Harada, M.
Osaka J. Math.
24 (1987), 373-389

GENERALIZATIONS OF NAKAYAMA RING V

(LEFT SERIAL RINGS WITH (*, 2))

MANABU HARADA

(Received February 3, 1986)

We have studied a left serial algebra over an algebraically closed field with (*, n) as right modules in [4] and further investigated an artinian left serial ring R with (*, 1) in [7], when eJ/eJ^2 is square-free for each primitive idempotent e, where J is the Jacobson radical of R. On the other hand, we have given a characterization of a certain artinian ring with (*, 3) in [6].

For a left serial ring R, we shall obtain, in the second section of this paper, a characterization of R with (*, 1) (Theorem 1), and one of R with (*, 2) (Theorem 2) in the third section. We shall study hereditary rings with (*, 2) in the forthcoming paper.

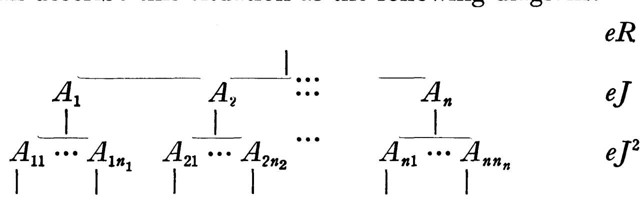
In order to give a complete study of a left serial ring with (*, 1), we need deep properties of a division ring (much more difficult than Artin problem, see (♯)).

We shall use the same terminologies given in [7] and every ring R is a both-sided artinian ring with identity, unless otherwise stated.

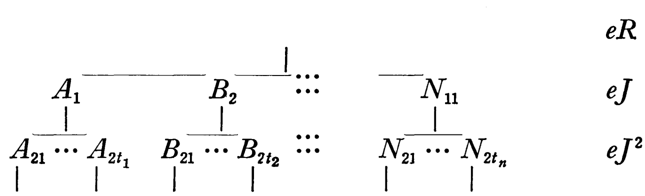
1. Left serial rings

In this section, we assume that R is a left serial ring. Then

$eJ^i=\sum_k \oplus A_k$, where the A_k are hollow right R-modules by [8], Corollary 4.2. We shall describe this situation as the following diagram:



or



where A, B, $\cdots$ are hollow modules. (cf. [3], §2).

Let e be a primitive idempotent and put $\Delta=eRe/eJe$, and for a submodule A of eR, $\Delta(A)=\{\bar{x}\mid x\in eRe,\ xA\subset A\}$, where $\bar{x}$ is the coset of x in Δ. Then $\Delta(A)$ is a division subring of Δ (see [1]). It is clear that $\Delta(A)=\Delta(\bar{A})=\{\bar{x}\mid x\in eRe,\ xA\subset A$ and $\bar{x}\bar{A}\subset\bar{A}\}$ provided A is hollow; $\bar{A}=A/\mathrm{J}(A)$.

Let $A_1\supset A_{i1}$ be as in the diagram above. We put $\tilde{R}=R/J^t$ $(t>i)$ and $\tilde{A}_{i1}=(A_{i1}+eJ^t)/eJ^t$. Then we can express $A_{i1}+eJ^t$ as a direct sum $A_{i1}\oplus C$, where $C\subset eJ^t-A_{i1}$ (see the diagram above). Let p and q be the projections of $A_{i1}+eJ^t$ to A_{i1} and C respectively. We can define $\Delta(A_{i1})$ and $\Delta(\tilde{A}_{i1})$. Since $eRe/eJe\approx(eRe/eJ^te)/(eJe/eJ^te)$, $\Delta(A_{i1})$ is canonically contained in $\Delta(\tilde{A}_{i1})$. Conversely, let $\bar{x}$ be an element in Δ such that $x(A_{i1}+eJ^t)\subset A_{i1}+eJ^t$. Put $f=qx_l\mid A_{1i}$ and f is in $\mathrm{Hom}_R(A_{i1}, eJ^t)$, where x_l means the left-sided multiplication of x. Let $A_{i1}=aR$ and $ag=a$ for some primitive idempotent g. Since $b=f(a)=f(a)g$, there exists d in eJe such that $da=b$ (note $i>t$), since R is left serial. Then $x_l\mid A_{i1}=(px_l+qx_l)\mid A_{i1}=px_l\mid A_{i1}+f=px_l\mid A_{i1}+d_l\mid A_{i1}$ and $px_l\mid A_{i1}\in\mathrm{Hom}_R(A_{i1}, A_{i1})$. Hence $(\bar{x}-\bar{d})=\bar{x}\in\Delta(A_{i1})$. Thus we have (from now on A_{ij} means always a hollow module in the diagram above)

Lemma 1. *Let R be a left serial ring, and let A_{i1} and $\tilde{A}_{i1}$ be as above. Then* $\Delta(A_{i1})=\Delta(\tilde{A}_{i1})$.

Lemma 2. *Let R be a left serial ring. Let A_{i1} contain A_{j1} and A_{jk}. Then* $\Delta(A_{j1})\subset\Delta(A_{i1})$, *and if f: $A_{j1}\approx A_{jk}$, there exists a unit δ in eRe which induces f and* $\delta A_{i1}=A_{i1}$.

Proof. Assume f: $A_{j1}\approx A_{jk}$. There exists a unit x in eRe such that $xA_{j1}=A_{jk}$ from [7], Lemma 2, and x_l induces f, since R is left serial. For x, we employ the similar argument given in the proof of Lemma 1. Let $eJ^i=A_{i1}\oplus E$ and p, q the projections. Consider $qx_l\mid A_{i1}$ $(=g)$. Since $g(A_{j1})=qxA_{j1}=qA_{jk}=0$, g is not a monomorphism. Hence $g=d_l$ for some d in eJe and so $(x-d)A_{i1}\subset A_{i1}$. Hence $(x-d)_l$ induces f. If we put $k=1$ in the above, we obtain the first half of the lemma.

2. (*, 1)

First we recall the definition of (*, n)

(*, n) *Every maximal submodule of a direct sum of n hollow modules is also a direct sum of hollow modules* [5].

We shall study, in this section, left serial rings R with (*, 1). We obtained a characterization of a left serial ring with (*, 1), when eJ/eJ^2 is squarefree, i.e., $\bar{A}_1\not\approx\bar{B}_1\not\approx\cdots\not\approx\bar{N}_1$ in [7], Theorem. Hence we may consider eR satisfying $A_1\approx B_1$.

Now we shall study such a ring with $(*, 1)$.

Lemma 3. *Let R be left serial. Assume that $A_1 \approx B_1$ and $(*, 1)$ holds. Then, for any submodules $C_i \supset D_i$ in A_1 such that C_i/D_i is simple and f; $C_1/D_1 \approx C_2/D_2$, f or f^{-1} is extendible to an element g in* $\mathrm{Hom}_R(A_1/D_1, A_1/D_2)$ *or* $\mathrm{Hom}_R(A_1/D_2, A_1/D_1)$.

Proof. There exists a unit element u in eRe such that $B_1 = uA_1$. Put $C_2' = uC_2$, $D_2' = uD_2$ and $f' = u_l f$. Then f' (or $f^{-1}u_l^{-1}$) is extendible to an element g' in $\mathrm{Hom}_R(A_1/D_1, B_1/D_2')$ (or $\mathrm{Hom}_R(B_1/D_2', A_1/D_1)$) by [6], Theorem 4. Then $g = u_l^{-1}g'$ (or $g = g'u_l$) is the desired extension of f (or f^{-1}).

Proposition 1. *Let R, A_1 and B_1 be as in Lemma 3. If there are three non-zero hollow modules A_{i1}, A_{i2}, A_{i3} $(\subset A_1)$ for some i, they are isomorphic to one another.*

Proof. First we shall show $\bar{A}_{i1} \approx \bar{A}_{i2}$. Put $C_1 = A_{i1} \oplus A_{i3}$ and $C_2 = A_{i2} \oplus A_{i3}$. Considering R/J^{i+1} from [3], Lemma 1, we may assume that the A_{ij} are simple. Now f: $C_1/A_{i1} \approx A_{i3} \approx C_2/A_{i2}$. Then by Lemma 3, there exists an element x in eRe which induces f or f^{-1}, i.e., $f(a + A_{i1}) = xa + A_{i2}$ for $a \in A_1$. Since C_1, C_2 are contained in eJ^i but not in eJ^{i+1}, x is a unit, and $xA_{i1} = A_{i2}$ (or $xA_{i2} = A_{i1}$) from the argument of the proof of [4], Theorem 3. Therefore $\bar{A}_{i1} \approx \bar{A}_{i2}$. Since R is left serial and A_{ij} are hollow, $A_{i1} \approx A_{i2}$ from [7], Lemma 2.

Let $\Delta \supset \Delta_1$ be division rings. $[\ ]_r$ ($[\ ]_l$) means the dimension of Δ over Δ_1 as a right (left) Δ_1-module.

Proposition 2. *Let A_1, B_1 be as in Lemma 3. Then for $A_{i1} \supset A_{j1}$ $[\Delta(A_{i1}): \Delta(A_{j1})]_r = |A_{i1}J^{j-i}/A_{i1}J^{j-i+1}|$, except $A_{i1}J^{j-i} = A_{j1} \oplus A_{j2}$ and $A_{j1} \not\approx A_{j2}$ (in the exceptional case $\Delta(A_{i1}) = \Delta(A_{j1})$*, cf. Example 2 below).

