]\\ &-P_{kfm}[\Lambda\cap\{t_p<\tau_{r,\delta}\}]|+|P_{kfm}[\Lambda\cap\{t_p<\tau_{r,\delta}\}]-P_{kfm}[\Lambda]|\\ &\le P^n_{k_n f m_n}[t_p\ge\tau_{r,\delta}]+P_{kfm}[t_p\ge\tau_{r,\delta}]+|P^n_{k_n f m_n}[\Lambda\cap\{t_p<\tau_{r,\delta}\}]\\ &-P_{kfm}[\Lambda\cap\{t_p<\tau_{r,\rho}\}]|\,.\end{aligned}$$

Then, since Condition ii) of (3.10) and Theorem 5 in [2] imply that the last term of (3.11) tends to zero by letting n to infinity, we have the stated weak convergence. By combining Theorem 3.1 with some other statements on the semi-group convergence in [2] and [3], we can get the corresponding weak convergence statements.

4. Preliminary estimates

Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet space on $L^2(X,m)$ as in §2. We assume that the corresponding diffusion $\boldsymbol{M}=(P_x,X_t)$ is conservative. Set

$$\mathcal{F}_{\mathrm{loc},\alpha}=\{\rho\in\mathcal{F}_{\mathrm{loc,ac}};\ \Gamma(\rho)\le\alpha,\ m\text{-a.e.}\},\ \alpha>0\,,$$

and denote by $\mathcal{B}(X)$ the family of the Borel sets of X. Then, we have

Lemma 4.1. *For $A, B\in\mathcal{B}(X)$ and $\rho\in\mathcal{F}_{\mathrm{loc},\alpha}\cap C(X)$*

$$(4.1)\qquad P_m[X_0\in A,\ X_T\in B]\le 2(m(A)+m(B))\,l\Big(\frac{\rho(A,B)}{\sqrt{\alpha T}}\Big)\,.$$

Here, $\rho(A,B)=\inf\{\rho(x)-\rho(y);\ x\in A,\ y\in B\}\vee\inf\{\rho(y)-\rho(x);\ x\in A,\ y\in B\}$.

Proof. Suppose that $\rho(A,B)=\inf\{\rho(y)-\rho(x);\ x\in A, y\in B\}$. Then, since $\rho(X_T)-\rho(X_0)=\frac{1}{2}M^{[\rho]}_T-\frac{1}{2}M^{[\rho]}_T(r_T)$, P_m-a.e. by the formula (1.2) with $t=T$, we have

$$(4.2)\quad P_m[X_0\in A,\ X_T\in B] = P_m[X_0\in A,\ X_T\in B,\ \rho(X_T)-\rho(X_0)\geq\rho(A,B)]$$
$$\leq P_m[X_0\in A,\ X_T\in B,\ M_T^{[\rho]}\geq\rho(A,B)]+P_m[X_0\in A,\ X_T\in B,$$
$$-M_T^{[\rho]}(r_T)\geq\rho(A,B)]\,.$$

The first term of the right hand side of the inequality (4.2) is not greater than

$$(\chi_A,\ P_x[\sup_{0\leq s\leq T} M_s^{[\rho]}\geq\rho(A,B)])_m$$
$$=(\chi_A,\ P_x[\sup_{0\leq s\leq T}\boldsymbol{B}(\int_0^s\Gamma(\rho)(X_u)\,du)\geq\rho(A,B)])_m$$
$$\leq(\chi_A,\ P_x[\sup_{0\leq s\leq \alpha T}\boldsymbol{B}(s)\geq\rho(A,B)])_m$$
$$=2\,m(A)\,l\Big(\frac{\rho(A,B)}{\sqrt{\alpha T}}\Big),$$

where $\boldsymbol{B}(s)$ denotes a one-dimensional Brownian motion with respect to P_x for q.e. $x\in X$. Moreover, the second term of the right hand side of inequality (4.2) is equal to $P_m[X_0\in B,\ X_T\in A,\ -M_T^{[\rho]}\geq\rho(A,B)]$. Therefore, in the same manner as above, we see that it is not greater than $2\,m(B)\,l\Big(\frac{\rho(A,B)}{\sqrt{\alpha T}}\Big)$, and thus we obtain the inequality (4.1). Noting that the left hand side of (4.1) is equal to $P_m[X_0\in B,\ X_T\in A]$, we attain the estimate (4.1) in the case that $\rho(A,B)=\inf\{\rho(x)-\rho(y);\ x\in A, y\in B\}$ as well. q.e.d.

