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ON A RELATION BETWEEN HIGHER ORDER ASYMPTOTIC RISK SUFFICIENCY AND HIGHER ORDER ASYMPTOTIC SUFFICIENCY IN A LOCAL SENSE

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1. Introduction. In Takeuchi [4] higher order asymptotic risk sufficiency of maximum likelihood estimator has been discussed. In this paper we try to find some relations between asymptotic risk sufficiency with a special loss function and asymptotic sufficiency in a local sense.

Let $\mathcal{P}_n=\{P_{\theta,n};\theta\in\Theta\}$ be a family of probability distributions on a measurable space $(\mathcal{X},\mathcal{A}_n)$ with an index set Θ which is a subset of an Euclidean space with the usual norm $|\cdot|$. For a sub σ-field $\mathcal{C}$ of $\mathcal{A}_n$, real number $c\geqq 0$ and $\theta,\theta'\in\Theta$ let $r_n^{\mathcal{C}}(c:\theta,\theta')=\inf\,(1+c)^{-1}\{1-E_{P_{\theta,n}}(\phi)+cE_{P_{\theta',n}}(\phi);\ \phi$ are $\mathcal{C}$-measurable statistical test functions on $\mathcal{X}\}$. We note that $r_n^{\mathcal{C}}(c:\theta,\theta')$ means the Bayes risk of statistical problem of testing a hypothesis '$P_{\theta',n}$ is true' against an alternative '$P_{\theta,n}$ is true' with experiment $(\mathcal{X},\mathcal{C},\{P_{\theta',n},P_{\theta,n}\})$ relative to a prior probability distribution $(c/(1+c),1/(1+c))$ on $\{\theta',\theta\}$ provided that the loss function is simple.

Let $\{\mathcal{B}_n;n=1,2,\cdots\}$ be a sequence of sub σ-fields of $\{\mathcal{A}_n\}(\mathcal{B}_n\subset\mathcal{A}_n)$. In this paper we give a sufficient condition about the Bayes risk $r_n^{\mathcal{B}_n}$ for $\{\mathcal{B}_n\}$ to be higher order locally asymptotically sufficient sequence of σ-fields. More precisely our main result in this paper is the following: Under some conditions if for some positive number α $\sup_{c>0}\sup_{\theta^*\in K}\sup_{\theta:n^{1/2}|\theta-\theta^*|\leqq b}\{r_n^{\mathcal{B}_n}(c:\theta,\theta^*)-r_n^{\mathcal{A}_n}(c:\theta,\theta^*)\}=o(n^{-\alpha})$ for every $b>0$ and every compact subset K of Θ, then for every β satisfying $0<\beta<3^{-1}\alpha$ $\{\mathcal{B}_n\}$ is locally asymptotically sufficient for $\{\mathcal{P}_n\}$ with order $o(n^{-\beta})$ in the sense that for each $n=1,2,\cdots$ and each $\theta_0\in\Theta$ there exists a family $\{Q_{\theta,n}^{\theta_0};\theta\in\Theta\}$ of probability distributions on $(\mathcal{X},\mathcal{A}_n)$ for which $\mathcal{P}_n$ is sufficient σ-field and that for every $b>0$

$$\sup_{\theta:n^{1/2}|\theta-\theta_0|\leqq b}||P_{\theta,n}-Q_{\theta,n}^{\theta_0}||_{\mathcal{A}_n}=o(n^{-\beta})$$

uniformly in θ_0 over every compact subsets of Θ. Here $||\cdot||_{\mathcal{A}_n}$ means the total variation norm over $\mathcal{A}_n$.

We have discussed such a problem in the case $\alpha=\beta=0$ in Suzuki [3] under non-local situation. In LeCam [1], Chap. 5 he discusses some relations between insufficiency and deficiency in his terminology.

In Section 2 some auxiliary results about the order of asymptotic sufficiency are proved. The main theorem is stated and followed by some discussions about the asymptotic sufficiency in non-local sense in Section 3.