Proof. We may assume from Lemma 1 and [3], Lemma 1 that $J^{j+1} = 0$, and hence $A_{1i}J^{j-i+1} = 0$, and so A_{j1} is simple. Let $A_{j1} = aR$ and $\{\bar{e}, \bar{\delta}_2, \bar{\delta}_3, \cdots, \bar{\delta}_t\}$ be a linearly independent set in $\Delta_i = \Delta(A_{i1})$ over $\Delta_j = \Delta(A_{j1})$ such that $\delta_k A_{i1} \subset A_{i1}$ for all k. We shall show $A_{j1} + \delta_2 A_{j1} + \delta_3 A_{j1} + \cdots + \delta_t A_{j1} = A_{j1} \oplus \delta_2 A_{j1} \oplus \delta_3 A_{j1} \oplus \cdots \oplus \delta_t A_{j1}$. If $(A_{j1} + \delta_2 A_{j1} + \cdots + \delta_{t-1} A_{j1}) \cap \delta_t A_{j1} \neq 0$, $\delta_t A_{j1} \subset A_{j1} + \cdots + \delta_{t-1} A_{j1}$, since $\delta_t A_{j1}$ is simple. Then $\delta_t a = a_1 + \delta_2 a_2 + \cdots + \delta_{t-1} a_{t-1}$, where $a_j \in A_{j1}$. The mapping; $a \to a_i$ gives an endomorphism of A_{j1}. Hence $a_i = k_i a$ for some $\bar{k}_i \in \Delta_j$ by Lemma 2. Accordingly $\bar{\delta}_t = \bar{k}_1 + \bar{\delta}_2 \bar{k}_2 + \cdots + \bar{\delta}_{t-1} \bar{k}_{t-1}$, since $J^{j+1} = 0$, a contradiction. From the similar argument we can show that $\{A_{j1}, \delta_2 A_{j1}, \cdots, \delta_t A_{j1}\}$ is independent. Hence $[\Delta(A_{i1}): \Delta(A_{j1})]_r \leqslant |A_{i1}J^{j-i}|$. Assume $|A_{i1}J^{j-i}| \geqslant 3$. Then by Proposition 1 $A_{i1}J^{j-i} = A_{j1} \oplus A_{j2} \oplus \cdots \oplus A_{jp}$; $p \geqslant 3$ and $A_{j1} \approx A_{jk}$ for $2 \leqslant k \leqslant p$. There exists $\bar{x}_k$ in Δ_i ($x_k \in eRe$) such that $x_k A_{j1} = \bar{x}_k A_{j1} = A_{jk}$. We shall show that $\{\bar{e}, \bar{x}_2, \cdots, \bar{x}_p\}$ is linearly independent over Δ_j. Assume $\bar{x}_p=\bar{k}_1+\bar{x}_2\bar{k}_2+\cdots+\bar{x}_{p-1}\bar{k}_{p-1}$, where $\bar{k}_iA_{j1}\subset A_{j1}$ and $k_i\in eRe$. Since $JA_{j1}=0$, $A_{jp}=x_pA_{j1}=\bar{x}_pA_{j1}\subset\bar{k}_1A_{j1}+\bar{x}_2\bar{k}_2A_{j1}+\cdots+\bar{x}_{p-1}\bar{k}_{p-1}A_{j1}=\sum_{i=1}^{p-1}\oplus A_{jk}$, a contradiction. Hence $|A_{i1}J^{j-i}|\leqslant[\Delta(A_{i1})\colon\Delta(A_{j1})]_r$. Finally assume $|A_{i1}J^{j-i}|\leqslant 2$. If $A_{j1}\approx A_{j2}$, we have the same result. If $A_{j1}\not\approx A_{j2}$, $p\leqslant 2$ from Proposition 1, and $\Delta_i=\Delta_j$ from the initial argument. If $A_{j2}=\cdots=A_{jp}=0$, it is clear that $\Delta_i=\Delta_j$. Hence $[\Delta(A_{i1})\colon\Delta(A_{j1})]_r=1$.

We consider the situation in Proposition 2 and $J^{n+1}=0$. Let $A_{k1}J^{n-k}=\sum_{j=1}^{p}\oplus A_{nj}$. If $p\geqslant 3$, $A_{n1}\approx A_{nj}$ for all j by Proposition 1. Put $\Delta_k=\Delta(A_{k1})$ and $\Delta_n=\Delta(A_{n1})$. Then $[\Delta_k\colon\Delta_n]_r=p$ by Proposition 2. Further $A_{k1}J^{n-k}=A_{n1}\oplus\delta_2A_{n1}\oplus\cdots\oplus\delta_pA_{n1}=\Delta_na\oplus\delta_2\Delta_na\oplus\cdots\oplus\delta_p\Delta_na$, where $A_{n1}=aR$, and every simple submodule in $A_{k1}J^{n-k}$ is of a form $\delta\Delta_na$ for some δ in Δ_k. Now we shall identify $A_{k1}J^{n-k}=\Delta_na\oplus\delta_2\Delta_na\oplus\cdots\oplus\delta_p\Delta_na=(\Delta_k\oplus\delta_2\Delta_n\oplus\cdots\oplus\delta_p\Delta_n)a$ with $\Delta_k=\Delta_n\oplus\delta_2\Delta_n\oplus\cdots\oplus\delta_p\Delta_n$, i.e., $\mathrm{Hom}_R(A_{n1}, A_{k1}J^{n-k})\approx\Delta_k$ ($\Delta_ka=A_{k1}J^{n-k}$) as left Δ_k, right Δ_n-modules. Let $T_1\supset T_2$ and $S_1\supset S_2$ be submodules in $A_{k1}J^{n-k}$ such that $f\colon T_1/T_2\approx S_1/S_2$ and $|T_1|=|S_1|(|T_1|\leqslant|S_1|)$, $|T_1/T_2|=1$. Then f is extendible to an element h in $\mathrm{Hom}_R(A_1/T_2, A_1/S_2)$. Since S_1, T_1 are contained in $A_{k1}J^{n-k}$, h is given by a unit element x in eRe. As given in the proof of Lemma 2, $(x+j)_l|A_{k1}$ is in $\mathrm{Hom}_R(A_{k1}, A_{k1})$ for some j in eJe. Since $JT_2=0$, $x+j$ induces f, and $\overline{x+j}\in\Delta(A_{k1})$, which means $(x+j)T_2=S_2$ ($(x+j)T_2\subset S_2$) and $f(t_1+T_2)=(x+j)t_1+S_2$ for any t_1 in T_1. We translate the above fact to $\Delta_k=\mathrm{Hom}_R(A_{n1}, A_{k1}J^{n-k})$.

(#) *For any Δ_n-subspace V_1, V_2 in Δ_k with $|V_1|=|V_2|(|V_1|\leqslant|V_2|)$ and $v_1\Delta_n\oplus V_1$, $v_2\Delta_n\oplus V_2$ ($v_i\in\Delta_k$), there exists x in Δ_k such that $xV_1=V_2$ ($xV_1\subset V_2$) and $xv_1\equiv v_2$ (mod V_2).*

Lemma 4. *Let $\Delta\supset\Delta_1$ be division rings. Assume that (#) holds for Δ and Δ_1. Then $[\Delta\colon\Delta_1]_l\leqslant 2$.*

Proof. We may assume $\Delta\neq\Delta_1$. Let δ be a fixed element in $\Delta-\Delta_1$ and δ' an element in $\Delta-\Delta_1$. Put $V_1=V_2=\Delta_1$, $v_1=\delta$ and $v_2=\delta'y$ for any $y\in\Delta_1$ in (#). Then there exists x in Δ_1 such that $x\delta=\delta'y+z$ for some z in Δ_1. Hence $\delta'\Delta_1\subset\Delta_1\oplus\Delta_1\delta$. Since δ' is arbitrary, $\Delta=\Delta_1+\Delta_1\delta$, and so $[\Delta\colon\Delta_1]_l\leqslant 2$.

Proposition 3. *Let R, A_1 and B_1 be as in Lemma 3. Then for $A_{i1}\supset A_{j1}$, $\Delta(A_{i1})$ and $\Delta(A_{j1})$ satisfy (#) and so $[\Delta(A_{i1})\colon\Delta(A_{j1})]_l\leqslant 2$.*

Proof. It is clear by Proposition 2 that if $A_{j1}\not\approx A_{j2}$, $\Delta(A_{i1})=\Delta(A_{j1})$. If $A_{j1}\approx A_{j2}$, $A_{j1}\approx A_{j2}\approx\cdots\approx A_{jt}$ by Proposition 1, where $t=[\Delta(A_{i1})\colon\Delta(A_{j1})]_r$. Then $\Delta(A_{i1})$ and $\Delta(A_{j1})$ satisfy (#) from the remark before Lemma 4. Hence $[\Delta(A_{i1})\colon\Delta(A_{j1})]_l\leqslant 2$ from Lemma 4.

Corollary 4. *Let A_1 and B_1 be as above. Assume either $\Delta(A_1)$ is commutative or R is an algebra over a field with finite dimension. Then $A_1J^{i-1}=A_{i1}\oplus A_{i2}$ for all $i\geqslant 2$, i.e., $[\Delta(A_1)\colon \Delta(A_{i1})]_r\leqslant 2$.*

Proof. From the assumption and Proposition 3, $[\Delta(A_1)\colon \Delta(A_{i1})]_r\leqslant 2$.