For $\rho\in\mathcal{F}_{\mathrm{loc}}\cap C(X)$ we let $T_{r,\rho}=\{x\in X;\ r\leq\rho(x)<r+1\}$.

Corollary 4.2. *Let $\rho\in\mathcal{F}_{\mathrm{loc},\alpha}\cap C(X)$. Then, for $r>0$ and $A\in\mathcal{B}(X)$ such that $A\subset B_{R,\rho}$*

$$(4.3)\qquad P_{\chi_A m}[X_T\in T_{R+r,\rho}]\leq 2(m(A)+m(T_{R+r,\rho}))\,l\Big(\frac{r}{\sqrt{\alpha T}}\Big)\,.$$

Using (4.3), we have the next lemma.

Lemma 4.3. *Let $\rho\in\mathcal{F}_{\mathrm{loc},\alpha}\cap C(X)$ and $A\in\mathcal{B}(X)$ with $A\subset B_{R,\rho}$. Let $\Lambda\in\mathcal{F}_T$ with $P_x[\Lambda]\leq\gamma$, m-a.e.. Then, it holds that for $p, q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$,*

$$(4.4)\qquad P_{\chi_A m}[r_T\omega\in\Lambda]\leq\gamma^{1/p}\{m(B_{R,\rho})+2^{1/q}\sum_{k=0}^{\infty}m(T_{R+k,\rho})^{1/p}(m(A)$$
$$+m(T_{R+k,\rho}))^{1/q}\,l\Big(\frac{k}{\sqrt{\alpha T}}\Big)^{1/q})\}\,.$$

Proof. By Hölder's inequality, we have

$$P_{\chi_A m}[r_T\omega\in\Lambda]=P_m[\omega\in\Lambda,\ X_T\in A]$$
$$\leq\int_X P_x[\Lambda]^{1/p}\,P_x[X_T\in A]^{1/q}\,dm$$

$$\leq \gamma^{1/p} \int_X P_x[X_T \in A]^{1/q}\, dm$$

$$= \gamma^{1/p} \{\int_{B_{R,\rho}} P_x[X_T \in A]^{1/q}\, dm + \sum_{k=0}^{\infty} \int_{T_{R+k,\rho}} P_x[X_T \in A]^{1/q}\, dm\}$$

$$\leq \gamma^{1/p} \{m(B_{R,\rho}) + \sum_{k=0}^{\infty} m(T_{R+k,\rho})^{1/p} (\int_{T_{R+k,\rho}} P_x[X_T \in A]\, dm)^{1/q}\},$$

and therefore the proof is complete in view of Corollary 4.2. q.e.d.

5. A sample path property of symmetric diffusion processes

If one combines Lemma 4.3 with the first Borel-Cantelli lemma, one can prove several sample path properties of symmetric diffusion processes. For example, we have the next theorem.

Theorem 5.1. *Consider $\rho \in \mathcal{F}_{\mathrm{loc},\alpha} \cap C(X)$ such that $\rho \geq 0$.*
i) *if $m(\{r \leq \rho \leq r+1\}) \leq c\, r^D$ (c: positive constant, $D > -1$: constant),*

$$\varlimsup_{t\to\infty} \frac{\rho(X_t)}{\sqrt{\alpha(D+1)\, t \log t}} \leq 1,\ P_x\text{-a.e., } m\text{-a.e. } x\,. \tag{5.1}$$

ii) *if $m(X) < \infty$*

$$\varlimsup_{t\to\infty} \frac{\rho(X_t)}{\sqrt{2\alpha\, t \log\log t}} \leq 1,\ P_x\text{-a.e., } m\text{-a.e. } x\,. \tag{5.2}$$

REMARK 5.2. We do not know if the statement (5.2) also holds for $D = -1$.