2. Auxiliary results. For each $n \in N = \{1, 2, \cdots\}$ let $\mathcal{P}_n = \{P_{\theta,n};\ \theta \in \Theta\}$ be a family of probability distributions on a measurable space $(\mathcal{X}, \mathcal{A}_n)$ with an index set Θ. For a subset $U(\neq \phi)$ of Θ we shall denote by $\mathcal{P}_n^U$ the totality of $P_{\theta,n}$'s satisfying $\theta \in U$. We assume that for each $n \in N$ $\mathcal{P}_n$ is dominated by a σ-finite measure μ_n on $(\mathcal{X}, \mathcal{A}_n)$. The probability density function of $P_{\theta,n}$ relative to μ_n will be denoted by $p_n(x, \theta)$. Without loss of generality we assume in the following that μ_n is a probability measure on $(\mathcal{X}, \mathcal{A}_n)$. For each $\theta, \theta' \in \Theta$ let $S_n(\theta) = \{x;\ p_n(x, \theta) > 0\}$ and let $h_n(x; \theta, \theta') = p_n(x, \theta)/p_n(x, \theta')$ if $x \in S_n(\theta')$, $= +\infty$ if $x \in S_n(\theta) \cap S_n(\theta')^c$, $=1$ if $x \in S_n(\theta)^c \cap S_n(\theta')^c$. We put $\beta_n(\theta, \theta') = P_{\theta,n}\{S_n(\theta')^c\}$. For each $\theta, \theta' \in \Theta$ and real number $s \geqq 1$ we define

$$J_n(s: \theta, \theta') = E_{P_{\theta',n}}[\{h_n(x; \theta, \theta')\}^s] .$$

We note that $\beta_n(\theta, \theta') = 1 - J_n(1: \theta, \theta')$.

Let $\{U_n\}$ be a sequence of nonempty subsets of Θ. For $\{U_n\}$ we consider the following assumption.

ASSUMPTION 1. There exist a sequence $\{\theta_n^*\}_{n \in N} (\theta_n^* \in U_n)$ and a positive number γ such that

(a) For every $s \geqq 1$

$$\lim_{b \to \infty} \sup \sup_{\theta \in U_n} J_n(s: \theta, \theta_n^*) < \infty ,$$

(b) $$\sup_{\theta \in U_n} \beta_n(\theta, \theta_n^*) = o(n^{-\gamma}) .$$

For a sub σ-field $\mathcal{C}$ of $\mathcal{A}_n$ we denote by $\Phi(\mathcal{C})$ the family of $\mathcal{C}$-measurable statistical test functions on $\mathcal{X}$. For each $\theta, \theta' \in \Theta$ and each real number $c \geqq 0$ we define

$$r_n^{\mathcal{C}}(c: \theta, \theta') = \inf\, (1+c)^{-1}\{1 - E_{P_{\theta,n}}(\phi) + cE_{P_{\theta',n}}(\phi);\ \phi \in \Phi(\mathcal{C})\} .$$

Let $\{\mathcal{B}_n\}$ be a sequence of sub σ-fields of $\{\mathcal{A}_n\}$ $(\mathcal{B}_n \subset \mathcal{A}_n)$. For each $\theta \in \Theta$ define $\bar{p}_n(x, \theta) = E_{\mu_n}[p_n(x, \theta) \mid \mathcal{B}_n]$ the conditional expectation of $p_n(x, \theta)$ given $\mathcal{B}_n$ with respect to μ_n and put $S_n'(\theta) = \{x;\ \bar{p}_n(x, \theta) > 0\}$. For $\theta, \theta' \in \Theta$ define $g_n(x; \theta, \theta') = \bar{p}_n(x, \theta)/\bar{p}_n(x, \theta')$ if $x \in S_n'(\theta')$, $= +\infty$ if $x \in S_n'(\theta')^c \cap S_n'(\theta)$, $=1$ if $x \in S_n'(\theta')^c \cap S_n'(\theta)$. For $c > 0$ and $\delta > 0$ let $E_n(c, \theta, \delta) = \{x;\ g_n(x; \theta, \theta_n^*) < c < c + \delta \leqq h_n(x; \theta, \theta_n^*)\}$ and $E_n'(c, \theta, \delta) = \{x;\ g_n(x; \theta, \theta_n^*) > c > c - \delta > h_n(x; \theta, \theta_n^*)\}$.