Proposition 5. *Let A_1, B_1 be as in Lemma 3. Assume $\mathrm{J}(A_{i1})=A_{i+11}\oplus A_{i+12}\oplus\cdots\oplus A_{i+1p}$. If $p\geqslant 2$, A_{i+1k} is uniserial for all k.*

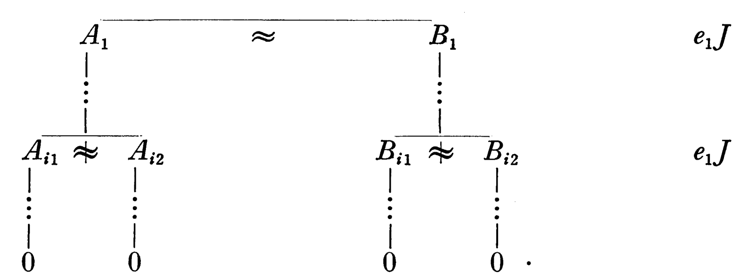
Proof. Assume that $\mathrm{J}(A_{j-11})$ is not uniserial, i.e., $\mathrm{J}(A_{j-11})=A_{j1}\oplus A_{j2}\oplus\cdots$ for $j>i+1$. We shall divide ourselves into two cases.

i) $A_{i+11}\not\approx A_{i+12}$. Then $p\leqslant 2$ by Proposition 1, and $A_{i+12}J^{j-i-1}=0$ by assumption: $A_1\approx B_1$, Proposition 1 and [7], Lemma 3. Put $D_1=A_{j1}\oplus \mathrm{J}(A_{j2})$, $D_2=A_{i+12}\oplus\mathrm{J}(A_{j2})$, $C_1=A_{j2}+D_1$ and $C_2=A_{j2}+D_2$. Then $f\colon C_1/D_1\approx\bar{A}_{j2}\approx C_2/D_2$. Since $(*, 1)$ is satisfies, f or f^{-1} is extendible to x_l for some x in eRe by Lemma 3. Being $f(A_{j2}+D_1)=A_{j2}+D_2$, x is a unit. Hence $xD_1\subset D_2$ or $xD_2\subset D_1$ (see the proof of [4], Theorem 3). However, by [7], Lemma 3, it is impossible.

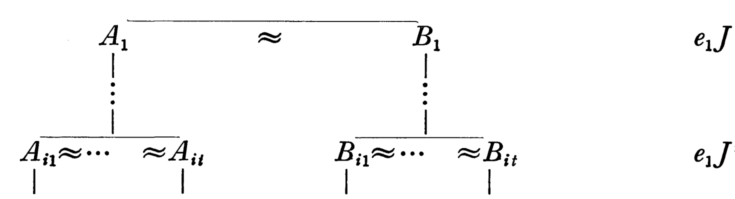
ii) $A_{i+11}\approx A_{i+12}\approx\cdots\approx A_{i+1p}$. Then $A_{j1}\approx A_{j2}$ by Proposition 1. Since $A_{i+11}\approx A_{i+12}$, $\Delta(A_1)\neq\Delta(A_{i+11})$ by Proposition 2. Similarly $\Delta(A_{j-11})\neq\Delta(A_{j1})$. Hence $[\Delta(A_1)\colon \Delta(A_{i+11})]_l=[\Delta(A_1)\colon \Delta(A_{j1})]_l=[\Delta(A_{j-11})\colon \Delta(A_{j1})]_l=2$ by Proposition 3 and Lemma 4. However $\Delta(A_1)\supset\Delta(A_{i+11})\supset\Delta(A_{j-11})\supset\Delta(A_{j1})$ by Lemma 2, which is impossible.

We shall give the structure of A_1. From Propositions 1 and 5 we obtain the following diagrams (a) and (b′).

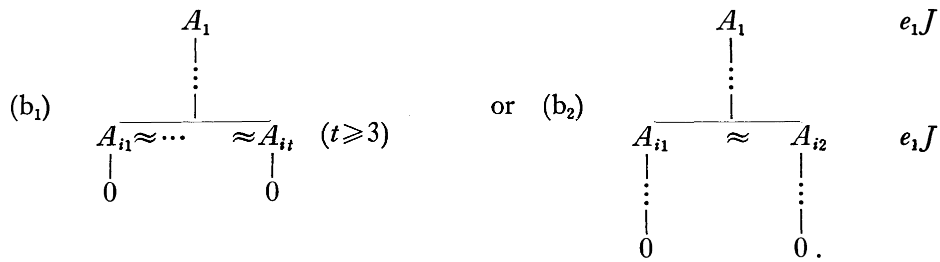
(a)



(b′)



Assume $t\geqslant 3$ and $\mathrm{J}(A_{i1})=A_{i+11}\neq 0$. Put $D_1=A_{i+11}\oplus A_{i2}$, $D_2=A_{i+11}\oplus A_{i+12}\oplus A_{i+13}$, $C_1=A_{i1}+D_1$ and $C_2=A_{i1}+D_2$. Then $C_1/D_1\approx\bar{A}_{i1}\approx C_2/D_2$. However, $xD_1\not\subset D_2$ $(xD_2\not\subset D_1)$. Hence we obtain a contradiction as above. Thus we have from Corollary 5



Lemma 5. *Let R be left serial. Then in the diagram* (a), *any two distinct simple sub-factor modules* (*e.g.* A_s/A_{s+1}, A_{t1}/A_{t+11}) *are not isomorphic to one another.*

Proof. Assume $\bar{A}_k\approx\bar{A}_{p2}$ for $k\leqslant i-1$ and $p\geqslant i$. Put $A_k=a_kR$, $A_{p2}=a_{p2}R$ and $a_kg=a_k$, $a_{p2}g=a_{p2}$ for a primitive idempotent g. Since $A_1\approx B_1$, $A_k\approx B_k$ and $A_{p2}\approx B_{p2}=b_{p2}R$; $b_{p2}g=b_{p2}$. Then there exists d in B_1 such that $da_k=b_{p2}$ by [7], Lemma 2, and $d\in \mathrm{T}(eJ^{p-k}e)$. Since $0\neq b_{p2}\in J^pg$, $db_k\in\mathrm{T}(eJ^pg)$. Let $db_k=x_1+x_2$; $x_j=x_jg\in B_{ij}$ $(j=1, 2)$. Assume $x_2\in\mathrm{T}(eJ^pg)$. Then $b_{p2}=x_2u$ for some unit u in gRg, and so $d(a_k-b_ku)=-x_1u$. Hence $-x_1u=-x_1ug\in\mathrm{T}(B_{p1})$. Accordingly, $B_{p1}\approx B_{p2}$, which contradicts [7], Lemma 3. Therefore $x_2\notin\mathrm{T}(eJ^pg)$, and so $x_1=x_1g\in\mathrm{T}(eJ^pg)$. Again we obtain the same contradiction from [7], Lemma 3. Thus $\bar{A}_k\not\approx\bar{A}_{p2}$. We can use the same argument for other cases (note that, for the case $\bar{A}_k\approx\bar{A}_{k'}$, $(k<k'<i-1)$, use [7], Lemma 7).

Lemma 6. *Assume that R is a left serial ring. Then in (b_1) we have the same situation as in Lemma* 5 *for simple sub-factor modules between A_1 and* $\mathrm{J}(A_{i-1})$. *Further* $\Delta(A_1)$ *and* $\Delta(A_{i1})$ *satisfy* (♯), *provided* (∗ 1) *holds. For* (b$_2$) *any two of simple sub-factor modules between A_1 and* $\mathrm{J}(A_{i-1})$ (*and of* A_{i1}) *are not isomorphic to one another, respectively.* (*Some simple sub-factor modules between A_1 and* $\mathrm{J}(A_{i-1})$ *may be isomorphic to one of* A_{i1}.)

Proof. The first halves of (b$_1$) and (b$_2$) are obtained from the argument similarly to Lemma 5. The last one of (b$_1$) is clear from Proposition 3.

Lemma 7. *Let R be left serial, and consider the diagram* (a). *Let $C_1\supset D_1$ and $C_2\supset D_2$ be submodules in A_1 such that f: $C_1/D_2\approx C_2/D_2$ and $|C_1/D_1|=1$. Then f or f^{-1} is extendible to an element in* $\mathrm{Hom}_R(A_1/D_1, A_1/D_2)$ *or* $\mathrm{Hom}_R(A_1/D_2, A_1/D_1)$.

Proof. We may assume $C_i=c_iR+D_i$ and $c_ig=c_i$ for $i=1, 2$. If $c_1\in\mathrm{T}(A_k)$ $(k\leqslant i-1)$, $C_1=A_k$ and $D_1=\mathrm{J}(C_1)=A_{k+1}$. Then $c_2\in\mathrm{T}(A_k)$ by Lemma 5. Hence there exists a unit d in eRe such that $dc_1=c_2$. We may assume $dA_1=A_1$ by Lemma 2. Then $dD_1=dA_kJ\subset C_2J=D_2$. Therefore d_l is an extension of f. Thus we may assume that $J(A_{i-1})$ contains C_1 and C_2. From Lemma 5 every submodule in $\mathrm{J}(A_{i-1})$ is standard (see the definition before Lemma 10 below). Let $C_1=A_{j1}\oplus A_{k2}$, $D_1=A_{j+1}\oplus A_{k2}$. Since $C_1/D_1\approx C_2/D_2$, $C_2=A_{j1}\oplus A_{k'2}$, $D_2=A_{j+11}\oplus A_{k'2}$. If $k\leqslant k'$ (resp. $k>k'$), f is extendible to an element d_l in $\mathrm{Hom}_R(A_1/D_2, A_1/D_1)(\mathrm{Hom}_R(A_1/D_1, A_1/D_2))$ as above by Lemmas 2 and 5.

Lemma 8. *Let R be left serial. In the diagram* (b$_1$), *we assume that* $\Delta(A_1)$ *and* $\Delta(A_{i1})$ *satisfy* (#). *Further we assume* $[\Delta(A_1)\colon \Delta(A_{i1})]_l=2$ *in* (b$_2$). *Then we obtain the same result as in Lemma* 7.