Proof of Theorem 5.1. In what follows, $c_1, c_2, \cdots$ will denote some positive constants.
i) Let $\delta > 0$ and set $\gamma(t) = \sqrt{\alpha(D+\delta+1)\, t \log t}$. By Corollary 4.2 we see that for $\theta > 1$ and $A \subset B_{R,\rho}$

$$P_{\chi_A m}[\rho(X_{\theta^n}) \geq \gamma(\theta^n)] \leq 2 \sum_{k=0}^{\infty} (m(A) + m(T_{\gamma(\theta^n)+k,\rho}))\, l\left(\frac{\gamma(\theta^n)+k-R}{\sqrt{\alpha\theta^n}}\right). \tag{5.3}$$

First, suppose that $D \geq 0$. Then, since for any p there exists a constant $c(p)$ such that

$$\int_a^{\infty} t^p\, e^{-t^2/2}\, dt \leq c(p)\, a^{p-1}\, e^{-a^2/2} \quad \text{for} \quad a > 0\,,$$

the right hand side of (5.3) is dominated by

$$c_1 \sum_{k=0}^{\infty} (\gamma(\theta^n)+k)^D\, l\left(\frac{\gamma(\theta^n)+k-R}{\sqrt{\alpha\theta^n}}\right) \leq c_2\, \theta^{n/2} \int_{\gamma(\theta^n)-R}^{\infty} t^{D-1}\, e^{-t^2/2\alpha\theta^n}\, dt$$

$$\leq c_3\, \theta^{1/2(D+1)n} \int_{\frac{\gamma(\theta^n)-R}{\sqrt{\alpha\theta^n}}}^{\infty} t^{D-1}\, e^{-t^2/2}\, dt$$
$$\leq c_4\, \theta^{1/2(D+1-(D-\delta'+1))\, n}\,,$$

where δ' is any constant such that $0<\delta'<\delta$. In case $-1<D<0$, the right hand side of (5.3) is dominated by

$$c_5 \sum_{k=0}^{\infty} l\Big(\frac{\gamma(\theta^n)+k-R}{\sqrt{\alpha\theta^n}}\Big) \leq c_6\, \theta^{-(\delta'/2)n}\,.$$

Hence, it holds that, for $D>-1$,

$$\sum_{n=0}^{\infty} P_{\chi_A m}[\rho(X_{\theta^n}) \geq \gamma(\theta^n)] < \infty\,. \tag{5.4}$$

Next, we will show that, for $D>-1$,

$$\sum_{n=0}^{\infty} P_{\chi_A m}[\sup_{\theta^n \leq t \leq \theta^{n+1}} (\rho(X_t)-\rho(X_{\theta^n})) \geq 2\gamma(\theta^n(\theta-1))] < \infty\,. \tag{5.5}$$

Since $\rho(X_t)-\rho(X_{\theta^n}) = \frac{1}{2}(M_t^{[\rho]}-M_{\theta^n}^{[\rho]}) + \frac{1}{2}\,(M_{\theta^{n+1}-t}^{[\rho]}(r_{\theta^{n+1}}) - M_{\theta^n(\theta-1)}^{[\rho]}(r_{\theta^{n+1}}))$, we obtain

$$\begin{aligned} &P_{\chi_A m}[\sup_{\theta^n \leq t \leq \theta^{n+1}} (\rho(X_t)-\rho(X_{\theta^n}) \geq 2\gamma(\theta^n(\theta-1))] \\ &\quad \leq P_{\chi_A m}[\sup_{\theta^n \leq t \leq \theta^{n+1}} (M_t^{[\rho]}-M_{\theta^n}^{[\rho]}) \geq 2\gamma(\theta^n(\theta-1))] \\ &\quad + P_{\chi_A m}[\sup_{\theta^n \leq t \leq \theta^{n+1}} (M_{\theta^{n+1}-t}^{[\rho]}(r_{\theta^{n+1}}) - M_{\theta^n(\theta-1)}^{[\rho]}(r_{\theta^{n+1}})) \geq 2\gamma(\theta^n(\theta-1))]\,. \end{aligned} \tag{5.6}$$

The first term of the right hand side of (5.6) is equal to $P_{\chi_A m}[\sup_{0 \leq t \leq \theta^n(\theta-1)} (M_{\theta^n+t}^{[\rho]} - M_{\theta^n}^{[\rho]}) \geq 2\gamma(\theta^n(\theta-1))]$, which is dominated by $P_{\chi_A m}[\sup_{0 \leq t \leq \alpha\theta^n(\theta-1)} \boldsymbol{B}(t) \geq 2\gamma(\theta^n(\theta-1))]$ $= 2\, m(A)\, l\Big(\frac{2\gamma(\theta^n(\theta-1))}{\sqrt{\alpha\theta^n(\theta-1)}}\Big)$.