Proposition. *Suppose that for some positive number* α *and a sequence* $\{\theta_n^*\}_{n\in N}$ $(\theta_n^*\in U_n)$

(2.1) $$\sup_{c>0}\sup_{\theta\in U_n}\{r_n^{\mathcal{B}_n}(c\colon\theta,\theta_n^*)-r_n^{\mathcal{A}_n}(c\colon\theta,\theta_n^*)\}=o(n^{-\alpha}).$$

Then we have

(2.2) $$\sup_{c>0,\delta>0}\delta(1+c)^{-1}\lambda_n(c,\delta)=o(n^{-\alpha}),\quad \textit{and}$$
$$\sup_{c>0,\delta>0}\delta(1+c)^{-1}\lambda_n'(c,\delta)=o(n^{-\alpha})$$

where $\lambda_n(c,\delta)=\sup_{\theta\in U_n}P_{\theta_n^*,n}(E_n(c,\theta,\delta))$ *and* $\lambda_n'(c,\delta)=\sup_{\theta\in U_n}P_{\theta_n^*,n}(E_n'(c,\theta,\delta))$.

This proposition can be proved in the same way as the proof of the first and second steps of Theorem 1 in Suzuki [3]. So we shall omit the proof of the proposition.

Theorem 1. *Suppose that Assumption* 1 *is satisfied with a sequence* $\{\theta_n^*\}_{n\in N}$ *and* $\gamma>0$, *and that* $\{\mathcal{B}_n\}$ *has the property* (2.1) *with* $\beta>0$. *Then for every* β *satisfying* $0<\beta<3^{-1}\alpha$ *and* $\beta\leqq\gamma$, $\{\mathcal{B}_n\}$ *is asymptotically sufficient for* $\{\mathcal{P}_n^{U_n}\}$ *with order* $o(n^{-\beta})$ *in the following sense: For each* $n\in N$ *there exists a family* $\{q_n(x;\theta,\theta_n^*);\theta\in\Theta\}$ *of probability density functions on* $(\mathcal{X},\mathcal{A}_n)$ *relative to* μ_n *such that*

(i) *each* q_n *can be factorized as follows:*

$$q_n(x;\theta,\theta_n^*)=r_n(x;\theta,\theta_n^*)\,p_n(x,\theta_n^*)$$

where r_n *is a* $\mathcal{B}_n$*-measurable function, and*

(ii) $$\sup_{\theta\in U_n}\int_{\mathcal{X}}|p_n(x,\theta)-q_n(x;\theta,\theta_n^*)|\,d\mu_n=o(n^{-\beta}).$$

Proof. We shall divide the proof into several steps.

The first step. Suppose that Assumption 1 is satisfied with a sequence $\{\theta_n^*\}_{n\in N}$ and $\gamma>0$, and that $\{\mathcal{B}_n\}_{n\in N}$ has the property (2.1) with $\alpha>0$. Let β be any number satisfying $0<\beta<3^{-1}\alpha$ and $\beta\leqq\gamma$. Take ε_1 be any number satisfying $0<\varepsilon_1<3^{-1}\cdot(\alpha-3\beta)$. Let $\alpha_n=n^{-\beta}(\log n)^{-1}$, $m_n=n^{\varepsilon_1}$ and $i_n=[m_n\alpha_n^{-1}]+1$ where $[a]$ means the maximum integer not exceeding a. Put $(\gamma_n=)\,\gamma_n(x;\theta,\theta_n^*)=|h_n(x;\theta,\theta_n^*)-g_n(x;\theta,\theta_n^*)|$, $(\gamma_n'=)\,\gamma_n'(x;\theta,\theta_n^*)=|h_n(x;\theta,\theta_n^*)-I_{W_n}(x)g_n(x;\theta,\theta_n^*)|$ and