Proof. Let c_j be as in the proof of Lemma 7. If c_j is in $\mathrm{T}(A_{s_j})$ $(s_j\leqslant i-1)$, then $C_1=C_2=A_{s_1}$ and $D_1=D_2=A_{s_1+1}$ by Lemma 6. Hence we can prove the lemma as in the proof of Lemma 7. Similarly if $C_1=A_{s_1}$ and C_2 is contained in $\mathrm{J}(A_{i-1})$, we can easily prove the lemma, since $D_1=\mathrm{J}(C_1)$. Therefore we may assume $\mathrm{J}(A_{i-1})$ contains C_1 and C_2.

(b$_1$) Since C_i is in $\mathrm{J}(A_{i-1})$, we have the lemma from (#).

(b$_2$) Let $\mathrm{J}(A_{i-1})=A_{i1}\oplus A_{i2}\supset C_1\supset D_1$ be submodules with $|C_1/D_1|=1$. Let p_j be the projection of $\mathrm{J}(A_{i-1})$ to A_{ij}. We shall show for C $(=C_1)$ and D $(=D_1)$ that there exists a unit x in eRe such that

(1) $xA_1=A_1$ and $xC=A_{k-11}\oplus A_{s2}\supset xD=A_{k1}\oplus A_{s2}$.

First we remark the following fact: for $C=A_{r1}\oplus A_{t2}$, there exists a unit y in eRe such that $yA_1=A_1$ and $yC=A_{t1}\oplus A_{r2}$.

i) $t\geqslant r$. There exists y in eRe such that $yA_1=A_1$ and $yA_{i1}=A_{i2}$ by Lemma 2. Since $yA_{i2}\neq A_{i2}$, $p_1(yA_{i2})\neq 0$, and so $p_1y(A_{i2})=A_{i1}$ by Lemma 6. Hence $yC=A_{t1}\oplus A_{r2}$.

ii) $t<r$. Take a unit y' such that $y'A_{i2}=A_{i1}$ and $y'A_1=A_1$.

Put $D_{(j)}=D\cap A_{ij}$ and $D^{(j)}=p_j(D)$ $(j=1, 2)$. Then $g'\colon D^{(1)}/D_{(1)}\approx D^{(2)}/D_{(2)}$. Let $D_{(1)}=A_{k1}$, $D_{(2)}=A_{s2}$, $D^{(1)}=A_{k-t1}$ and $D^{(2)}=A_{s-t2}$. We may assume $k\leqslant s$ from the remark (actually $k=s$ by Lemma 6). There exists x in eRe such that x_l induces g. Hence $xD_{(1)}\subset D_{(2)}$. Putting $\alpha=e+x$, $\alpha(D_{(1)}\oplus D_{(2)})\subset D_{(1)}\oplus D_{(2)}$ and $\alpha(A_{k-t1}+D_{(1)}\oplus D_{(2)})\subset\alpha A_{k-t1}+D_{(1)}\oplus D_{(2)}=D$. α is clearly a unit, and so $\alpha^{-1}D=A_{k-t1}+D_{(1)}\oplus D_{(2)}=A_{k-t1}\oplus A_{s2}$. Now $\alpha^{-1}C\supset\alpha^{-1}D=A_{k'1}\oplus A_{s2}$, where $k'=k-t$. Since $|C/D|=1$, $\alpha^{-1}C$ is one of the following: $A_{k'-11}\oplus A_{s2}$, $A_{k'1}\oplus A_{s-12}$ and $(e+y)A_{k'-11}\oplus\alpha^{-1}D$ (in the last case $k'=s$), where $y\in eRe$ and $yA_{k'-11}=A_{s-12}$. Noting $yA_{k'1}=A_{s2}$ and $k\leqslant s$, we obtain (1) from the initial remark.

Next we assume that $C_i\supset D_i$ are of the form (1). Put $C_i=A_{k_i-11}\oplus A_{s_i2}$ and $D_i=A_{k_i1}\oplus A_{s_i2}$ for $i=1, 2$. Since $f\colon C_1/D_1\approx C_2/D_2$, $k_1=k_2$ $(=k)$ by Lemma 6. We shall divide ourselves to the following cases:

(α) $k\leqslant\min(s_1, s_2)$. We may assume $s_1\geqslant s_2$. Let $A_{k-1}=aR$. Then there exists a unit z in eRe such that $f(a+D_1)=za+D_2$ and $zA_{k-11}=A_{k-11}$, $zA_1=A_1$ by Lemma 2. Since $k\leqslant s_2\leqslant s_1$, $zD_1=z(A_{k_11}\oplus A_{s_12})\subset A_{k1}\oplus A_{s_12}=D_2$. Hence z_l is an extension of f.

(β) $s_2\leqslant k\leqslant s_1$ ($s_1\leqslant k\leqslant s_2$). We obtain the same result as in (α). (Take f^{-1}.)

(γ) $k<\max(s_1, s_2)$. We may assume $s_1\geqslant s_2$. Let $A_{k-12}=aR$ and $\delta A_{i2}=A_{i1}$ ($\delta A_1=A_1$) for some unit δ by Lemma 2. Then $A_{k-11}=\delta aR$ and $f(\delta a+D_1)=\delta wa+D_2$ for some w with $wA_1=A_1$ and $wA_{k-12}=A_{k-12}$. Since $[\Delta(A_1)\colon \Delta(A_{i2})]_l=2$, there exist y_1 and y_2 in eRe such that $\bar{\delta}\bar{w}=\bar{y}_1+\bar{y}_2\bar{\delta}$ and $y_jA_{i2}=A_{i2}$, and $y_jA_1=A_1$ for $j=1, 2$, i.e., $\delta w=y_1+y_2\delta+j$; $j\in eJe$. Then $jA_1=(\delta w-y_1-y_2\delta)A_1\subset A_1$, and so $y_2(\delta a)=(\delta w-y_1-j)a=\delta wa-(y_1+j)a\equiv\delta wa \pmod{D_2}$ and $y_2D_1\subset D_2$, since $s_2\leqslant s_1\leqslant k$ and $j\in eJe$. Hence $(y_2)_l$ is an extension of f.

Finally we consider the general case. Let $f\colon C_1/D_1\to C_2/D_2$ be as before. Then there exist u_1, u_2 in eRe as in (1). Take

$$f'\colon (A_{k_1-11}\oplus A_{s_12})/(A_{k_11}\oplus A_{s_12}) \xrightarrow{u_1^{-1}} C_1/D_1 \xrightarrow{f} C_2/D_2 \xrightarrow{u_2} (A_{k_2-11}\oplus A_{s_22})/(A_{k_21}\oplus A_{s_22})\,.$$

Applying the above argument to f', we can find v in eRe such that v_l induces f' (or f'^{-1}) and $vA_1=A_1$. Therefore $(u_1vu_2^{-1})_l$ $((u_2vu_1^{-1})_l)$ induces f (or f^{-1}).

Thus we obtain

Theorem 1. *Let R be a left serial ring, and $eJ=A_1\oplus B_1\oplus\cdots\oplus N_1$ a direct sum of hollow modules. Then* (*, 1) *holds for any hollow right R-module if and only if the following conditions are satisfied*:

1) *If $A_1\approx B_1$, A_1 has the structure of* (a), (b_1) *or* (b_2) *such that* (♯) *holds for $\Delta(A_1)$ and $\Delta(A_{i1})$ if $t\geqslant 3$ in* (b_1), *and $[\Delta(A_1)\colon\Delta(A_{i1})]_l=2$ if $t=2$ in* (b_1) *and* (b_2).

2) *The condition in* [7], *Theorem is satisfied.*

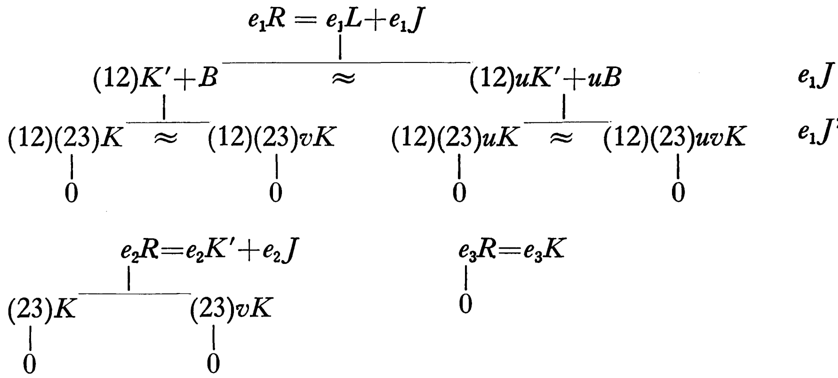
Proof. If $A_1\not\approx B_1$, we obtain 2). Assume $A_1\approx B_1$. We have studied an isomorphism $f\colon C_1/D_2\approx C_2/D_2$ for submodules $C_i\supset D_i$ in A_1. If C_2 is a submodule of B_1, xC_2 is a submodule in A_1, where $xB_1=A_1$ for some unit x. Then using the manner given in the proof of Lemma 8, we can extend f to an element in $\mathrm{Hom}_R(A_1/D_1, B_1/D_2)$ or $\mathrm{Hom}_R(B_1/D_2, A_1/D_1)$.

Proposition 6. *Let R be as above. Assume $A_1\approx B_1\approx\cdots\approx N_1$ for each primitive idempotent. Then* (*, 1) *holds for any hollow right R-module if and only if* 1) *in Theorem* 1 *holds.*

REMARK. If R is left serial, eR has the structure in § 1. Under this assumption, for a fixed primitive idempotent e, we have studied a problem: when is eJ/K a direct sum of hollow modules for any submodule K? Hence Theorem 1 gives a characterization of such e, provided R is left serial. This remark is applicable to the next section, in particular to Proposition 7 below.