On the other hand, it holds that

$$\begin{aligned} &P_x[\sup_{\theta^n \leq t \leq \theta^{n+1}} (M_{\theta^{n+1}-t}^{[\rho]} - M_{\theta^n(\theta-1)}^{[\rho]}) \geq 2\gamma(\theta^n(\theta-1))] \\ &\quad = P_x[\sup_{0 \leq t \leq \theta^n(\theta-1)} M_t^{[\rho]} - M_{\theta^n(\theta-1)}^{[\rho]} \geq 2\gamma(\theta^n(\theta-1))] \\ &\quad \leq 2\, P_x[\sup_{0 \leq t \leq \alpha\theta^n(\theta-1)} \boldsymbol{B}(t) \geq \gamma(\theta^n(\theta-1))] \\ &\quad = 4\, l\Big(\frac{\gamma(\theta^n(\theta-1))}{\sqrt{\alpha\theta^n(\theta-1)}}\Big). \end{aligned}$$

According to Lemma 4.3 we see that the second term of (5.6) is not greater than

$$4^{1/p}\, l\Big(\frac{\gamma(\theta^n(\theta-1))}{\sqrt{\alpha\theta^n(\theta-1)}}\Big)^{1/p} \{m(B_{R,\rho}) + 2^{1/q} \sum_{k=0}^{\infty} m(T_{R+k,\rho})^{1/p}(m(T_{R+k,\rho})+m(A))^{1/q}$$

$$l\left(\frac{k}{\sqrt{\alpha\theta^{n+1}}}\right)^{1/q}\}.$$

Beside, if $D\geq 0$, the inside of { } in the above expression is dominated by

$$\begin{aligned}
&c_7\int_0^\infty (R+t)^D\, l\left(\frac{t}{\sqrt{\alpha\theta^{n+1}}}\right)^{1/q} dt \\
&\leq c_8\,\theta^{(n+1)/2q}\int_0^\infty t^{D-(1/q)}\, e^{-(t^2)/2q\alpha\theta^{n+1}}\, dt \\
&\leq c_9\,\theta^{\frac{n+1}{2q}+\left(D-\frac{1}{q}\right)\frac{n+1}{2}+\frac{n+1}{2}}\int_0^\infty t^{D-(1/q)}\, e^{-t^2/2}\, dt \\
&\leq c_{10}\,\theta^{1/2(D+1)n},
\end{aligned}$$

and if $-1<D<0$, it is dominated by

$$\begin{aligned}
&c_{11}\int_0^\infty t^{D/q}\, l\left(\frac{t}{\sqrt{\alpha\theta^{n+1}}}\right)^{1/q} dt \\
&\leq c_{12}\,\theta^{\frac{n+1}{2q}+\left(\frac{D}{p}-\frac{1}{q}\right)\frac{n+1}{2}+\frac{n+1}{2}} \\
&\leq c_{12}\,\theta^{\left(\frac{D}{2p}+\frac{1}{2}\right)n}.
\end{aligned}$$

Hence, if we choose a constant p such that $\frac{1}{2}(D+1)-\frac{D+1+\delta}{2p}<0$ in case $D\geq 0$ and $\frac{D}{2p}+\frac{1}{2}-\frac{D+1+\delta}{2p}<0$ in case $-1<D<0$, we can conclude that the statement (5.5) is true.