$$\rho_n(\theta,\theta_n^*)=\int_{\mathcal{X}}\gamma_n'(x;\theta,\theta_n^*)\,dP_{\theta_n^*,n}$$

where $W_n=W_n(\theta,\theta_n^*)=\{x;g_n(x;\theta,\theta_n^*)\leqq m_n\}$ and I_{W_n} means the indicator function of W_n.

We have

$$\sup_{\theta\in U_n}\rho_n(\theta,\theta_n^*)\leqq \sup_{\theta\in U_n}\int_{W_n}\gamma_n\,dP_{\theta_n^*,n}+\sup_{\theta\in U_n}\int_{W_n^c}h_n(x;\theta,\theta_n^*)\,dP_{\theta_n^*,n}=J_n^*+J_n^{**},$$

(2.3) and $J_n^*=\sup_{\theta\in U_n}\int_{W_n}\gamma_n\,dP_{\theta_n^*,n}\leqq\alpha_n+\sup_{\theta\in U_n}\int_{D_n\cap W_n}\gamma_n\,dP_{\theta_n^*,n}=\alpha_n+I_n$

$(D_n=\{x;\gamma_n\geqq\alpha_n\})$.

Furthermore we have

$$I_n=\sup_{\theta\in U_n}\int_{D_n\cap W_n}\gamma_n\,dP_{\theta_n^*,n}\leqq\sup_{\theta\in U_n}\int_{D_n\cap\widetilde{W}_n}\gamma_n\,dP_{\theta_n^*,n}+\sup_{\theta\in U_n}\int_{W_n^*}\gamma_n\,dP_{\theta_n^*,n}$$

where $W_n'=\{x;h_n(x;\theta,\theta_n^*)\leqq m_n\}$, $\widetilde{W}_n=W_n\cap W_n'$ and $W_n^*=W_n\cap(W_n')^c$.

The second step. It holds that

$$\begin{aligned}I_n'&=\sup_{\theta\in U_n}\int_{D_n\cap\widetilde{W}_n}\gamma_n\,dP_{\theta_n^*,n}\leqq\sum_{i=1}^{2i_n-2}\sup_{\theta\in U_n}\int_{B_i}\gamma_n\,dP_{\theta_n^*,n}\\&\quad+\sum_{i=0}^{2i_n-3}\sup_{\theta\in U_n}\int_{C_i}\gamma_n\,dP_{\theta_n^*,n}\\&=I_{n,1}'+I_{n,2}'\end{aligned}\tag{2.4}$$

where $B_j=\widetilde{W}_n\cap\{x;h_n(x;\theta,\theta_n^*)\geqq 2^{-1}(i+1)\,\alpha_n,\,g_n(x;\theta,\theta_n^*)<2^{-1}i\,\alpha_n\}$ and $C_i=\widetilde{W}_n\cap\{x;h_n(x;\theta,\theta_n^*)<2^{-1}(i+1)\,\alpha_n,\,g_n(x;\theta,\theta_n^*)\geqq 2^{-1}(i+2)\,\alpha_n\}$. Using the property (2.2) in Proposition we can evaluate $I_{n,i}'(i=1,2)$ as follows. Taking account of $3\varepsilon_1<\alpha-3\beta$ we have