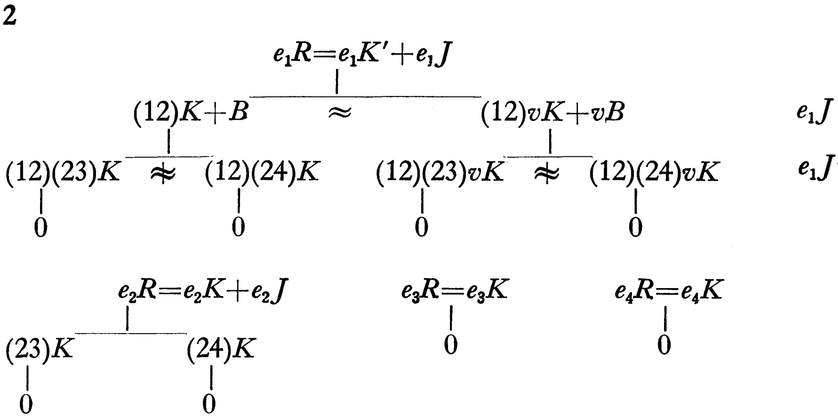
We shall give some algebras concerning Theorem and Propositions.

1 Let $L\supset K'\supset K$ be fields with $[L: K']=[K': K]=2$. Let $L=K'+K'u$ and $K'=K+Kv$. We construct a similar example to ones in [4].



where $B=(12)(23)K\oplus(12)(23)vK$ and $le_1=e_1l$ for any l in L, $k'e_2=e_2k'$ for any k' in K'. Then $R=\sum_{i=1}^{3}\oplus e_iR$ is a left serial algebra. Further we can show from Theorem 1 that $(*, 1)$ holds for any hollow right R-module $((12)(23)K\approx(12)(23)vK\approx(12)(23)uK)$. This example shows that [7], Lemma 6 is not true if $i=j$.

2



where $B=(12)(23)K\oplus(12)(24)K$ and $k'e_1=e_1k'$ for any k' in K'. Then $R=\sum_{i=1}^{4}\oplus e_iR$ is a left serial algebra with $(*, 1)$ $((12)(23)K\not\approx(12)(24)K)$.

3 In Example 1, we replace K' by an extension K'_0 over K ($K'_0=K(v)$ and $[K'_0: K]\geqslant 3$). We add further semisimple modules $(12)(23)v^2K\oplus(12)(23)v^3K\oplus\cdots$ to B and $(23)v^2K\oplus(23)v^3K\oplus\cdots$ to e_2R. Then $(*, 1)$ does not hold by Corollary 4.

3 (*, 2)

We shall give a characterization of left serial rings with (*, 2).

Proposition 7. *Let R be a right artinian ring and e a fixed primitive idempotent. Assume that (*, 2) holds for any two hollow modules of form eR/K. Then eJ is a direct sum of uniserial modules.*

Proof. Since $eR\oplus eJ$ is a maximal submodule of $eR\oplus eR$, $eJ=\sum_{i=1}^{u}\oplus A_i$ by assumption, where the A_i are hollow. We shall show by induction that A_i/A_iJ^k is uniserial for all i. If $k=0$, $A_i/A_iJ^0=0$. Assume that A_i/A_iJ^n is uniserial for all i. Let $A_mJ^n/A_mJ^{n+1}=B_{m1}\oplus B_{m2}\oplus\cdots\oplus B_{ms_m}$, where the $\bar{B}_{mj}$ are simple. We shall show $s_m=1$. Otherwise, $\bar{B}_{m1}\neq 0$ and $\bar{B}_{m2}\neq 0$. Put $B_j^*=\sum_{i=1}^{m-1}\oplus A_iJ^n\oplus B_j$, where $A_mJ^{n+1}\subset B_j\subset A_mJ^n$ for $j=1, 2$ and $B_1/A_mJ^{n+1}=\bar{B}_{m2}\oplus B_{m3}\oplus\cdots\oplus\bar{B}_{ms_m}$, $B_2/A_mJ^{n+1}=\bar{B}_{m1}\oplus\bar{B}_{m3}\oplus\cdots\oplus\bar{B}_{ms_m}$, and $D=eR/B_1^*\oplus eR/B_2^*$. We shall show, in this case, that D does not satisfy (*, 2). Contrarily assume that D satisfies (*, 2). Then D contains a maximal submodule M with a direct summand M_1 isomorphic to $\widetilde{eR}=eR/(B_1^*\cap(e+j)B_2^*)$ where $j\in eJe$ by [3], Lemma 3. Since $eJ^{n+1}\supset B_2^*\supset eJ^{n+2}$ and $jB_2^*\subset eJ^{n+2}$, $(e+j)B_2^*=B_2^*$. Hence $M_1\approx eR/(B_1^*\cap B_2^*)$ $(=\widetilde{eR})$. We shall denote A_i/A_iJ^n $(i\neq m)$ and A_m/B_3' by $\tilde{A}_i$ and $\tilde{A}_m$, respectively, where $B_3'/A_mJ^{n+1}=\sum_{j\geq 3}\oplus\bar{B}_{mj}$. Let $M=M_1\oplus M^*$ and $|\tilde{A}_i|=n_i$ and $|\tilde{A}_m|=n_m+1$, where $n_i\leqslant n_m$ and $n_m=n+1$. Then $|\widetilde{eR}|=|M_1|=\sum_{i=1}^{m}n_i+2$ and $|D|=2\sum_{i=1}^{m}n_i+2$. Put $\bar{D}=D/\mathrm{J}(D)\supset\bar{M}=M/\mathrm{J}(D)$. We note that $\bar{M}=(\bar{e}+\bar{e})eR/eJ$ in $\bar{D}$ (see [3], Lemma 3). Since $|\bar{D}|=2$, $\bar{M}$ is a simple module. Now $M^*=\sum_{i\geq 2}\oplus M_i$; M_i are hollow by (*, 2). If $\bar{M}_2=(M_2+\mathrm{J}(D))/\mathrm{J}(D)=\bar{M}$, eR/B_1^* is an epimorphic image of M_2 by the remark above. Then $|M_2|\geqslant|\widetilde{eR}|-1$ and so $|M|\geqslant|M_1|+|M_2|\geqslant|D|$, a contradiction. Hence $M^*\subset\mathrm{J}(D)$. Let φ be the given isomorphism of $\widetilde{eR}$ to M_1. It is clear that $\varphi(\widetilde{eJ})\subset\mathrm{J}(D)$, and hence

$$\mathrm{J}(D)=\varphi(\widetilde{eJ})\oplus M^* \tag{2}$$

(note $M\supset\mathrm{J}(D)$). Put $Q=\tilde{A}_1\oplus\cdots\oplus\tilde{A}_{m-1}$, and $\widetilde{eJ}=Q\oplus\tilde{A}_m$. Then

$$\mathrm{J}(D)=Q_1\oplus L_1\oplus Q_2\oplus L_2\,, \tag{3}$$

where $Q_1\approx Q_2\approx Q$, $L_1=\tilde{A}_m/\bar{B}_{m1}$ and $L_2=\tilde{A}_m/\bar{B}_{m2}$. From (3) $\varphi(Q)=\{q+0+q+0\,|\,q\in Q\}$. Hence

$$\mathrm{J}(D)=\varphi(Q)\oplus L_1\oplus Q_2\oplus L_2\,. \tag{4}$$

On the other hand, $\mathrm{Soc}(\varphi(\tilde{A}_m))=\mathrm{Soc}(L_1)\oplus\mathrm{Soc}(L_2)$, and $\mathrm{Soc}(\varphi(\widetilde{eJ}))=\mathrm{Soc}(\varphi(Q))$

$\oplus\mathrm{Soc}(\varphi(\tilde{A}_m))$. Let p be the projection of $\mathrm{J}(D)$ onto Q_2 in (4). Then $p|\mathrm{Soc}(M^*)$ is a monomorphism from the above observation (note $\mathrm{soc}(M^*)\cap\mathrm{Soc}(\varphi(\widetilde{eJ}))=0$), and hence so is $p|M^*$. Hence $|M^*|\leqslant|Q_2|=\sum_{i=1}^{m}n_i$. Therefore $|M|=|M_1|+|M^*|\leqslant\sum_{i=1}^{m}n_i+2+\sum_{i=1}^{m-1}n_i=2\sum_{i=1}^{m}n_i+2-n_m<2\sum_{i=1}^{m}n_i+1=|D|-1$ (note $n_m=n+1\geqslant 2$), which is a contradiction. Hence A_mJ^n/A_mJ^{n+1} is simple.

The following lemma is substantially due to T. Sumioka [9].

Lemma 9. *Let R be left serial and eJ a direct sum of uniserial modules A_i and A_i', i.e., $eJ=\sum\oplus A_i=\sum\oplus A_i'$. Let d' be an element in eJe such that $d'A_{1\alpha}=A_{1\beta}'$, for $A_{1\alpha}\subset A_1$ and $A_{1\beta}'\subset A_1'$. Then there exists d in $A_1'\cap eJe$ such that $d_l|A_{1\alpha}=d_l'|A_{1\alpha}$. Further for such d $dA_i=0$ $(i\neq 1)$.*

Proof. Put $A_{1\alpha}=a_\alpha R$, $A_1=a_1R$ and $A_{1\beta}'=a_\beta'R$ $(d'a_\alpha=a_\beta')$. Assume that $a_\alpha g=a_\alpha$ and $a_\beta' g=a_\beta'$ for a primitive idempotent g. Let $d'=\sum d_r'$; $d_r'\in A_r'$. Since $A_1'\supset A_{1\beta}'\ni a_\beta'=d'a_\alpha=\sum d_r'a_\alpha$, $a_\beta'=d_1'a_\alpha$. Put $d=d_1'\in A_1'\cap eJe$. Since $da_\alpha=a_\beta'$, $d\in\mathrm{T}(J^{\beta-\alpha}g)$. Assume $da_i\neq 0$ for some $A_i=a_iR$ $(i\neq 1)$. Then da_1 $(\neq 0)$ and da_i are elements in $\mathrm{T}(A_{1\beta-\alpha+1}')$, which is a contradiction to [7], Lemma 7. Hence $dA_i=0$ for $i\neq 1$.