By virtue of (5.4), (5.5) and first Borel-Cantelli lemma, it holds that, for $P_{\chi_A m}$-a.e. ω, there exists $N(\omega)$ such that for $n\geq N(\omega)$ and $\theta^n\leq t\leq\theta^{n+1}$

$$\begin{aligned}
\rho(X_t) &= \rho(X_{\theta^n})+(\rho(X_t)-\rho(X_{\theta^n})) \\
&\leq\sqrt{\alpha(D+\delta+1)\,\theta^n\log\theta^n}+2\sqrt{\alpha(D+\delta+1)\,\theta^n(\theta-1)\log\theta^n(\theta-1)} \\
&= \sqrt{\alpha(D+\delta+1)\,\theta^n\log\theta^n}\,\left(1+2\sqrt{\theta-1}\sqrt{\frac{\log\theta^n(\theta-1)}{\log\theta^n}}\right) \\
&\leq\sqrt{\alpha(D+1)\,t\log t}\sqrt{\frac{D+\delta+1}{D+1}}\left(1+2\sqrt{\theta-1}\sqrt{\frac{n\log\theta+\log(\theta-1)}{n\log\theta}}\right).
\end{aligned}$$

Consequently

$$\varlimsup_{t\to\infty}\frac{\rho(X_t)}{\sqrt{\alpha(D+1)\,t\log t}}\leq\sqrt{\frac{D+\delta+1}{D+1}}\,(1+2\sqrt{\theta-1}),\ P_{\chi_A m}\text{-a.e.}.$$

By letting $\delta\downarrow 0$ and $\theta\downarrow 1$, we get (5.1).

ii) Let $\delta>0$ and set $\gamma(t)=\sqrt{(2+\delta)\,\alpha t\log\log t}$. Then, for $A\subset B_{R,\rho}$

$$P_{\chi_A m}[\rho(X_{\theta^n})\geq\gamma(\theta^n)]\leq 2(m(A)+m(\{\rho(x)\geq\gamma(\theta^n)\}))\, l\left(\frac{\gamma(\theta^n)-R}{\sqrt{\alpha\theta^n}}\right)$$

$$\leq c_{13} \frac{1}{\sqrt{(2+\delta') \log \log \theta^n}} e^{-\frac{(2+\delta') \log \log \theta^n}{2}}$$
$$\leq c_{14} \frac{1}{n^{\frac{2+\delta'}{2}}} \quad \text{for} \quad 0<\delta'<\delta ,$$

and

$$P_m[\sup_{\theta^n \leq t \leq \theta^{n+1}} (\rho(X_t)-\rho(X_0)) \geq 2\gamma(\theta^n(\theta-1))]$$
$$\leq 2\, m(X)\, l\Big(\frac{2\gamma(\theta^n(\theta-1))}{\sqrt{\alpha\theta^n(\theta-1)}}\Big)+4\, m(X)\, l\Big(\frac{\gamma(\theta^n(\theta-1))}{\sqrt{\alpha\theta^n(\theta-1)}}\Big)$$
$$\leq c_{15} \frac{1}{n^{\frac{2+\delta'}{2}}} \quad \text{for} \quad 0<\delta'<\delta .$$

Therefore, we can prove the statement (5.2) by the same argument as above. q.e.d.

6. An extension of Lyons-Zheng's formula

In this section, we shall extend the formula (1.2) in the case of special non-symmetric Dirichlet spaces.

Let $\mathcal{E}$ be a non-symmetric bilinear form on $L^2(\boldsymbol{R}^d, m)$ written as

$$\mathcal{E}(u, v) = \frac{1}{2} \sum_{i,j=1}^{d} \int_{\boldsymbol{R}^d} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dm + \sum_{i=1}^{d} \int_{\boldsymbol{R}^d} b_i \frac{\partial u}{\partial x_i} v\, dm , \quad \text{for} \quad u, v \in C_0^\infty(\boldsymbol{R}^d) . \tag{6.1}$$

where a_{ij}, b_i are bounded measurable functions which satisfy the following conditions:

(6.2)
i) $a_{ij} = a_{ji}$

ii) there exists a constant $\delta>0$ such that

$$\delta|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j , \quad \xi \in \boldsymbol{R}^d$$

iii) $\displaystyle\sum_{i=1}^{d} \int_{\boldsymbol{R}^d} b_i \frac{\partial \phi}{\partial x_i} dm = 0$ for any $\phi \in C_0^\infty(\boldsymbol{R}^d)$.