$$\begin{aligned}I_{n,1}'&=\sum_{i=1}^{2i_n-2}\sup_{\theta\in U_n}\int_{B_i}\gamma_n\,dP_{\theta_n^*,n}\\&\leqq 2i_n\,m_n[\sup_{1\leqq i\leqq 2i_i-2}\ \sup_{\theta\in U_n}P_{\theta_n^*,n}\{x;h_n(x;\theta,\theta_n^*)\geqq 2^{-1}(i+1)\,\alpha_n,\\&\quad g_n(x;\theta,\theta_n^*)<2^{-1}i\,\alpha_n\}]\\&\leqq 2i_n\,m_n[\sup_{1\leqq i\leqq 2i_n-2}\lambda_n(2^{-1}i\,\alpha_n,\,2^{-1}\alpha_n)]\\&\leqq 4i_n\,m_n[\sup_{1\leqq i\leqq 2i_n-2}\alpha_n^{-1}(1+2^{-1}i\,\alpha_n)\,n^{-\alpha}\,\eta_n']\ (\eta_n'=o(1))\\&\leqq 4i_n^2\,m_n\,n^{-\alpha}\,\eta_n'\\&\leqq A_1\cdot n^{-(\alpha-2\beta-3\varepsilon_1)}(\log n)^2\,\eta_n'\quad(A_1\text{ is a constant})\\&=o(n^{-\beta}).\end{aligned}\tag{2.5}$$

Similarly we have

$$I_{n,2}'=o(n^{-\beta}).\tag{2.6}$$

Thus from (2.4) and (2.5) we have

$$I_n'=o(n^{-\beta}).\tag{2.7}$$

The third step. Next we evaluate I_n' as follows. For every $s>1$ we have

$$I_n'' = \sup_{\theta\in U_n} \int_{W_n^*} \gamma_n \, dP_{\theta_q^*,n} \leq \sup_{\theta\in U_n} \int_{\{h_n>m_n\}} h_n(x;\theta,\theta_n^*)\, dP_{\theta_n^*,n}$$
$$\leqq (m_n)^{1-s} \sup_{\theta\in U_n} J_n(s:\theta,\theta_n^*)\,.$$

Hence we have

$$I_n'' \leqq A_2(s)\,(m_n)^{1-s} = A_2(s)\, n^{(1-s)\varepsilon_1}$$

where $A_2(s)$ is some constant depending only on s. We can choose $s>1$ large enough so that

$$I_n'' = o(n^{-\beta})\,. \tag{2.8}$$

From (2.7) and (2.8) we have

$$I_n = o(n^{-\beta})\,.$$

Hence from (2.3) we have

$$J_n^* = o(n^{-\beta})\,.$$

Put $W_n''=\{x;\, h_n(x;\theta,\theta_n^*)<2^{-1} m_n\}$. Then we have

$$\begin{aligned} J_n^{**} &= \sup_{\theta\in U_n} \int_{W_n^c} h_n(x;\theta,\theta_n^*)\, dP_{\theta_n^*,n} \\ &\leq \sup_{\theta\in U_n} \int_{W_n^c\cap W_n''} h_n(x;\theta,\theta_n^*)\, dP_{\theta_n,n} + \sup_{\theta\in U_n} \int_{W_n^c\cap (W_n'')^c} h_n(x;\theta,\theta_n^*)\, dP_{\theta_n^*,n} \\ &\leqq 2^{-1} m_n\, \lambda_n'(m_n, m_n/2) + (m_n/2)^{1-s} \sup_{\theta\in U_n} J_n(s:\theta,\theta_n^*)\,. \end{aligned} \tag{2.9}$$