Let $M=\sum_{i=1}^{t}\oplus M_i$. For $N_i\subset M_i$, $i=1,2,\cdots,t$, we call $\sum_{i=1}^{t}\oplus N_i$ a *standard submodule* of M (with respect to the decomposition $\sum_{i=1}^{t}\oplus M_i$).

Lemma 10 ([9], Lemma 3.3) *Let R be a left serial ring such that eJ is a direct sum of uniserial modules A_i. Then every submodule in eJ is a standard submodule with respect to some direct decomposition of eJ, whose direct summands are all uniserial.*

Proposition 8. *Let R be left serial and eJ a direct sum of uniserial modules. Then $(*, 2)$ holds for any direct sum of two hollow modules of form eR/K.*

Proof. We may consider a maximal submodule M' in $D'=eR/E_1\oplus eR/E_2$, where E_i are submodules in eJ. There exists a maximal submodule M in $D=eR\oplus eR$ such that $M\supset E_1\oplus E_2$ and $M/(E_1\oplus E_2)=M'$. From [0], Theorem 2 there exists a decomposition $D=eR(f)\oplus eR$ such that $M=eR(f)\oplus eJ$, where $f\in\mathrm{Hom}_R(eR,eR)$. Since $E_2\subset 0\oplus eJ$, $D/E_2=eR(f)\oplus eJ/E_2$. Hence $M'=M/(E_1\oplus E_2)=(eR(f)\oplus eJ/E_2)/\varphi(E_1)$, where φ; $E_1\to eR(f)\oplus eJ/E_2$ is the natural mapping. Accordingly, since $eR\approx eR(f)$, we may show for submodules X_i in eJ $(i=1,2)$ and Y in $D^*=eR/X_1\oplus eJ/X_2$

(5) D^*/Y is a direct sum of hollow modules.

First assume $X_1\subsetneqq eJ$. Let S' be a submodule in $eJ\oplus eJ$ such that $(Y\supset)S'\supset X_1\oplus X_2$ and $S'/(X_1\oplus X_2)$ $(=S)$ is simple. We shall show

$$(6)\qquad D^*/S \approx eR/X_1' \oplus eJ/X_2',$$
$$\text{where } X_1' \subset eR \text{ and } X_2' \subset eJ.$$

Put $X_1 = A_{\alpha_1} \oplus \cdots \oplus A_{m\alpha_m}$, $X_2 = A'_{1\beta_1} \oplus \cdots \oplus A'_{m\beta_m}$ by Lemma 10, where $eJ = \sum_{i=1}^{m} \oplus A_i = \sum_{i=1}^{m} \oplus A'_i$, $A_{i\alpha_i} \subset A_i$ and $A'_{j\beta_j} \subset A'_j$. Then $S \subset A_1/A_{1\alpha_1} \oplus \cdots \oplus A_m/A_{m\alpha_m} \oplus A'_1/A'_{1\beta_1} \oplus \cdots \oplus A'_m/A'_{m\beta_n}$. If $S \subset \sum_{i=1}^{m} \oplus A'_i/A'_{i\beta_i}$, $D^*/S = eR/X_1 \oplus eJ/S'$. Since eJ/S' is a direct sum of uniserial modules by Lemma 10, D^*/S is a direct sum of hollow modules. We obtain the same result for a case $S \subset \sum_{i=1}^{m} \oplus A_i/A_{i\alpha_i}$. Let p_i: $eJ/X_1 \oplus eJ/X_2 \to A_i/A_{i\alpha_i}$ and q_j: $eJ/X_1 \oplus eJ/X_2 \to A'_j/A'_{j\beta_j}$ be the projections. We shall show (6) by induction on t, where t=(the number of $\{p_i$ and $q_j \mid p_i(S) \neq 0$ and $q_j(S) \neq 0\}$). If $t=1$, we are done from the observation above. Now we may assume that $S = \{s_1 + f_2(s_1) + \cdots + f_m(s_1) + f'_1(s_1) + \cdots + f'_m(s_1) \mid s_1 \in A_{1\alpha_1-1}/A_{1\alpha_1}, f_i \in \mathrm{Hom}_R(A_{1\alpha_1-1}/A_{1\alpha_1}, A_{i\alpha_i-1}/A_{i\alpha_i})$ and $f'_j \in \mathrm{Hom}_R(A_{1\alpha_1-1}/A_{1\alpha_1}, A'_{j\beta_j-1}/A'_{j\beta_j})\}$. From the above assumption, we may assume $f'_1 \neq 0$. If $\alpha_1 = \beta_1$, then there exists a unit x in eRe such that $x_l \mid A'_{1\beta_1-1}/A'_{1\beta_1} \to A_{1\alpha_1-1}/A_{1\alpha_1} = f_1'^{-1}$. Accordingly $xA'_{1\beta_1} = A_{1\alpha_1}$, and so

$$(7)\qquad x_l\ (=h) \in \mathrm{Hom}_R(A'_1/A'_{1\beta_1}, eR/X_1).$$

Next assume $\alpha_1 > \beta_1$ or $\alpha_1 < \beta_1$. In the former case we obtain d in eJe as the above x. Let $\alpha_1 < \beta_1$. Then there exists d' in eJe such that $d'_l \mid A_{1\alpha_1-1}/A_{1\alpha_1}$ induces f'_1. From Lemma 9, we may assume $d' \in A'_1$ and $d'A_k = 0$ for $k \neq 1$. Further, since $d'(eR) \subset A'_1$

$$(8)\qquad d'_l\ (=h') \in \mathrm{Hom}_R(eR/X_1, A'_1/A'_{1\beta_1}).$$

Case (7)

$$(9)\qquad eR/X_1 \oplus eJ/X_2 = eR/X_1 \oplus (A'_1/A'_{1\beta_1})(h) \oplus \sum_{j\geqslant 2} \oplus A'_j/A'_{j\beta_j}.$$

Then $S \subset (\sum_{k\neq 1} p'_k + \sum q'_j)(S)$, where p'_i and q'_j are the projections of (9). It is clear that (the number of $\{p'_k, q'_i\}$)=(the number of $\{p_i, q_j\}$)-1.
Case (8)

$$(10)\qquad eR/X_1 \oplus eJ/X_2 = (eR/X_1)(h') \oplus eJ/X_2.$$

Then $S \subset (\sum p'_i + \sum_{1\neq j} q'_j)(S)$. Hence we obtain the same situation. If $X_1 = eJ$, eR/X_1 is simple. This is a special case in the above argument. In case (9), since $(A'_1/A'_{1\beta_1})(h) \approx A'_1/A'_{1\beta_1}$, we obtain the isomorphism f_1: $eR/X_1 \oplus (A'_1/A'_{1\beta_1})(h) \oplus \sum_{j\geqslant 2} \oplus A'_j/A'_{j\beta_j} \to eR/X_1 \oplus eJ/X_2$. Similarly in case (10) we have f_2: $(eR/X_1)(h') \oplus eJ/X_2 \to eR/X_1 \oplus eJ/X_2$. Then (the number of $\{p_i, q_j \mid p_i(f_k(S)) \neq 0, q_j(f_k(S)) \neq 0\}$)

=(the number of $q_j, p_i | \{p_i(S)\neq 0, q_j(S)\neq 0\}$)$-1$ for $k=1, 2$ (note $f(J((eR/X_1)(h'))=J(eR/X_1))$. Further $D^*/S \approx f_k(D^*)/f_k(S)=D^*/f_k(S)$. Therefore (6) holds by induction on t. If we take a chain $Y=S'_{p+1}\supset S'_p\supset\cdots\supset S'_1\supset X_1\oplus X_2=S'_0$ such that S'_i/S'_{i+1} is simple, we can show (5).

From the above proof and Proposition 7 we have

Theorem 2. *Let R be a left serial ring and e a primitive idempotent. Then the following conditions are equivalent:*

1) *$(*, 2)$ holds for a direct sum of any two hollow right R-modules of form eR/K.*

2) *eJ is a direct sum of uniserial modules.*

3) *Every factor module of $eR\oplus eJ$ is a direct sum of hollow modules (direct sum of a hollow module and uniserial modules).*

4) *Every factor module of $eR\oplus eJ^{(n)}$ is a direct sum of hollow modules, where $eJ^{(n)}$ is a direct sum of n-copies of eJ.*

We shall study further structures of R with $(*, 2)$ when $\overline{eJ}$ is square-free.

Lemma 11. *Let R be a left serial ring. Let $\alpha=e+d$ $(d\in eJe)$ be a unit in eRe. Assume $\bar{A}_i\not\approx\bar{A}_j$ if $i\neq j$. Then if $\alpha A_1\neq A_1$, $\alpha A_i=A_i$ for $i\neq 1$, where $eJ=\sum\oplus A_i$ and the A_i are uniserial.*

Proof. From [7], Lemma 5 $d\in A_j$ for some j. Since $\alpha A_1\neq A_1$, $j\neq 1$, and so $dA_1\neq 0$. Therefore $dA_k=0$ for $k\neq 1$ by Lemma 9.