We set $\mathcal{E}^{(s)}(u, v)=\frac{1}{2} \sum_{i,j=1}^{d} \int_{\boldsymbol{R}^d} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dm$, and suppose that the symmetric form $\mathcal{E}^{(s)}$ is closable in $L^2(\boldsymbol{R}^d, m)$. If we denote by $\mathcal{F}$ the closure of $C_0^\infty(\boldsymbol{R}^d)$, the pair $(\mathcal{E}, \mathcal{F})$ becomes a non-symmetric Dirichlet space and we get a diffusion process $\boldsymbol{M}=(\Omega, X_t, P_x)$ through the Dirichlet space $(\mathcal{E}, \mathcal{F})$ (see S. Carrilo Menendez [4]). Here, we set $\Omega=C([0, \infty)\to\boldsymbol{R}^d)$ and define $X_t(\omega)$ as the position of $\omega\in\Omega$: $X_t(\omega)$

$=\omega(t)$. We can also define the adjoint Dirichlet form by $\hat{\mathcal{E}}(u, v)=\mathcal{E}(v, u)$ and in this case

$$\hat{\mathcal{E}}(u, v)=\frac{1}{2}\sum_{i,j=1}^{d}\int_{R^d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dm-\sum_{i=1}^{d}\int_{R^d} b_i\frac{\partial u}{\partial x_i}v\, dm$$

by the condition iii) of (6.2). Then, it was shown in Kim [8] that for $u\in\mathcal{F}$ the AF $A_t^{[u]}=\tilde{u}(X_t)-\tilde{u}(X_0)$ has the decomposition

$$A_t^{[u]}=M_t^{[u]}+N_t^{[u]},\ P_x\text{-a.e. q.e. } x\,, \tag{6.3}$$

where $M_t^{[u]}$ is a martingale AF of finite energy and $N_t^{[u]}$ is a continuous AF of zero energy. But, unlike symmetric cases, the energy of an AF A is defined as

$$\mathrm{e}(A)=\lim_{\alpha\to\infty}\alpha^2 E_m[\int_0^\infty e^{-\alpha t}(A_t)^2 dt]\,. \tag{6.4}$$

Denote by $\hat{\boldsymbol{M}}=(\Omega, X_t, \hat{P}_x)$ the diffusion process associated with the adjoint Dirichlet space $(\hat{\mathcal{E}}, \mathcal{F})$. Then, using the notions corresponding to the adjoint Dirichlet space $(\hat{\mathcal{E}}, \mathcal{F})$, the AF $A^{[u]}$ is also decomposed as

$$A_t^{[u]}=\hat{M}_t^{[u]}+\hat{N}_t^{[u]},\ \hat{P}_x\text{-a.e. q.e. } x\,. \tag{6.5}$$

Now we assume that diffusion processes $\boldsymbol{M}$ and $\hat{\boldsymbol{M}}$ are conservative. Then, the basic measure m becomes an invariant measure and the following relation holds: for $\mathcal{F}_T$-measurable function F

$$E_m[F(r_T\omega)]=\hat{E}_m[F]\,. \tag{6.6}$$

Lemma 6.1. *It holds that for* $u\in\mathcal{F}$

$$\hat{N}_t^{[u]}-2\sum_{i=1}^{d}\int_0^t (b_i\frac{\partial u}{\partial x_i})(X_s)\,ds=N_t^{[u]},\ P_m\text{-a.e.}\,. \tag{6.7}$$

For the proof we need the next proposition due to Ôshima which is an extension of Theorem 5.3.1 of [4] to non-symmetric case.