The first term on the right hand side is of order $o(n^{-\beta})$ by Proposition. The similar consideration as the evaluation of I_n'' implies that the second term of (2.9) is also of order $o(n^{-\beta})$ for sufficiently large number s. Thus we have

$$J_n^{**} = o(n^{-\beta})\,.$$

Hence it follows from (2.3) that

$$\sup_{\theta\in U_n} \rho_n(\theta,\theta_n^*) = o(n^{-\beta})\,. \tag{2.10}$$

The fourth step. Let $a_n(\theta,\theta_n^*)=[\int_{\mathcal{X}} I_{W_n}(x)\, g_n(x;\theta,\theta_n^*)\, dP_{\theta_n^*,n}]^{-1} (\leqq\infty)$ and let $r_n(x;\theta,\theta_n^*)=a_n(\theta,\theta_n^*)\, I_{W_n}(x)\, g_n(x;\theta,\theta_n^*)$ if $a_n(\theta,\theta_n^*)<\infty$,$=1$ otherwise. Define $q_n(x;\theta,\theta_n^*)=r_n(x;\theta,\theta_n^*)\, p_n(x,\theta_n^*)$ and let $Q_{\theta,n}^{\theta_n^*}$ be the probability distribution on $(\mathcal{X}, \mathcal{A}_n)$ with density $q_n(x;\theta,\theta_n^*)$ relative to μ_n. We note that $\mathcal{B}_n$ is sufficient σ-field for the family $\{Q^{\theta_n^*}_{\theta,n};\,\theta\in\Theta\}$ by the factorization theorem. It follows from (2.10) that there exists n_0 such that $a_n(\theta,\theta_n^*)<\infty$ for every $n\geqq n_0$ and every $\theta\in U_n$. Therefore we can assume without loss of generality that $a_n(\theta,\theta_n^*)<\infty$ for every $\theta\in U_n$ and every $n\geqq 1$.

Under this circumstances we have

$$\begin{aligned}
\|P_{\theta,n}-Q^{\theta_n^*}_{\theta,n}\|_{\mathcal{A}_n} &= \int_{\mathcal{X}}|p_n(x,\theta)-q_n(x;\theta,\theta_n^*)|\,d\mu_n \\
&\leqq \int_{S_n(\theta_n^*)} |h_n(x;\theta,\theta_n^*)-a_n(\theta,\theta_n^*)\,I_{W_n}(x)\,g_n(x;\theta,\theta_n^*)| \\
&\qquad \cdot p_n(x,\theta_n^*)\,d\mu_n+\beta_n(\theta,\theta_n^*) \\
&= \rho_n(\theta,\theta_n^*)+|1-a_n(\theta,\theta_n^*)^{-1}|+\beta_n(\theta,\theta_n^*) \\
&\leqq 2\,\rho_n(\theta,\theta_n^*)+2\,\beta_n(\theta,\theta_n^*)\,.
\end{aligned}$$

Here $\|\nu\|_{\mathcal{A}_n}$ means the total variation norm of a signed measur ν on $(\mathcal{X},\mathcal{A}_n)$. From Assumption 1, (b) and (2.10) we have

$$\sup_{\theta\in U_n}\|P_{\theta,n}-Q^{\theta_n^*}_{\theta,n}\|_{\mathcal{A}_n}=o(n^{-\beta})\,.$$

This completes the proof of the theorem.

3. The order of local asymptotic sufficiency. In this section the index set Θ is assumed to be a subset of p-dimensional Euclidean space R^p. We denote by $|\cdot|$ the usual Euclidean norm in R^p. For $\theta\in\Theta$ and $b>0$ let $U_n(\theta,b)=\{\theta'\in\Theta;\,n^{1/2}|\theta'-\theta|\leqq b\}$.

Let $\{\mathcal{B}_n\}_{n\in N}$ be the sequence of sub σ-fields $\mathcal{B}_n\subset\mathcal{A}_n$ as in the previous section. We consider the following assumption which will be used to prove our main theorem, Theorem 2.

ASSUMPTION 2. For every compact subset K of Θ and $b>0$

(a) $\displaystyle\limsup_{n\to\infty}\ \sup_{\theta^*\in K}\ \sup_{\theta\in U_n(\theta^*,b)} J_n(s:\theta,\theta^*)<\infty\quad(\forall\, s>1)$, and

(b) $\displaystyle\sup_{\theta^*\in K}\ \sup_{\theta\in U_n(\theta^*,b)} \beta_n(\theta,\theta^*)=o(n^{-\gamma})$.