Proposition 9. *Let R be left serial. Assume that eJ is a direct sum of uniserial modules A_i: $eJ=\sum_{i=1}^{m}\oplus A_i$ and that $\overline{eJ}$ is square-free. Let X be a submodule of eJ. Then there exist uniquely k and k' (not depending on X) such that $X=\alpha\ (\sum_{j=1}^{m}\oplus A_{ji_j})=A_{1i_1}\oplus\cdots\oplus A_{k-1i_{k-1}}\oplus\alpha A_{ki_k}\oplus A_{k+1i_{k+1}}\oplus\cdots\oplus A_{ni_n}$, where $A_{ji_j}\subset A_j$, and $\alpha A_k\subset A_k\oplus A_{k'}$. Further all A_i except A_k are characteristic and the number of hollow modules of form eR/K is finite up to isomorphism*

Proof. Let $eJ=\sum_{i=1}^{m}\oplus A_i$ be as in the proposition. Assume that a sub-factor module of A_1 is isomorphic to one of A_2. Then from [7], Lemma 2 there exists d in A_2 (or A_1) which induces this isomorphism. If we have the same situation between A_i and A_j, we obtain d' in A_i (or A_j). Then $i=2$ by assumption and [7], Lemma 4. Since A_2 is uniserial, $\mathrm{Soc}(A_2)\approx A_{1k}/A_{1k+1}\approx A_{js}/A_{js+1}$ for some k and s. Hence $j=1$ by [7], Lemmas 2 and 4. Therefore, for $j\neq 1, 2$, any sub-factor modules of A_j are not isomorphic to any one of A_k for all $k\neq j$. Put $F_1=A_1\oplus A_2$ and $F_2=\sum_{j\geqslant 3}^{m}\oplus A_j$. Then we can easily show by induction on m that every submodule of F_2 is standard. Further from the argument after (1) in the proof of Lemma 8, every submodule of F_1 is of a form $\alpha(A_{1i_1}\oplus A_{2i_2})$; $\alpha=e+d$, $d\in A_2$. Let p_i be the projection of eJ onto F_i, and X a submodule of eJ. Put $X^{(j)}=p_j(X)$ and $X_{(j)}=X\cap F_j$. Assume $X^{(1)}\neq X_{(1)}$, and $X_{(1)}=\alpha(A_{1k_1}\oplus A_{2k_2})$. $A_1\oplus A_2=\alpha^{-1}(A_1\oplus A_2)\supset\alpha^{-1}X^{(1)}\supset\alpha^{-1}X_{(1)}=A_{1k_1}\oplus A_{2k_2}$. Hence some simple sub-factor module T of $X^{(1)}/X_{(1)}$ is isomorphic to one of A_1 or A_2. Since $X^{(1)}/X_{(1)}\approx X^{(2)}/X_{(2)}$, T is isomorphic to a sub-factor module of $X^{(2)}/X_{(2)}$. On the other hand, every submodule of F_2 is standard, and so T is isomorphic to a sub-factor module of some A_j ($j\geqslant 3$), which is impossible from the initial observation. Hence $X^{(1)}=X_{(1)}$, and $X=X_{(1)}\oplus X_{(2)}=\alpha(A_{1k_1}\oplus A_{2k_2})\oplus\sum_{j\geqslant 3}\oplus A_{jk_j}=\alpha(\sum_{i=1}^{m}\oplus A_{ik_1})$ by Lemma 11. The remaining part is clear from the above.

Lemma 12. *Let R be a right artinian ring with $(*, 2)$. Let D be a direct sum of two hollow modules and M a maximal submodule of D. Then M has the following decomposition: $M=M_1\oplus M_2$; M_1 is a hollow module not contained in $J(D)$ and $J(D)=J(M_1)\oplus M_2$.*

Proof. Let $D=eR/E\oplus e'R/E'$. If $eR\not\approx e'R$, $M=eR/E\oplus e'J/E'$ (or $eJ/E\oplus e'R/E'$). If $eR\approx e'R$, we can obtain the lemma for any M similarly to (2) in the proof of Proposition 7.

For two integers $\alpha(1)$ and $\alpha(2)$, we denote $\max\{\alpha(1), \alpha(2)\}$ (resp. $\min\{\alpha(1), \alpha(2)\}$) by $\underline{\alpha}$ (resp. $\bar{\alpha}$). If R is a right artinian ring with $(*, 2)$,

$$(11)\qquad eJ=\sum_{i=1}^{m}\oplus A_i;\ \textit{the } A_i \textit{ are uniserial}$$

from Proposition 7.

Proposition 10. *Let R be a left serial ring with $(*, 2)$ and let eJ and A_i be as above. We assume that $\overline{eJ}$ is square-free. Put $E_i=A_{1\alpha_1(i)}\oplus\cdots\oplus A_{n\alpha_n(i)}$; $A_{k\alpha_k(i)}\subset A_k$ for $i=1, 2$ and all k. Then every maximal submodule M of $D=eR/E_1\oplus eR/E_2$ is isomorphic to $eR/(A_{\underline{\alpha}_1}\oplus A_{\underline{\alpha}_2}\oplus\cdots\oplus A_{\underline{\alpha}_n})\oplus A_1/A_{\bar{\alpha}_1}\oplus A_2/A_{\bar{\alpha}_2}\oplus\cdots\oplus A_n/A_{\bar{\alpha}_n}$, unless $M\approx eR/E_1\oplus eJ/E_2$ or $\approx eJ/E_1\oplus eR/E_2$.*

Proof. We may assume that R is basic. Assume $\bar{M}=(\bar{e}+\bar{e}\alpha)eRe/eJe$, $0\neq\bar{\alpha}\in eRe/eJe$. Then $(A_1/A_{1\alpha_1(1)}\oplus\cdots\oplus A_n/A_{n\alpha_n(1)})\oplus(A_1/A_{1\alpha_1(2)}\oplus\cdots\oplus A_n/A_{n\alpha_n(2)})=J(D)\approx eJ/(E_1\cap(\alpha+j)E_2)\oplus M_2$ by Lemma 12 and [3], Lemma 3. On the other hand, $E_1\cap(\alpha+j)E_2=\gamma(A_{1\alpha_1(3)}\oplus\cdots\oplus A_{n\alpha_n(3)})$ by Proposition 9. Hence $eJ/(E_1\cap(\alpha+j)E_2)\approx A_1/A_{1\alpha_1(3)}\oplus\cdots\oplus A_n/A_{n\alpha_n(3)}$. Since $\overline{eJ}$ is square-free, either $A_1/A_{1\alpha_1(3)}\approx A_1/A_{1\alpha_1(1)}$ or $A_1/A_{1\alpha_1(2)}$. Therefore $\alpha_i(3)=\alpha_i(1)$ or $\alpha_i(2)$. Further $A_{1\alpha_1(1)}\oplus\cdots\oplus A_{n\alpha_n(1)}\supset\gamma(A_{1\alpha_1(3)}\oplus\cdots\oplus A_{n\alpha_n(3)})$ implies $\gamma A_{i\alpha_i(3)}\subset A_{1\alpha_1(1)}\oplus\cdots\oplus A_{n\alpha_n(1)}$. Considering the projection of eJ to A_i, we obtain $\alpha_i(3)\geqslant\alpha_i(1)$ (note $A_i\approx\gamma A_i$

$\subset eJ$). Similarly $\alpha_i(3) \geqslant \alpha_i(2)$, and so $\alpha_i(3) = \underline{\alpha}_i$. Therefore $M_2 \approx \sum_{i=1}^{n} \oplus A_i/A_{i\bar{\alpha}i}$.

Corollary 11. *Let R be as above. Then the number of isomorphism classes of maximal submodules in a direct sum of (fixed) two hollow modules is at most three.*

REMARK. Assume in (11) that $\overline{eJ}$ is not square-free. Then we can show, by direct computation, the following fact:

Let $D = eR/E_1 \oplus eR/E_2$ be a direct sum of hollow modules eR/E_i. Then the number of isomorphism classes of maximal submodules in D at most three for any E_1 and E_2 if and only if one of the following occurs.

i) $m=2$, $A_1 \approx A_2$ and $|A_1| \leqslant 2$.

ii) $m=3$, $A_1 \approx A_2 \approx A_3$ and $|A_1| = 1$.

iii) $m=3$, $A_1 \approx A_2 \not\approx A_3$ and $|A_1| = 1$.

For example, $m=2$, $A_1 \approx A_2$ and $|A_1| \geqslant 3$: $D = eR/A_1 \oplus eR/(A_{12} \oplus A_{23})$. Then D contains the following maximal submodules:

$eJ/A_1 \oplus eR/(A_{12} \oplus A_{23})$, $eR/A_1 \oplus eJ/(A_{12} \oplus A_{23})$, $eR/A_{12} \oplus A_1/A_{13}$ and $eR/A_{13} \oplus A_1/A_{12}$ (cf. the proof of [6], Lemma 3). Therefore Corollary 11 characterizes almost left serial rings with (*, 2) and $\overline{eJ}$ being square-free.