Proposition 6.2. (Ôshima)* *Let A be an AF. Then, the following two conditions are equivalent.*

i) $A=N^{[u]}$, $u\in\mathcal{F}$

ii) *A is a continuous AF such that* $\mathrm{e}(A)=0$, $\lim_{\alpha\to\infty}\alpha\, E_x[\int_0^\infty e^{-\alpha t}A_t\,dt]=0$, *q.e. x, and*

$$\lim_{\alpha\to\infty}\alpha^2 E_{vm}[\int_0^\infty e^{-\alpha t}A_t\,dt]=-\mathcal{E}(u, v)\quad \textit{for any}\quad v\in\mathcal{F}\,.$$

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*private communication

Proof of Lemma 6.1. Denote by $\hat{A}$ the generator associated with $(\hat{\mathcal{E}}, \mathcal{F})$ and $\mathcal{D}(\hat{A})$ the domain of $\hat{A}$. We first prove the lemma for $u \in \mathcal{D}(\hat{A})$. Note that $\hat{N}_t^{[u]} = \int_0^t \hat{A}u(X_s)\, ds$ and so $\hat{N}_t^{[u]}(r_t) = \hat{N}_t^{[u]}$. Then, we see that for $v \in \mathcal{F}$

$$\begin{aligned} E_{vm}[\hat{N}_t^{[u]}] &= \hat{E}_m[\hat{N}_t^{[u]}(r_t)\, \tilde{v}(X_t)] \\ &= \hat{E}_m[\hat{N}_t^{[u]}\, \tilde{v}(X_t)] \\ &= \hat{E}_m[\hat{N}_t^{[u]}\, \tilde{v}(X_t)] + \hat{E}_m[\hat{N}_t^{[u]}(\tilde{v}(X_t) - \tilde{v}(X_0))]\,. \end{aligned}$$

But since

$$\begin{aligned} &\alpha^2 |\hat{E}_m[\int_0^\infty e^{-\alpha t}\, \hat{N}_t^{[u]}(\tilde{v}(X_t) - \tilde{v}(X_0))\, dt]| \\ &\leq (\alpha^2 \hat{E}_m[\int_0^\infty e^{-\alpha t} (\hat{N}_t^{[u]})^2\, dt])^{1/2} (\alpha^2 \hat{E}_m[\int_0^\infty e^{-\alpha t} (\tilde{v}(X_t) - \tilde{v}(X_0))^2\, dt])^{1/2} \\ &\to \hat{\mathrm{e}}(\hat{N}^{[u]})^{1/2} \cdot \hat{\mathrm{e}}(A^{[v]})^{1/2} = 0\,, \quad \text{as} \quad \alpha \to \infty\,, \end{aligned}$$

it follows from Proposition 6.2

$$\lim_{\alpha \to \infty} \alpha^2 E_{vm}[\int_0^\infty e^{-\alpha t}\, \hat{N}_t^{[u]}\, dt] = \lim_{\alpha \to \infty} \alpha^2 \hat{E}_{vm}[\int_0^\infty e^{-\alpha t}\, \hat{N}_t^{[u]}\, dt] = -\hat{\mathcal{E}}(u, v)\,.$$

Hence, by the equality that

$$-\hat{\mathcal{E}}(u, v) - 2 \sum_{i=1}^d \int_{R^d} b_i \frac{\partial u}{\partial x_i} v\, dm = -\mathcal{E}(u, v)$$

we have

$$\begin{aligned} \text{(6.8)} \qquad &\lim_{\alpha \to \infty} \alpha^2 E_{vm}[\int_0^\infty e^{-\alpha t} (\hat{N}_t^{[u]} - 2 \sum_{i=1}^d \int_0^t (b_i \frac{\partial u}{\partial x_i})(X_s)\, ds)\, dt] \\ &= \lim_{\alpha \to \infty} \alpha^2 E_{vm}[\int_0^\infty e^{-\alpha t}\, N_t^{[u]}\, dt]\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{(6.9)} \qquad \mathrm{e}(\hat{N}^{[u]}) &= \lim_{\alpha \to \infty} \alpha^2 E_m[\int_0^\infty e^{-\alpha t} (\hat{N}_t^{[u]})^2\, dt] \\ &= \lim_{\alpha \to \infty} \alpha^2 \hat{E}_m[\int_0^\infty e^{-\alpha t} (\hat{N}_t^{[u]})^2\, dt] \\ &= \hat{\mathrm{e}}(\hat{N}^{[u]}) \\ &= 0\,, \end{aligned}$$

and hence $\hat{N}_t^{[u]} - 2 \sum_{i=1}^d \int_0^t (b_i \frac{\partial u}{\partial x_i})(X_s)\, ds$ is an AF of $\boldsymbol{M}$ of zero energy. (6.8), (6.9) and Proposition 6.2 lead us to the desired equality (6.7).