Let α be a given positive number. We state a result about higher order locally asymptotic sufficiency of $\{\mathcal{B}_n\}$ for $\{\mathcal{P}_n\}$.

Theorem 2. *Suppose that Assumption 2 is satisfied with $\gamma>0$, and that for every compact subset K of Θ and every $b>0$*

(3.1) $$\sup_{c>0}\ \sup_{\theta^*\in K}\ \sup_{\theta\in U_n(\theta^*,b)}\{r_n^{\mathcal{B}_n}(c:\theta,\theta^*)-r_n^{\mathcal{A}_n}(c:\theta,\theta^*)\}=o(n^{-\alpha})\,.$$

Then for every positive number β satisfying $\beta<3^{-1}\alpha$ and $\beta\leqq\gamma$ $\{\mathcal{B}_n\}_{n\in N}$ is locally asymptotically sufficient for $\{\mathcal{P}_n\}$ with order $o(n^{-\beta})$ in the following sense: For each

$n\in N$ and each $\theta_0\in\Theta$ there exists a family $Q_n^{\theta_0}=\{Q_{\theta,n}^{\theta_0};\theta\in\Theta\}$ of probability distributions on $(\mathcal{X},\mathcal{A}_n)$ such that

(i) *$\mathcal{B}_n$ is sufficient for $Q_n^{\theta_0}$, and*

(ii) *for every compact subset K of Θ and every $b>0$*

$$\sup_{\theta\in K}\sup_{\theta\in U_n(\theta,b)}\|P_{\theta,n}-Q_{\theta,n}^{\theta_0}\|_{\mathcal{A}_n}=o(n^{-\beta}).$$

Since the above result follows directly from Theorem 1 we shall omit the proof.

It is open problem whether non-local version of Theorem 2 still holds or not, i.e., whether any conditions such as in Theorem 2 imply the followings or not: There exists a sequence $Q_n=\{Q_{\theta,n};\theta\in\Theta\}$ of probability distributions on $(\mathcal{X},\mathcal{A}_n)$ such that $\mathcal{B}_n$ is sufficient for Q_n, and that for every compact subset K of Θ

$$\sup_{\theta\in K}\|P_{\theta,n}-Q_{\theta,n}\|_{\mathcal{A}_n}=o(n^{-\beta}). \tag{3.2}$$

The case of $\alpha=\beta=0$ has been discussed in Suzuki [3] in such a non-local situation.

It is well known that under some regularity conditions there exist a sequence $\{\hat{\theta}_n\}_{n\in N}$ of estimators of θ, a positive number γ and a number $v\geqq 1$ having the following property: For every compact subset K of Θ there corresponds $a(K)$ such that

$$\sup_{\theta\in K}P_{\theta,n}\{n^{1/2}|\hat{\theta}_n(x)-\theta|\geqq a(K)(\log n)^{v/2}\}=o(n^{-\gamma})$$

(c.f. Matsuda [2], Chap. 3).

Using such an estimator $\{\hat{\theta}_n\}$ we may be able to construct $\{Q_{\theta,n};\theta\in\Theta\}$ satisfying the property (3.2), and for which $\mathcal{B}_n$ is sufficient.

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References

[1] L. LeCam: Asymptotic methods in statistical decision theory. Springer-Verlag, New York, 1986.

[2] T. Matsuda: Tokeiteki suisoku no zenkinriron (Asymptotic theory of statistical inference). (In Japanese), Hakuto Shobo, Tokyo, 1985.

[3] T. Suzuki: *A characterization of approximate sufficiency*. Memoirs of the School

of Science & Engineering Waseda Univ. **45** (1981), 113–122.
[4] K. Takeuchi: *Higher order asymptotic efficiency of estimators in decision procedures.* Statistical Decision Theory and Related Topics III. S.S. Gupta and J.O. Berger, eds. Academic, New York, 1982.

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