Lemma 13. *Let R be a left serial ring. Assume that $\overline{eJ}$ is square-free and eJ is a direct sum of uniserial modules; $eJ = \sum_{i=1}^{m} \oplus A_i$. Let x be a unit in eRe and $xA_1 \neq A_1$. Then there exists d in eJe such that $(x+d)A_i = A_i$ for all i.*

Proof. Let p_i be the projection of eJ onto A_i, and $A_j = a_jR$ for $j=1, 2, \cdots, m$. Since $\overline{eJ}$ is square-free, $p_i x A_1 \subset \mathrm{J}(A_i)$ for $i \neq 1$. Hence $p_i x_l | A_1 = (d_i)_l$ for some d_i in $\mathrm{J}(A_i)$ by [7], Lemma 2. By assumption and [7], Lemma 4, only one d_i, say d_2, is non-zero, since $xA_1 \neq A_1$. Similarly for $j \neq 1, 2$ and $i \neq j$, $p_i x_l | A_j = (d_{ji})_l$ for some $d_{ji} \in \mathrm{J}(A_i)$. Then $d_{jk} = 0$ $(k \neq 2)$ by [7], Lemma 4. Assume $d_{j2} \neq 0$, and so $d_{j2} a_j \neq 0$. Since $d_2 \neq 0$, $0 \neq d_2 a_1 R \subset d_{j2} a_j R$ (or $d_{j2} a_j R \subset d_2 a_1 R$). Let $d_2 a_1 = d_{j2} a_j r$ (and $a_1 g = a_1$ and $rg = r$ for a primitive idempotent g). Hence there exist non-zero three elements $a_1 g$, $a_j r g$ and $d_2 a_1 g$. This is a contradiction to [7], Lemma 5. Hence $xA_j = A_j$ $(j \neq 1, 2)$. If $xA_2 \neq A_2$, we obtain again a contradiction to [7], Lemmas 2 and 4. Finally, since $0 \neq d_2 A_1 \subset A_2$, $d_2 A_j = 0$ for $j \neq 1$ from Lemma 9. Therefore $(x - d_2)A_i = A_i$ for all i.

From Proposition 10 we know the form of maximal submodules in $eR/E_1 \oplus eR/E_2$ up to isomorphism, provided (*, 2) holds and $\overline{eJ}$ is square-free. We shall show explicitly such an isomorphism. Let $eJ = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ be a direct sum of uniserial submodules. Put $E_i = A_{1\alpha_1(i)} \oplus A_{2\alpha_2(i)} \oplus \cdots \oplus A_{n\alpha_n(i)}$ for $i=1, 2$, where $A_{j\alpha_j(i)} \subset A_j$. Set $D = eR/E_1 \oplus eR/E_2$ and let M be a maximal submodule in D. Put $M^* = eR/(A_{1\underline{\alpha}_1} \oplus A_{2\underline{\alpha}_2} \oplus \cdots \oplus A_{n\underline{\alpha}_n}) \oplus A_1/A_{1\bar{\alpha}_1} \oplus A_2/A_{2\bar{\alpha}_2} \oplus \cdots$

$\oplus A_n/A_{n\bar{\alpha}_n}$ and $\bar{D}=D/J(D)\supset\bar{M}=M/J(D)$. We may assume $\bar{M}=(\bar{e}+\bar{e}\bar{k})\Delta$ (cf. [2], p. 93), where $\bar{k}\neq 0\in\Delta$ (R is basic). From Lemma 13, we may assume $kA_i=A_i$ for all i. We define a mapping φ: $M^*\to D$ by setting for $x\in eR$, $a_i\in A_i$,

$$(12)\qquad \begin{aligned}&\varphi(x+(A_{1\underline{\alpha}_1}\oplus\cdots\oplus A_{n\underline{\alpha}_n})+(a_1+A_{1\bar{\alpha}_1})+\cdots+(a_n+A_{n\bar{\alpha}_n}))\\ &=(x+a_1\delta_{\bar{\alpha}_1\alpha_1(1)}+\cdots+a_n\delta_{\bar{\alpha}_n\alpha_n(1)})+(A_{1\alpha_1(1)}\oplus\cdots\oplus A_{n\alpha_n(1)})\\ &\quad+(kx+a_1\delta'_{\bar{\alpha}_1\alpha_1(2)}+\cdots+a_n\delta'_{\bar{\alpha}_n\alpha_n(2)})+(A_{1\alpha_1(2)}\oplus\cdots\oplus A_{n\alpha_n(2)}),\end{aligned}$$

where the δ, δ' are Kronecker deltas such that $\delta'_{\bar{\alpha}_i\alpha_i(2)}=0$ provided $\alpha_i(1)=\alpha_i(2)$. Since $(A_{1\alpha_1(1)}\oplus\cdots\oplus A_{n\alpha_n(1)})\cap(A_{1\alpha_1(2)}\oplus\cdots\oplus A_{n\alpha_n(2)})=A_{1\underline{\alpha}_1}\oplus\cdots\oplus A_{n\underline{\alpha}_n}$, φ is an R-homomorphism. $(\varphi(M^*)+J(D))/J(D)=\bar{M}$ means $\varphi(M^*)\subset M$, and so $\varphi(M^*)=M$, since $|M^*|=|S|-1=|M|$.

Finally we shall give a property of a right artinian ring with (*, 2). Put $P=\sum_{k=1}^{i}\oplus A_k$ and $Q=\sum_{k=i+1}^{m}\oplus A_k$ in (11). Assume $\bar{A}_k\not\approx\bar{A}_{k'}$ for all k, k' such that $k\leqslant i<k'$.

Proposition 12. *Let R, P and Q be as above. Let L be a direct summand of eJ such that $L/LJ\approx P/PJ$. Then there exists a unit $\alpha=e+j$ ($j\in eJe$) such that $\alpha P=L$.*

Proof. From the assumption $L/LJ\approx P/PJ$ and Krull-Remak-Schmidt theorem, $L\approx P$. We apply the exchange property of L to $eJ=P\oplus Q$. Then $eJ=L\oplus P'\oplus Q'$, where $P'\subset P$ and $Q'\subset Q$. Since no one of indecomposable direct summands of L is isomorphic to any one in Q, $eJ=L\oplus Q$. Put $D=eR/P\oplus eR/L$. We shall employ the similar argument to the proof of Proposition 7. From [3], Lemma 3 and its proof, D contains a maximal submodule M such that $M=M_1\oplus M^*$ with $M_1\approx eR/K$, where $K=P\cap\alpha L$, $\alpha=e+j$. Now

$$(13)\qquad \mathrm{J}(D)=Q_1\oplus Q_2\,,\qquad \text{where } Q_i\approx Q\,.$$

Further, as in the proof of Proposition 7,

$\mathrm{J}(D)=\varphi(eJ/K)\oplus M^*$, φ: $eR/K\to D$ is the given injection. On the other hand, $\varphi((Q+K)/K)=Q_1(f)$, where f: $Q_1\to Q_2$. Hence

$$(14)\qquad \mathrm{J}(D)=\varphi((Q+K)/K)\oplus Q_2\quad\text{and}\quad\varphi(P/K)\subset Q_2\,.$$

Let p be the projection of $\mathrm{J}(D)$ onto Q_2 in (14), and x an element in $p(\mathrm{Soc}(M^*))\cap\varphi(P/K)$; $x=p(y)$ for some y in $\mathrm{Soc}(M^*)$. Then $y=(1-p)y+py$ and $(1-p)y\in\varphi((Q+K)/K)$. Hence $y\in\varphi(eJ/Q)\cap M^*=0$, and so $x=0$. Similarly, we know $p\,|\,\mathrm{Soc}(M^*)$ is a monomorphism. Hence

$$(15)\qquad p(M^*)\oplus(P/K)\subset Q_2\quad\text{and}\quad p(M^*)\approx M^*\,.$$

Now $|M|=|M_1|+|M^*|=|eR/K|+|M^*|=1+|Q|+|P/K|+|M^*|\leqslant$

$1+|Q|+|Q_2|=|D|-1=|M|$ from (15). Hence $p(M^*)\oplus\varphi(P/K)=Q_2=\sum_{k=i+1}^{m}\oplus A_k$, and so $\varphi(P/K)$ is isomorphic to a direct sum of some A_k $(k\geqslant i+1)$ by Krull-Remak-Schmidt theorem. On the other hand, $\bar{A}_s\not\approx\bar{A}_k$ for $s\leqslant i<k$, and hence $P=K=P\cap\alpha L$. Therefore $\alpha L=P$.

EXAMPLE 4. Let Q be the field of rationals. We regard $Q(\sqrt[4]{-1})$ $(=L)$ as a Q-space. Then we can directly compute that $V=Q\oplus Q(\sqrt{-1}+\sqrt[4]{-1})$ is not transferred to a standard submodule of $L=Q\oplus Q\alpha\oplus Q\alpha^2\oplus Q\alpha^3$ by a unit, where $\alpha=\sqrt[4]{-1}$. Hence

$$\begin{pmatrix} L & L \\ 0 & Q \end{pmatrix}$$

is a left serial ring with $(*, 2)$ by [3], Proposition 3, however $(0, V)$ is not transferred to a standard submodule of a decomposition $eJ=(0, Q)\oplus(0, Q\alpha)\oplus(0, Q\alpha^2)\oplus(0, Q\alpha^3)$, (cf. Lemma 10 and Proposition 9).

References

[0] M. Harada: *On lifting property on direct sums of hollow modules,* Osaka J. Math. **17** (1980), 783–791.

[1] ———: *Serial rings and direct decompositions,* J. Pure Appl. Algebra **41** (1984), 55–61.

[2] ———: *On maximal submodules of a finite direct sum of hollow modules* I, Osaka J. Math. **21** (1984), 649–670.

[3] ———: *On maximal submodules of a finite direct sum of hollow modules* III, Osaka J. Math. **22** (1985), 81–98.

[4] ———: *Note on left serial algebras,* Tsukuba J. Math. **10** (1986), 193–197.

[5] ———: *Generalizations of Nakayama ring* I, Osaka J. Math. **23** (1986), 181–200.

[6] ———: *Generalizations of Nakayama ring* II, Osaka J. Math. **23** (1986), 509–521.

[7] ——— and Y. Baba: *Generalizations of Nakayama ring* IV (*Left serial rings with* (*, 1)), Osaka J. Math. **24** (1987), 139–145.

[8] T. Sumioka: *Tachikawa's theorem on algebras of left colocal type,* Osaka J. Math. **21** (1984), 629–648.

[9] ———: *On Artinian rings of right local type,* to appear.

Department of Mathematics
Osaka City University
Sugimoto-3, Sumiyoshi-ku
Osaka 558, Japan