Next, for a general $u \in \mathcal{F}$ there exists a sequence $u_n \in \mathcal{D}(\hat{A})$ such that u_n converges to u with respect to $\mathcal{E}_1^{(s)}$ and for q.e. x

$$\hat{P}_x[\hat{\Gamma}_T] = 1$$

where $\hat{\Gamma}_T = \{\omega \in \Omega;\ \hat{N}_t^{[u_n]}(\omega)$ converges to $\hat{N}_t^{[u]}(\omega)$ uniformly in t on an interval $[0, T]\}$. Since

$$\hat{N}_t^{[u_n]}(r_T\omega) = \int_{T-t}^{T} \hat{A}u_n(X_s(\omega))\, ds = \hat{N}_T^{[u_n]}(\omega) - \hat{N}_{T-t}^{[u_n]}(\omega)\,, \tag{6.10}$$

the set $\hat{\Gamma}_T$ is r_T-invariant, i.e., $\{r_T\omega \in \hat{\Gamma}_T\} = \hat{\Gamma}_T$, and consequently the complement of $\hat{\Gamma}_T$ ($\hat{\Gamma}_T^C$ in notation) is also r_T-invariant. Hence, we have that

$$\begin{aligned} P_m[\hat{\Gamma}_T^C] &= \hat{P}_m[r_T\omega \in \hat{\Gamma}_T^C] \\ &= \hat{P}_m[\hat{\Gamma}_T^C] \\ &= 0\,, \end{aligned}$$

and consequently we can attain (6.7) for the present u by the approximation method. q.e.d.

Now, we obtain

Theorem 6.3. *For $u \in \mathcal{F}_{\text{loc}}$*

$$\begin{aligned} \tilde{u}(X_t) - \tilde{u}(X_0) &= \frac{1}{2} M_t^{[u]} - \frac{1}{2}(\hat{M}_T^{[u]}(r_T) - \hat{M}_{T-t}^{[u]}(r_T)) \\ &\quad - \sum_{i=1}^{d} \int_0^t (b_i \frac{\partial u}{\partial x_i})(X_s)\, ds \quad 0 \le t \le T,\ P_m\text{-a.e.}\,. \end{aligned} \tag{6.11}$$

Proof. By operating r_T to the formula (6.5), we have

$$\tilde{u}(X_{T-t}) - \tilde{u}(X_T) = \hat{M}_t^{[u]}(r_T) + \hat{N}_t^{[u]}(r_T),\ P_m\text{-a.e.}\,. \tag{6.12}$$

Since by the approximation method the relation (6.10) extends to $u \in \mathcal{F}_{\text{loc}}$, namely,

$$\hat{N}_t^{[u]}(r_T) = \hat{N}_T^{[u]}(r_T) - \hat{N}_{T-t}^{[u]},\ P_m\text{-a.e.}\,, \tag{6.14}$$

the right hand side of (6.11) is equal to

$$\begin{aligned} &\frac{1}{2}(\tilde{u}(X_t) - \tilde{u}(X_0) - N_t^{[u]}) - \frac{1}{2}(\tilde{u}(X_0) - \tilde{u}(X_t) + \hat{N}_{T-t}^{[u]}(r_T) - \hat{N}_T^{[u]}(r_T)) \\ &\quad - \sum_{i=1}^{d} \int_0^t (b_i \frac{\partial u}{\partial x_i})(X_s)\, ds \\ &= (\tilde{u}(X_t) - \tilde{u}(X_0)) + \frac{1}{2}(\hat{N}_t^{[u]} - N_t^{[u]} - 2\sum_{i=1}^{d} \int_0^t (b_i \frac{\partial u}{\partial x_i})(X_s)\, ds),\ P_m\text{-a.e.}\,. \end{aligned}$$

Therefore, by Lemma 6.1 the proof is complete. q.e.d.

REMARK 6.4. Using Theorem 6.3 we can obtain the results corresponding to Lemma 4.1, Corollary 4.2 and Lemma 4.3 in the present non-symmetric situation.

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Department of Mathematics
Himeji Institute of Technology
Shosya 2167, Himeji 671–22
Japan